C^1 ANALYSIS OF HERMITE SUBDIVISION SCHEMES ON MANIFOLDS*

CAROLINE MOOSMÜLLER†

Abstract. We propose two adaptations of linear Hermite subdivisions schemes to operate on manifold-valued data based on a Log-exp approach and on projection, respectively. Furthermore, we introduce a new proximity condition, which bounds the difference between a linear Hermite subdivision scheme and its manifold-valued analogue. Verification of this condition gives the main result: The manifold-valued Hermite subdivision scheme constructed from a C^1- convergent linear scheme is also C^1, if certain technical conditions are met.

Key words. Hermite subdivision, manifold subdivision, proximity, C^1 smoothness

AMS subject classifications. 65D05, 65D17, 41A05, 41A25

1. Introduction. In this paper we continue recent work on adapting linear subdivision schemes to operate on manifold-valued data. We are treating Hermite schemes, which are iterative methods for refining discrete point-vector data in order to obtain, in the limit, a function together with its derivatives.

Linear Hermite schemes are widely studied, see e.g. [4, 7, 8, 10, 13, 14, 21, 24, 25] and others. The C^1 analysis of linear Hermite schemes is often related to the convergence analysis of scalar-valued or vector-valued stationary subdivision schemes, which are easier to handle. This approach was first suggested by [13, 14]. In this paper we are particulary interested in the approach of [25]: The authors introduce the *Taylor operator* and the *Taylor scheme* for linear Hermite schemes, which play the same role as the forward difference operator and derived scheme, respectively, in the analysis of ordinary subdivision schemes [1, 11, 12]. We use the Taylor operator to define a *smoothness condition* for linear Hermite schemes (inspired by a similar condition in [29]), which is sufficient for C^1 convergence.

We present two adaptations of linear Hermite schemes to the manifold setting. The first one is based on the Log-exp approach of [20, 30]; the second one is a so-called projection analogue as suggested by [18, 31]. The C^1 convergence of these nonlinear Hermite schemes is established from their linear counterparts by means of a new proximity condition, following the ideas of [29]. Like almost all previous work on subdivision in general Riemannian manifolds we show convergence only for dense enough input data, and we show C^1 smoothness of all limits which exist. There has been some progress in showing convergence for any input data, see e.g. [15, 16, 17, 28].

Our results imply C^1 convergence of nonlinear analogues of linear Hermite schemes, in particular analogues of the examples listed in [24].

The paragraph above mentioned previous work on (non-Hermite) subdivision in manifolds, but there is also previous work on nonlinear Hermite subdivision: The paper [5] gives a detailed discussion of shape-preserving subdivision on the basis of linear Hermite schemes. Here the dependence of the limit function on the data is nonlinear, but this topic is different from the one studied in the present paper.

The paper is organised as follows. In Section 2 we give a short survey on linear Hermite subdivision and introduce notation used throughout the text. Section 3 presents our adaptation of linear Hermite schemes to the manifold setting. The Log-

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exp approach as well as the projection analogue are discussed in detail. The Taylor operator and the Taylor scheme are defined in Section 4, which, together with convergence results of Section 6.1, are important ingredients for the C^1 analysis of linear and nonlinear Hermite schemes. In Section 6 we prove C^1 convergence results, first in the linear Hermite case by means of a smoothness condition and then in the manifold case using a proximity condition. Section 7 concludes the paper by proving that a proximity condition applies to both the Log-exp analogue and the projection analogue.

We would like to mention that some results concerning linear Hermite subdivision in Sections 4 and 6 are already presented in the literature. We reprove these results in order to extend them more easily to the manifold-valued case.

2. Linear Hermite subdivision. We begin by introducing the notation and recalling some known facts about linear Hermite subdivision. The data to be refined by a linear Hermite subdivision scheme consists of a point-vector sequence, where we consider both the point and vector component to have values in the same real vector space V. In the course of our analysis we also encounter refinement of vector-data, which cannot be interpreted as points. To cover all cases we therefore use the notation f for elements in V^2, where f can be interpreted as point and vector, vector and vector, or as point and point. If the particular interpretation of the components is of interest, we use $\left(\begin{smallmatrix} p \\ v \end{smallmatrix}\right)$ for point-vector, $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$ for vector-vector, and $\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)$ for point-point.

Sequences of elements in V^2 are denoted by boldface letters, that is $\mathbf{f} = \{f(\alpha) : \alpha \in \mathbb{Z}\}$. If we are interested in the components, we use the notation $\left(\begin{smallmatrix} \mathbf{p} \\ \mathbf{v} \end{smallmatrix}\right) = \{\left(\begin{smallmatrix} p(\alpha) \\ v(\alpha) \end{smallmatrix}\right) : \alpha \in \mathbb{Z}\}$, etc. The space of all sequences with values in V^2 is denoted by $\ell(V^2)$.

We also consider the space $\ell(L(V)^{2\times 2})$, where $L(V)$ is the vector space of linear functions on V. Uppercase letters A are used for elements in $L(V)^{2\times 2}$ and boldface uppercase letters $\mathbf{A}$ for elements in $\ell(L(V)^{2\times 2})$.

A finitely supported sequence $\mathbf{A} \in \ell(L(V)^{2\times 2})$ is called *mask* and with it we associate a *linear subdivision operator* $S_{\mathbf{A}} : \ell(V^2) \to \ell(V^2)$ by

$$(S_{\mathbf{A}}\mathbf{f})(\alpha) = \sum_{\beta\in\mathbb{Z}} A(\alpha - 2\beta) f(\beta), \qquad \alpha \in \mathbb{Z}, \quad \mathbf{f} \in \ell(V^2). \tag{2.1}$$

We associate two types of linear schemes to a linear subdivision operator $S_{\mathbf{A}}$:

- A *linear Hermite subdivision scheme* is the procedure of constructing $\mathbf{f}^1, \mathbf{f}^2, \ldots$ from $\mathbf{f}^0 \in \ell(V^2)$ by the rule

  $$D^n \mathbf{f}^n = S_{\mathbf{A}}^n \mathbf{f}^0,$$

  where $D \in L(V)^{2\times 2}$ is the block-diagonal dilation operator

  $$D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

  Here a constant c is to be understood as $c\,\mathrm{id}_V$. This notation will be used throughout the text.
- The procedure of constructing $\mathbf{g}^1, \mathbf{g}^2, \ldots$ from $\mathbf{g}^0 \in \ell(V^2)$ by the rule

  $$\mathbf{g}^n = S_{\mathbf{A}}^n \mathbf{g}^0$$

  will be called a *linear point subdivision scheme*. This is because here the two components of $\mathbf{g}^n \in \ell(V^2)$ are not interpreted as a point-vector sequence, but as a point-point sequence.

