10. Calculation of π_1 for CW complexes (25 November)

We consider a path-connected CW complex K and use the notation $K^{(j)}$ for its j-skeleton.

Problem 1. Prove the following statements.

a) Any loop in K is homotopic to some loop in $K^{(1)}$.

b) **(Extra.)** Two loops in $K^{(1)}$ are homotopic in K if and only if they are homotopic in $K^{(2)}$. (This mean that a homotopy between these two loops can be deformed to the 2-skeleton).

c) **(Extra.)** Any continuous map of an n-dimensional compact CW complex L^n to K is homotopic to some map from L^n to $K^{(n)}$.

Problem 2. Consider a continuous map of some compact space X to K. Prove that the image of X is contained in finitely many cells of K.

Problem 3. Prove that $\pi_1(K) = \pi_1\big(K^{(2)}\big)$. (*Hint: Use Problems 1a, 1b.*)

Problem 4. (Generators.) Consider a wedge product of n circles. Let a_i be a (chosen arbitrary) loop corresponding to the i-th circle. Prove that any loop in the wedge product is homotopy equivalent to some finite concatenation of loops a_i and a_i^{-1}.

Problem 5. (Relations.) Let $K^{(1)}$ be a wedge product of n circles. Suppose that $K^{(2)}$ contains only one 2-dimensional cell D.

a) Find some relation for the group $\pi_1(K)$ related to the disc D. (*Hint: The answer can be obtained from observation of the boundary of D*).

b) **(Extra.)** Find $\pi_1\big(K^{(2)}\big)$. (*Hint: Here you need to prove that there is no other relations except the one obtained in a).*)

Problem 6. (Extra.) Describe $\pi_1(K)$ in terms of generations and relations.

Problem 7. Prove that any finite path-connected CW complex K is homotopy equivalent to a finite CW complex $\tilde{K}$ with $\tilde{K}^{(1)}$ being homeomorphic to a wedge product of a finite number of circles.

Problem 8. a) Find a CW complex for the sphere with g handles such that its 1-skeleton is the wedge product of $2g$ circles.

b) Calculate the fundamental group of the sphere with g handles.