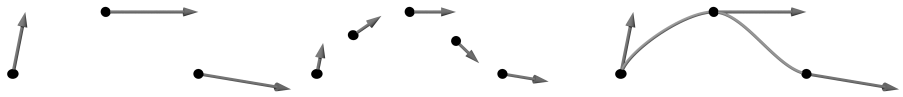

Fig. 1: The point subdivision scheme of Example 1

Note that if $\mathbf{f}^0 = \mathbf{g}^0$, then the two schemes are related via $D^n\mathbf{f}^n = \mathbf{g}^n$. Therefore, the refined sequences $\mathbf{f}^n$ and $\mathbf{g}^n$ only differ in the second component by the factor 2^n.

A linear Hermite subdivision scheme is called *interpolatory* if the mask satisfies $A(0) = D$ and $A(2\alpha) = 0$ for all $\alpha \in \mathbb{Z}\backslash 0$.

We always assume a linear subdivision operator $S_\mathbf{A}$ of a linear Hermite scheme to reproduce a degree 1 polynomial and its derivative

$$\mathbf{f} = \left\{ \begin{pmatrix} v + \alpha w \\ w \end{pmatrix} : \alpha \in \mathbb{Z} \right\} \quad \text{for } v, w \in V,$$

apart from a parameter shift. This means that we require that there is $\varphi \in \mathbb{R}$ such that the shifted sequence $\mathbf{f}_\varphi = \left\{ \begin{pmatrix} v+(\alpha+\varphi)w \\ w \end{pmatrix} : \alpha \in \mathbb{Z} \right\}$ satisfies

$$(S_\mathbf{A}\mathbf{f}_\varphi)(\alpha) = \begin{pmatrix} v+\frac{\alpha+\varphi}{2}w \\ \frac{1}{2}w \end{pmatrix}, \quad \text{for } v, w \in V, \alpha \in \mathbb{Z}.$$

This condition is called the *spectral condition* and is equivalent to the requirement that there is $\varphi \in \mathbb{R}$ such that both the constant sequence $\mathbf{k_0} = \{ \begin{pmatrix} w \\ 0 \end{pmatrix} : \alpha \in \mathbb{Z}\}$ and the linear sequence $\boldsymbol{\ell} = \{ \begin{pmatrix} (\alpha+\varphi)w \\ w \end{pmatrix} : \alpha \in \mathbb{Z}\}$ for $w \in V$ obey the rule

$$S_\mathbf{A}\mathbf{k_0} = \mathbf{k_0}, \quad S_\mathbf{A}\boldsymbol{\ell} = \tfrac{1}{2}\boldsymbol{\ell}. \tag{2.2}$$

The spectral condition has been introduced in [10] and is crucial for the C^1 analysis of linear Hermite subdivision schemes.

By means of the components of the mask $\mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$ the spectral condition reads

$$\sum_{\beta\in\mathbb{Z}} a(\alpha - 2\beta) = 1, \quad \sum_{\beta\in\mathbb{Z}} c(\alpha - 2\beta) = 0, \tag{2.3}$$

$$\sum_{\beta\in\mathbb{Z}} a(\alpha - 2\beta)\beta + b(\alpha - 2\beta) = \frac{1}{2}(\alpha - \varphi), \quad \sum_{\beta\in\mathbb{Z}} c(\alpha - 2\beta)\beta + d(\alpha - 2\beta) = \frac{1}{2}, \tag{2.4}$$

for some φ which indicates a parameter transform, and for all $\alpha \in \mathbb{Z}$.

Example 1. As a model example we consider one of the interpolatory linear Hermite subdivision schemes introduced in [24], see Figure 1. Its mask is given by

$$A(-1) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{1}{8} \end{pmatrix}, \quad A(0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad A(1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} \\ -\frac{3}{4} & -\frac{1}{8} \end{pmatrix}.$$

It is easy to see that it satisfies the spectral condition (2.2). It is well known that this scheme produces the piecewise cubic interpolant of given point-vector input data. ♢

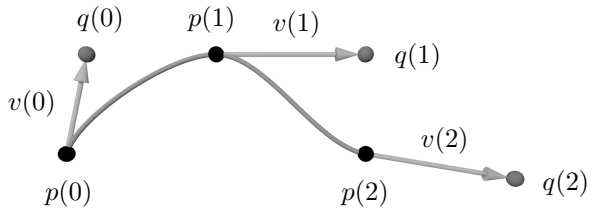

Fig. 2: Transformation of input data from point-vector data $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)$ to point-point data $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right)$, where $\mathbf{q} = \mathbf{p} + \mathbf{v}$.

2.1. Transformation of input data. To a subdivision operator $S_{\mathbf{A}}$ we associate a subdivision operator $S_{\tilde{\mathbf{A}}}$ by transformation of input data. We change the point-vector input data $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)$ to point-point input data $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right)$ via the transformation $\mathbf{q} = \mathbf{p} + \mathbf{v}$, see Figure 2. Hence $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right) = \mathcal{T}\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)$, where $\mathcal{T} = \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$. Then we let

$$S_{\tilde{\mathbf{A}}}\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right) = \mathcal{T} S_{\mathbf{A}} \mathcal{T}^{-1}\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right),$$

i.e., the mask $\tilde{\mathbf{A}}$ is computed from the mask $\mathbf{A}$ by the relation

$$\tilde{A}(\alpha) = \mathcal{T} A(\alpha) \mathcal{T}^{-1},$$

for $\alpha \in \mathbb{Z}$.

Note that $\mathbf{A}$ satisfies (2.3) if and only if $\tilde{\mathbf{A}} = \left(\begin{smallmatrix}\tilde{\mathbf{a}} & \tilde{\mathbf{b}}\\ \tilde{\mathbf{c}} & \tilde{\mathbf{d}}\end{smallmatrix}\right)$ satisfies

$$\sum_{\beta\in\mathbb{Z}} \tilde{a}(\alpha - 2\beta) + \tilde{b}(\alpha - 2\beta) = 1, \quad \sum_{\beta\in\mathbb{Z}} \tilde{c}(\alpha - 2\beta) + \tilde{d}(\alpha - 2\beta) = 1. \tag{2.5}$$

This is the reproduction property $S_{\tilde{\mathbf{A}}}\mathbf{k_2} = \mathbf{k_2}$, where $\mathbf{k_2}$ is the constant sequence $\mathbf{k_2} = \{\left(\begin{smallmatrix}w\\ w\end{smallmatrix}\right) : \alpha \in \mathbb{Z}\}$ for $w \in V$. In Section 3 the subdivision operator $S_{\tilde{\mathbf{A}}}$ as well as property (2.5) will be useful.

In Example 1, the mask $\tilde{\mathbf{A}}$ associated to $\mathbf{A}$ is given by

$$\tilde{A}(-1) = \begin{pmatrix} \frac{5}{8} & -\frac{1}{8} \\ \frac{3}{2} & -\frac{1}{4} \end{pmatrix}, \quad \tilde{A}(0) = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \tilde{A}(1) = \begin{pmatrix} \frac{3}{8} & \frac{1}{8} \\ -\frac{1}{4} & 0 \end{pmatrix}. \tag{2.6}$$

3. Hermite subdivision on manifolds. This section presents two methods for deriving nonlinear Hermite subdivision schemes from linear ones. The first one is an intrinsic construction. It works on any manifold that has an exponential mapping. The main instances of manifolds we consider here are finite-dimensional Riemannian manifolds, Lie groups and symmetric spaces. The second method invokes a projection and can be defined on submanifolds. We start by generalising the notions of *subdivision operator*, *point subdivision scheme*, and *Hermite subdivision scheme*.

Definition 2 (Subdivision operator). *A subdivision operator is a map U which takes as argument a sequence $\mathbf{f}$ and produces a new sequence $U\mathbf{f}$. It must satisfy*

(i) $L^2U = UL$, where L is the left shift operator, and

(ii) U has compact support, that is, there exists $N \in \mathbb{N}$ such that both $U\mathbf{f}(2\alpha)$ and $U\mathbf{f}(2\alpha+1)$ only depend on $f(\alpha-N), \dots, f(\alpha+N)$ for all $\alpha \in \mathbb{Z}$ and sequences $\mathbf{f}$.

Note that a linear subdivision operator with finitely supported mask satisfies these conditions.

While for linear subdivision schemes both point-point and point-vector data can be taken from the same vector space V^2, this is no longer the case in the manifold setting. On a manifold M, point-point data is sampled from the space M^2. For point subdivision we therefore consider the associated sequence space $\ell(M^2)$.

DEFINITION 3 (Point subdivision scheme). *Let U be a subdivision operator which takes arguments in $\ell(M^2)$ and again produces a sequence in M^2. We associate a point subdivision scheme to U:*

A point subdivision scheme is the procedure of constructing $\mathbf{g}^1, \mathbf{g}^2, \dots$ from input data $\mathbf{g}^0 \in \ell(M^2)$ by the rule

$$\mathbf{g}^n = U^n \mathbf{g}^0.$$

The derivative in a point of a manifold-valued curve $c : \mathbb{R} \to M$ lies in a tangent space of M, namely $c'(t) \in T_{c(t)}M$. Therefore tangent vectors serve as point-vector input data for Hermite subdivision. Let $TM = \bigcup_{p \in M} T_pM$ be the tangent bundle of M and $\ell(TM)$ its associated sequence space. We consider an element of $\ell(TM)$ as a point-vector pair $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)$, where $\mathbf{p}$ is a sequence in M and $\mathbf{v}$ a sequence in the appropriate tangent space, that is $v(\alpha) \in T_{p(\alpha)}M$ for all $\alpha \in \mathbb{Z}$ (strictly speaking $v(\alpha)$ carries the information which tangent space it is contained in, but we want to retain the analogy to the linear case). In this notation let $D : \ell(TM) \to \ell(TM)$ be the dilation operator

$$\begin{pmatrix}\mathbf{p}\\ \mathbf{v}\end{pmatrix} \mapsto \begin{pmatrix}\mathbf{p}\\ \frac{1}{2}\mathbf{v}\end{pmatrix},$$

which is an analogue of the block-diagonal operator D defined in Section 2.

DEFINITION 4 (Hermite subdivision scheme). *Let U be a subdivision operator which takes arguments in $\ell(TM)$ and again produces a sequence in TM. We associate an Hermite subdivision scheme to U:*

An Hermite subdivision scheme is the procedure of constructing $\mathbf{f}^1, \mathbf{f}^2, \dots$ from input data $\mathbf{f}^0 \in \ell(TM)$ by the rule

$$D^n \mathbf{f}^n = U^n \mathbf{f}^0.$$

An Hermite subdivision scheme is called *interpolatory* if $(U\mathbf{f})(2\alpha) = (D\mathbf{f})(\alpha)$ for all $\mathbf{f}$ and $\alpha \in \mathbb{Z}$. Note that if U is a linear subdivision operator, this is equivalent to the definition given in Section 2.

3.1. The Log-exp analogue of a linear subdivision scheme. The idea of using the exponential mapping for transferring linear operations to manifold-valued data has been proposed by [6, 26]. Analysis of subdivision schemes has been done by [19, 20, 30, 32, 33] and others.

Constructing a subdivision rule by the Log-exp method requires operations $q = p \oplus v$ and $v = q \ominus p$, which are similar to point-vector addition and the difference vector of points. We recall their definition which is found e.g. in [30].

Let N be a Riemannian manifold. For $p \in N$ and a tangent vector $v \in T_pN$, the exponential mapping $\exp_p(v)$ gives the endpoint of the geodesic line of length $\|v\|$

emanating from p in direction v. It is a local diffeomorphism around $0 \in T_pN$ and hence posses a local inverse $\exp_p^{-1}$. We define

$$p \oplus v = \exp_p(v) \quad \text{and} \quad q \ominus p = \exp_p^{-1}(q). \tag{3.1}$$

While $\oplus$ is always smooth, and is often defined for all tangent vectors v (for example, this is the case on complete Riemannian manifolds, see [22, Theorem 10.3]), $\ominus$ in general is definable as a smooth mapping only for p, q close to each other. Since the convergence and smoothness analysis of Section 6.3 only considers "dense enough" input data, we may assume that $\ominus$ is always smooth.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra, which is the tangent space at the identity element. By $\exp : \mathfrak{g} \to G$ we denote the exponential mapping in the group [22, §2, Chapter II]. Define $\exp_p : T_pG \to G$ by

$$\exp_p(v) = p \exp(p^{-1} \cdot v), \quad p \in G, v \in T_pG.$$

Here $p^{-1} \cdot v$ is the transfer of v to the vector space $\mathfrak{g}$ by left multiplication with p^{-1}. The map $\exp_p$ is a local diffeomorphism, hence possesses a local inverse $\exp_p^{-1}$. Define

$$p \oplus v = \exp_p(v) \quad \text{and} \quad q \ominus p = \exp_p^{-1}(q), \quad \text{for } p, q \in G \text{ and } v \in T_pG. \tag{3.2}$$

Note that $\exp_p^{-1}(q) = p \cdot \exp^{-1}(p^{-1}q)$. On Lie groups $\oplus$ is globally smooth, while this is generally not the case for $\ominus$.

Definition (3.2) is invariant with respect to left translations in G, that is, for $g, p, q \in G$ and $v \in T_pG$, we have $gp \oplus (g \cdot v) = g(p \oplus v)$ and $(gq) \ominus (gp) = g \cdot (q \ominus p)$.

We follow the construction of [30] to define an exponential mapping on symmetric spaces. Let $X = G/K$ be a symmetric space, which is a Lie group G factorised by a closed subgroup K meeting certain conditions which ensure that an exponential mapping can be defined. By $\exp_p : T_pX \to X$ we denote its exponential mapping. We define

$$p \oplus v = \exp_p(v) \quad \text{and} \quad q \ominus p = \exp_p^{-1}(q), \tag{3.3}$$

where $p, q \in X$ and $v \in T_pX$. As in the Lie group case, $\oplus$ is globally smooth, while $\ominus$ is a diffeomorphism in a neighbourhood of p. Furthermore, (3.3) is invariant with respect to the action of the group G.

Let M be a Riemannian manifold, Lie group or symmetric space. For $p, q \in M$ we define their *mean* by

$$\operatorname{mean}(p, q) := p \oplus \frac{1}{2}(q \ominus p).$$

Again in the case of Lie groups and symmetric spaces we have invariance of the mean with respect to the action of the group, i.e. $\operatorname{mean}(gp, gq) = g \operatorname{mean}(p, q)$.

In the following, we derive a subdivision operator $U_{\mathbf{A}}$ on TM from a linear subdivision operator $S_{\mathbf{A}}$ which satisfies the spectral condition (2.2).

We write $S_{\mathbf{A}}$ in the form

$$S_{\mathbf{A}} \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} (\alpha) = \sum_{\beta \in \mathbb{Z}} \begin{pmatrix} a(\alpha - 2\beta) & b(\alpha - 2\beta) \\ c(\alpha - 2\beta) & d(\alpha - 2\beta) \end{pmatrix} \begin{pmatrix} p(\beta) \\ v(\beta) \end{pmatrix},$$

where $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)$ is point-vector input data. In Section 2.1 we associated a linear subdivision operator $S_{\tilde{\mathbf{A}}}$ to $S_{\mathbf{A}}$ by changing the input data from point-vector form to point-point form:

$$S_{\tilde{\mathbf{A}}}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}(\alpha) = \sum_{\beta\in\mathbb{Z}}\begin{pmatrix}\tilde{a}(\alpha-2\beta) & \tilde{b}(\alpha-2\beta)\\ \tilde{c}(\alpha-2\beta) & \tilde{d}(\alpha-2\beta)\end{pmatrix}\begin{pmatrix}p(\beta)\\ q(\beta)\end{pmatrix} = \begin{pmatrix}\sum_{\beta\in\mathbb{Z}}\tilde{a}(\alpha-2\beta)p(\beta)+\tilde{b}(\alpha-2\beta)q(\beta)\\ \sum_{\beta\in\mathbb{Z}}\tilde{c}(\alpha-2\beta)p(\beta)+\tilde{d}(\alpha-2\beta)q(\beta)\end{pmatrix}, \tag{3.4}$$

where $\mathbf{q}=\mathbf{p}+\mathbf{v}$ and $\tilde{\mathbf{A}} = \left(\begin{smallmatrix}\tilde{\mathbf{a}} & \tilde{\mathbf{b}}\\ \tilde{\mathbf{c}} & \tilde{\mathbf{d}}\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\\ -1 & 1\end{smallmatrix}\right)$. By using the reproduction property (2.5), the operator (3.4) can be equivalently written as

$$S_{\tilde{\mathbf{A}}}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}(\alpha) = \begin{pmatrix}m_0(\alpha)+\sum_{\beta\in\mathbb{Z}}\tilde{a}(\alpha-2\beta)(p(\beta)-m_0(\alpha))+\tilde{b}(\alpha-2\beta)(q(\beta)-m_0(\alpha))\\ m_1(\alpha)+\sum_{\beta\in\mathbb{Z}}\tilde{c}(\alpha-2\beta)(p(\beta)-m_1(\alpha))+\tilde{d}(\alpha-2\beta)(q(\beta)-m_1(\alpha))\end{pmatrix}, \tag{3.5}$$

for any base point sequences $\mathbf{m_0}, \mathbf{m_1}$. This definition is useful for transferring the linear refinement rule to the manifold setting, since it consists of point-vector addition and point-point subtraction. As we have seen in the beginning of this section, these operations are defined on manifolds. We therefore use (3.5) to define a subdivision operator $U_{\mathbf{A}}$ for sequences of tangent vectors:

Consider input data $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right) \in \ell(TM)$ and define $\mathbf{r}\in\ell(M)$ by $\mathbf{r} = \{p(\alpha)\oplus v(\alpha) : \alpha\in\mathbb{Z}\}$. As to base point sequences, the following possibilities are an obvious choice:

(i) $\mathbf{m_0}=\mathbf{p}$ or $\mathbf{m_0} = \{\mathrm{mean}(p(\alpha), p(\alpha+1)) : \alpha\in\mathbb{Z}\}$,

(ii) $\mathbf{m_1}=\mathbf{r}$, $\mathbf{m_1} = \{\mathrm{mean}(r(\alpha), r(\alpha+1)) : \alpha\in\mathbb{Z}\}$ or $\mathbf{m_1}=\mathbf{m_0}$.

Define $\mathbf{s}_1, \mathbf{r}_1 \in \ell(M)$ using (3.5):

$$\begin{pmatrix}s_1(\alpha)\\ r_1(\alpha)\end{pmatrix} = \begin{pmatrix}m_0(\alpha)\oplus\sum_{\beta\in\mathbb{Z}}\tilde{a}(\alpha-2\beta)(p(\beta)\ominus m_0(\alpha))+\tilde{b}(\alpha-2\beta)(r(\beta)\ominus m_0(\alpha)),\\ m_1(\alpha)\oplus\sum_{\beta\in\mathbb{Z}}\tilde{c}(\alpha-2\beta)(p(\beta)\ominus m_1(\alpha))+\tilde{d}(\alpha-2\beta)(r(\beta)\ominus m_1(\alpha))\end{pmatrix},$$

for $\alpha\in\mathbb{Z}$. Therefore, the operator $U_{\mathbf{A}}$ defined by

$$U_{\mathbf{A}}\begin{pmatrix}\mathbf{p}\\ \mathbf{v}\end{pmatrix}(\alpha) = \begin{pmatrix}s_1(\alpha)\\ r_1(\alpha)\ominus s_1(\alpha)\end{pmatrix}, \tag{3.6}$$

is a subdivision operator on TM. In Section 6 we show that the Hermite scheme $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right), D^{-1}U_{\mathbf{A}}\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right), D^{-2}U_{\mathbf{A}}^2\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right), \ldots$ converges to a curve and its derivative.

In the case of symmetric spaces and Lie groups, $U_{\mathbf{A}}$ is invariant with respect to the group action.

Remark 5. While for the smoothness analysis of manifold-valued Hermite schemes it does not matter if we use base point sequences (i) or (ii) from above, there is a preferred choice if the linear Hermite scheme has additional properties, such as symmetry. A linear Hermite scheme $S_{\mathbf{A}}$ is called *symmetric*, if its mask satisfies $A(\gamma-\alpha) = A(\gamma+\alpha)$ (case 1) or $A(\gamma-\alpha) = A(\gamma+1-\alpha)$ (case 2), for a fixed index γ and for all $\alpha\in\mathbb{Z}$. Then the following relation is fullfilled

$$S_{\mathbf{A}}M = L^2MS_{\mathbf{A}}, \tag{3.7}$$

where L is the left shift operator and M operates on sequences $\mathbf{q}$ by $(M\mathbf{q})(\alpha) = q(-\alpha)$ for all $\alpha\in\mathbb{Z}$. A natural question addresses the preservation of symmetry: Is the

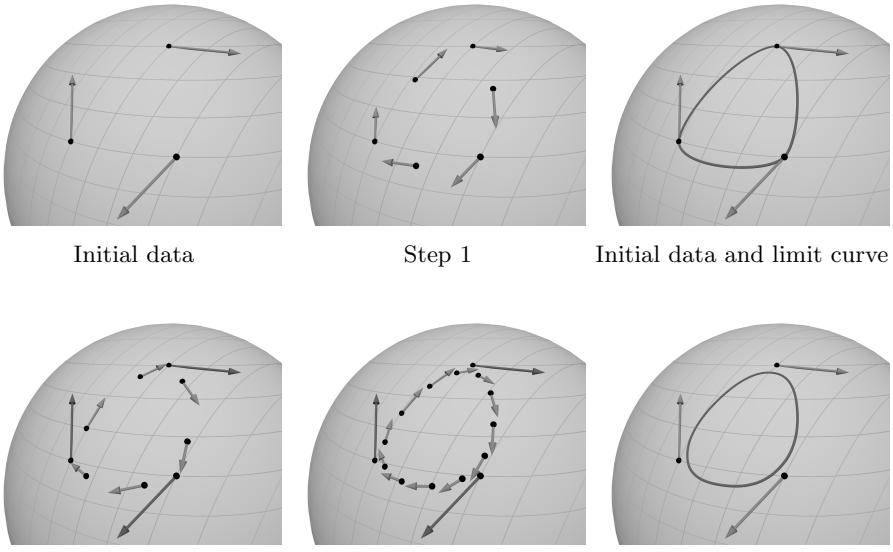

Fig. 3: Log-exp analogue of an interpolatory Hermite scheme (first row) vs. a non-interpolatory one (second row). Both schemes are applied to the same initial data on the sphere. *First row*: Initial data, first step and limit curve of the interpolatory scheme presented in Example 6. *Second row*: Initial data together with the first step, second step and limit curve of a non-interpolatory scheme. This scheme is the log-exp analogue of a linear Hermite scheme constructed as the de Rham transform [9] of a scheme proposed by [24]. The mask is the special case $\lambda = -\frac{1}{8}, \mu = \frac{1}{2}$ in [9, Section 4].

manifold-valued analogue of a symmetric linear scheme also symmetric? That is, does it satisfy (3.7)? Depending on whether we consider case 1 or case 2, the symmetry is preserved if the base points are chosen as the data points ($\mathbf{m_0} = \mathbf{p}, \mathbf{m_1} = \mathbf{r}$) or as the geodesic midpoints ($\mathbf{m_0} = \{\text{mean}(p(\alpha), p(\alpha+1)) : \alpha \in \mathbb{Z}\}$, $\mathbf{m_1} = \{\text{mean}(r(\alpha), r(\alpha+1)) : \alpha \in \mathbb{Z}\}$), respectively. For more details, including examples, see [20]. $\diamondsuit$

Example 6. Consider the 2-dimensional sphere S^2 in $\mathbb{R}^3$. It is a Riemannian manifold with metric induced from the ambient space $\mathbb{R}^3$. The operations $\oplus, \ominus$ on $p, q \in S^2$ and $v \in T_pS^2 = \{w \in \mathbb{R}^3 : \langle w, p\rangle = 0\}$ are given by

$$p \oplus v = \cos(\|v\|)p + \sin(\|v\|)\frac{v}{\|v\|}, \quad q \ominus p = \arccos(\langle q, p\rangle)\frac{q - \langle q, p\rangle p}{\|q - \langle q, p\rangle p\|}. \tag{3.8}$$

We consider the Log-exp analogue of the linear interpolatory Hermite scheme introduced in Example 1 on the sphere. In Figure 3 the input data, the first step and the limit curve of the Log-exp analogue are shown.

For input data $\binom{\mathbf{p}}{\mathbf{v}} \in \ell(TS^2)$ and $\mathbf{r} = \{p(\alpha) \oplus v(\alpha) : \alpha \in \mathbb{Z}\}$, define the base point

sequences $\mathbf{m_0}, \mathbf{m_1}$ by

$$m_0(2\alpha) = m_0(2\alpha+1) = \operatorname{mean}(p(\alpha+1), p(\alpha)) = p(\alpha+1) \oplus \Big(\frac{1}{2} p(\alpha) \ominus p(\alpha+1)\Big),$$
$$m_1(2\alpha) = m_1(2\alpha+1) = \operatorname{mean}(r(\alpha+1), r(\alpha)) = r(\alpha+1) \oplus \Big(\frac{1}{2} r(\alpha) \ominus r(\alpha+1)\Big).$$

Furthermore, we introduce the sequences $\mathbf{v}^0, \mathbf{v}^1, \mathbf{w}^0, \mathbf{w}^1$:

$$v^0_{\alpha,\beta} = p(\alpha) \ominus m_0(\beta), \qquad v^1_{\alpha,\beta} = p(\alpha) \ominus m_1(\beta),$$
$$w^0_{\alpha,\beta} = r(\alpha) \ominus m_0(\beta), \qquad w^1_{\alpha,\beta} = r(\alpha) \ominus m_1(\beta),$$

for $\alpha, \beta \in \mathbb{Z}$. Therefore, the Log-exp version of the Hermite subdivision scheme $\left(\begin{smallmatrix} \mathbf{s}_1 \\ \mathbf{r}_1 \ominus \mathbf{s}_1 \end{smallmatrix}\right) = U_{\mathbf{A}} \left(\begin{smallmatrix} \mathbf{p} \\ \mathbf{v} \end{smallmatrix}\right)$ is given by the sequences

$$s_1(2\alpha) = p(\alpha),$$
$$r_1(2\alpha) = m_1(2\alpha) \oplus \Big(\frac{1}{2} v^1_{\alpha,2\alpha} + \frac{1}{2} w^1_{\alpha,2\alpha}\Big),$$
$$s_1(2\alpha+1) = m_0(2\alpha+1) \oplus \Big(\frac{5}{8} v^0_{\alpha+1,2\alpha+1} + \frac{3}{8} v^0_{\alpha,2\alpha+1} - \frac{1}{8} w^0_{\alpha+1,2\alpha+1} + \frac{1}{8} w^0_{\alpha,2\alpha+1}\Big),$$
$$r_1(2\alpha+1) = m_1(2\alpha+1) \oplus \Big(\frac{3}{2} v^1_{\alpha+1,2\alpha+1} - \frac{1}{4} v^1_{\alpha,2\alpha+1} - \frac{1}{4} w^1_{\alpha+1,2\alpha+1}\Big),$$

where the coefficients are taken from Equation (2.6).

The sphere is also a symmetric space, namely $S^2 = \mathrm{SO}_3/\mathrm{SO}_2$. The exponential map of S^2 as a Riemannian manifold coincides with the exponential map of S^2 as a symmetric space [23, Chapter XI, Theorem 3.3] and therefore the above calculations are valid in both cases. Furthermore, it can be checked immediately that (3.8) is invariant with respect to SO_3, which implies the invariance of $\mathbf{s}_1$ respectively $\mathbf{r}_1$. $\diamondsuit$

3.2. The projection analogue of a linear subdivision scheme. Projections onto submanifolds have been used to create subdivision schemes for manifold-valued data by various authors, see [29, 27, 18, 31]. We generalise this method to the Hermite case.

Let M be a submanifold of Euclidean space $\mathbb{R}^n$. A *projection* π is a smooth mapping onto M defined in a neighbourhood of M such that $\pi|_M = \mathrm{id}$. Its derivative is denoted by $d\pi$.

Let $S_{\mathbf{A}}$ be a linear subdivision operator. Define a subdivision operator $U_{\mathbf{A}}$ that operates on $\ell(TM)$ by

$$U_{\mathbf{A}}\mathbf{f}(\alpha) = d\pi(S_{\mathbf{A}}\mathbf{f}(\alpha)), \quad \text{for } \alpha \in \mathbb{Z}.$$

In Section 6 we show that the sequence of refined data $\mathbf{f}, D^{-1}U_{\mathbf{A}}\mathbf{f}, D^{-2}U^2_{\mathbf{A}}\mathbf{f}, \ldots$ converges to a curve and its derivative.

4. Derived schemes and factorisation results. For the convergence and C^1 analysis of linear subdivision schemes, the derived schemes with respect to the forward difference operator Δ are of importance, see for example [1, 12]. In the Hermite case, the Taylor operator T is the natural analogue of Δ, see [25]. It is introduced as follows:

DEFINITION 7. *We define operators T, Δ_0, Δ_1 acting on $\ell(V^2)$ in block operator notation by*

$$T := \begin{pmatrix} \Delta & -1 \\ 0 & 1 \end{pmatrix}, \quad \Delta_0 := \begin{pmatrix} \Delta & 0 \\ 0 & 1 \end{pmatrix}, \quad \Delta_1 := \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix},$$

where Δ is the forward difference operator $(\Delta \mathbf{f})(\alpha) = f(\alpha+1) - f(\alpha)$. T is called Taylor operator.

The next proposition considers derived schemes of $S_{\mathbf{A}}$ with respect to the operators $\Delta_0, T, \Delta_1 T$. In order to keep an overview of these derived schemes, we introduce the notation $\partial_T S_{\mathbf{A}}$ to mean the derived scheme of $S_{\mathbf{A}}$ with respect to the operator T.

PROPOSITION 8. *Let $\mathbf{A}$ be a mask that satisfies the spectral condition (2.2). Then there exist derived schemes of $S_{\mathbf{A}}$ with respect to $\Delta_0, T, \Delta_1 T$, i.e. subdivision operators $\partial_T S_{\mathbf{A}}, \partial_{\Delta_0} S_{\mathbf{A}}$ and $\partial_{\Delta_1 T} S_{\mathbf{A}}$ which satisfy*
(i) $2TS_{\mathbf{A}} = (\partial_T S_{\mathbf{A}})T$,
(ii) $\Delta_0 S_{\mathbf{A}} = (\partial_{\Delta_0} S_{\mathbf{A}})\Delta_0$ *and* $2\Delta_1 T S_{\mathbf{A}} = (\partial_{\Delta_1 T} S_{\mathbf{A}})\Delta_1 T$.
Furthermore, if we define a mask $\mathbf{E}$ by

$$E(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E(-1) = E(1) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad E(\alpha) = 0 \quad \text{for } \alpha \neq -1, 0, 1,$$

then there exist subdivision operators $S_{\mathbf{B}}, S_{\mathbf{C}}$ such that
(iii) $S_{\mathbf{A}} - S_{\mathbf{E}} = S_{\mathbf{B}}\Delta_0$,
(iv) $\partial_T S_{\mathbf{A}} - S_{\mathbf{E}} = S_{\mathbf{C}}\Delta_1$.

Part (i) is a result of [25]. The second factorisation of (ii) is also proved in [25] (in this paper $\Delta_1 T$ is called *complete Taylor operator*), but will follow from results we derive here. The proof of Proposition 8 can be found at the end of this section.

The essential ingredient in the proof is the reproduction of constants property (2.2): $S_{\mathbf{A}}\mathbf{k_0} = \mathbf{k_0}$, where $\mathbf{k_0}$ is the constant sequence $\mathbf{k_0} = \{\binom{w}{0} : \alpha \in \mathbb{Z}\}$ for $w \in V$. The operator $\partial_T S_{\mathbf{A}}$ satisfies a different reproduction property, namely

LEMMA 9. *Let $\mathbf{A}$ be a mask that satisfies the spectral condition (2.2) and let $\partial_T S_{\mathbf{A}}$ be the derived scheme of $S_{\mathbf{A}}$ w.r.t. T (which exists by [25]). Denote by $\mathbf{k_1}$ the constant seqeunce $\mathbf{k_1} = \{\binom{0}{w} : \alpha \in \mathbb{Z}\}$ for $w \in V$. Then we have*

$$(\partial_T S_{\mathbf{A}})\mathbf{k_1} = \mathbf{k_1}. \tag{4.1}$$

Proof. By property (2.2), there exists $\varphi \in \mathbb{R}$ such that $\boldsymbol{\ell} = \{\binom{(\alpha+\varphi)w}{w} : \alpha \in \mathbb{Z}\}$ is reproduced: $S_{\mathbf{A}}\boldsymbol{\ell} = \frac{1}{2}\boldsymbol{\ell}$. Furthermore for all $\alpha \in \mathbb{Z}$

$$(T\boldsymbol{\ell})(\alpha) = T\binom{(\alpha+t)w}{w} = \begin{pmatrix} \Delta & -1 \\ 0 & 1 \end{pmatrix}\binom{(\alpha+t)w}{w} = \binom{(\alpha+1+t)w-(\alpha+t)w-w}{w} = \binom{0}{w},$$

hence $T\boldsymbol{\ell} = \mathbf{k_1}$. This implies $(\partial_T S_{\mathbf{A}})\mathbf{k_1} = (\partial_T S_{\mathbf{A}})T\boldsymbol{\ell} = 2TS_{\mathbf{A}}\boldsymbol{\ell} = T\boldsymbol{\ell} = \mathbf{k_1}$ □

Note that the mask $\mathbf{E}$ defined in Proposition 8 reproduces both $\mathbf{k_0}$ and $\mathbf{k}_1$, since

$$S_{\mathbf{E}}\mathbf{k_0}(2\alpha) = E(0)\binom{w}{0} = k_0(2\alpha), \quad S_{\mathbf{E}}\mathbf{k_0}(2\alpha+1) = (E(-1)+E(1))\binom{w}{0} = k_0(2\alpha+1)$$

and similarly for $\mathbf{k}_1$. Therefore, in order to prove Proposition 8 we are going to study masks which reproduce either $\mathbf{k}_0$ or $\mathbf{k}_1$ or both.

The mask of a linear subdivision scheme is often analysed by considering its symbol, which is a vector-valued or matrix-valued Laurent polynomial. We recall some basic facts concerning symbols, as they frequently appear in the sequel.

To a sequence $\mathbf{f}$ we associate its *symbol* by

$$\mathbf{f}^*(z) := \sum_{\alpha \in \mathbb{Z}} f(\alpha) z^\alpha.$$

By (2.1) we have the following identities:

$$(\Delta\mathbf{f})^*(z) = (z^{-1}-1)\mathbf{f}^*(z), \quad (S_\mathbf{A}\mathbf{f})^*(z) = \mathbf{A}^*(z)\mathbf{f}^*(z^2), \tag{4.2}$$

where $\mathbf{A}$ has finite support. The operators of Definition 7, acting from the left on symbols $\mathbf{f}^*(z)$, are given by

(4.3)
$$\Delta_0^*(z) = \begin{pmatrix} z^{-1}-1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Delta_1^*(z) = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1}-1 \end{pmatrix}, \quad T^*(z) = \begin{pmatrix} z^{-1}-1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, we have the following well known result for a finitely supported symbol:

$$\sum_{\beta\in\mathbb{Z}} f(\alpha-2\beta) = 0 \quad \forall\alpha\in\mathbb{Z} \quad \text{implies} \quad \mathbf{f}^*(z) = (z^{-2}-1)\mathbf{h}^*(z), \tag{4.4}$$

for a symbol $\mathbf{h}^*(z)$. This follows from the fact that $\sum_{\beta\in\mathbb{Z}} f(\alpha-2\beta)=0$ for all $\alpha\in\mathbb{Z}$ implies $\sum_{\beta\in\mathbb{Z}} f(1-2\beta)=0$ and $\sum_{\beta\in\mathbb{Z}} f(2\beta)=0$. Therefore -1 as well as 1 are zeros of $\mathbf{f}^*(z)$. This yields that $1-z^2$ is a factor of $\mathbf{f}^*(z)$.

We say that a mask $\mathbf{M}$ *annihilates* $\mathbf{k_0}$ (resp. $\mathbf{k_1}$) if $S_\mathbf{M}\mathbf{k_0}=0$ (resp. $S_\mathbf{M}\mathbf{k_1}=0$).

PROPOSITION 10. *Let $\mathbf{M}$ be a mask that annihilates $\mathbf{k_0}$ (resp. $\mathbf{k_1}$). Then there exists a linear subdivision operator $S_\mathbf{N}$ such that*

$$S_\mathbf{M} = S_\mathbf{N}\Delta_0 \quad (\textit{resp. } S_\mathbf{M} = S_\mathbf{N}\Delta_1).$$

Proof. We prove the case where $\mathbf{k_0}$ is annihilated, the other case is analogous. In terms of symbols, we want to prove that $(S_\mathbf{M}\mathbf{f})^*(z) = (S_\mathbf{N}\Delta_0\mathbf{f})^*(z)$. By (4.2) and (4.3), this is equivalent to

$$\mathbf{M}^*(z)\mathbf{f}^*(z^2) = \mathbf{N}^*(z)(\Delta_0\mathbf{f})^*(z^2) = \mathbf{N}^*(z)\begin{pmatrix} z^{-2}-1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{f}^*(z^2).$$

Let $\mathbf{M} = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right)$. Annihilation of $\mathbf{k_0}$ implies

$$\sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)=0 \quad \text{and} \quad \sum_{\beta\in\mathbb{Z}} c(\alpha-2\beta)=0.$$

Thus (4.4) implies that there exist symbols $\tilde{\mathbf{a}}^*(z), \tilde{\mathbf{c}}^*(z)$ such that $\mathbf{a}^*(z) = (z^{-2}-1)\tilde{\mathbf{a}}^*(z)$ and $\mathbf{c}^*(z) = (z^{-2}-1)\tilde{\mathbf{c}}^*(z)$. Therefore

$$\mathbf{M}^*(z) = \begin{pmatrix} \mathbf{a}^*(z) & \mathbf{b}^*(z) \\ \mathbf{c}^*(z) & \mathbf{d}^*(z) \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{a}}^*(z) & \mathbf{b}^*(z) \\ \tilde{\mathbf{c}}^*(z) & \mathbf{d}^*(z) \end{pmatrix}\begin{pmatrix} z^{-2}-1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The result follows with $\mathbf{N}^*(z) := \left(\begin{smallmatrix} \tilde{\mathbf{a}}^*(z) & \mathbf{b}^*(z) \\ \tilde{\mathbf{c}}^*(z) & \mathbf{d}^*(z) \end{smallmatrix}\right)$. □

The proof of Proposition 8 follows from Proposition 10:

Proof. (Proposition 8) Let $\mathbf{A}$ be a mask that satisfies the spectral condition (2.2) and $\partial_T S_\mathbf{A}$ the derived scheme w.r.t. T. We prove (ii)–(iv) of Proposition 8.

(ii): The subdivision operator $S_\mathbf{A}$ reproduces $\mathbf{k_0}$ and $\Delta_0\mathbf{k_0}=0$. Therefore $\Delta_0 S_\mathbf{A}\mathbf{k_0} = 0$ and by Proposition 10 there exists an operator $\partial_{\Delta_0} S_\mathbf{A}$ such that

$\Delta_0 S_{\mathbf{A}} = (\partial_{\Delta_0} S_{\mathbf{A}})\Delta_0$. Similarly, $\partial_T S_{\mathbf{A}}$ reproduces $\mathbf{k_1}$ (by Lemma 9) and $\Delta_1 \mathbf{k_1} = 0$. Thus there exists $\partial_{\Delta_1}\partial_T S_{\mathbf{A}}$ such that $\Delta_1 \partial_T S_{\mathbf{A}} = (\partial_{\Delta_1}\partial_T S_{\mathbf{A}})\Delta_1$ and hence

$$2\Delta_1 T S_{\mathbf{A}} = \Delta_1(\partial_T S_{\mathbf{A}})T = (\partial_{\Delta_1}\partial_T S_{\mathbf{A}})\Delta_1 T.$$

Therefore $\partial_{\Delta_1 T} S_{\mathbf{A}}$ exists and is given by $\partial_{\Delta_1 T} S_{\mathbf{A}} = \partial_{\Delta_1}\partial_T S_{\mathbf{A}}$.

(iii): The equation $(S_{\mathbf{A}} - S_{\mathbf{E}})\mathbf{k_0} = 0$ and Proposition 10 implies that there exists a subdivision operator $S_{\mathbf{B}}$ such that $(S_{\mathbf{A}} - S_{\mathbf{E}}) = S_{\mathbf{B}}\Delta_0$.

(iv): is proved analogously to (iii). □

5. Norms. In the following sections we will encounter different types of norms, which are collected here. On a finite-dimensional vector space V we choose a norm $\|\cdot\|$. From this norm, we induce a norm on V^2 by

$$\|\binom{v}{w}\| = \max\{\|v\|, \|w\|\}, \quad \text{for } v, w \in V. \tag{5.1}$$

The norm of an element $A \in L(V)^{2\times 2}$ is the operator norm

$$\|A\| = \sup\{\|A\binom{v_0}{v_1}\|, \text{ where } \|\binom{v_0}{v_1}\| = 1\}.$$

For simplicity we use the same symbol in all three cases; it will be clear from the context which one is used.

We write $\ell^\infty(V)$ resp. $\ell^\infty(V^2)$ resp. $\ell^\infty(L(V)^{2\times 2})$ for the Banach spaces of all bounded sequences equipped with the norms

$$\|\mathbf{p}\|_\infty = \sup_{\alpha\in\mathbb{Z}} \|p(\alpha)\| \text{ resp. } \|\mathbf{f}\|_\infty = \sup_{\alpha\in\mathbb{Z}} \|f(\alpha)\| \text{ resp. } \|\mathbf{A}\|_\infty = \sup_{\alpha\in\mathbb{Z}} \|A(\alpha)\|.$$

It is easy to see that if $\mathbf{f} \in \ell^\infty(V^2)$ has the components $\mathbf{f} = \binom{\mathbf{v}}{\mathbf{w}}$, then $\|\mathbf{f}\|_\infty = \max\{\|\mathbf{v}\|_\infty, \|\mathbf{w}\|_\infty\}$.

A linear subdivision operator $S_{\mathbf{A}}$ as defined in (2.1) maps $\ell(V^2)$ to $\ell(V^2)$. Since the linear combination in (2.1) is finite, it is clear that $\|S_{\mathbf{A}}\mathbf{f}\|_\infty \le d\|\mathbf{f}\|_\infty$ for some $d > 0$. It is not difficult to see that $d \le \sup_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} \|A(\alpha - 2\beta)\| \le N\|\mathbf{A}\|_\infty$, where N is a positive integer such that the support of $\mathbf{A}$ is contained in the interval $[-N, N]$. Therefore, $S_{\mathbf{A}}$ restricts to an operator $\ell^\infty(V^2) \to \ell^\infty(V^2)$, hence has an induced operator norm $\|S_{\mathbf{A}}\|_\infty$.

Also the operators of Definition 7 restrict to operators $\ell^\infty(V^2) \to \ell^\infty(V^2)$, since it is easy to see that

$$\|\Delta_0\|_\infty = \|\Delta_1\|_\infty = 2 \quad \text{and} \quad \|T\|_\infty = 3. \tag{5.2}$$

We choose the norm (5.1) on V^2 only for technical reasons. We would like to mention that the results presented in the next sections are valid for *any* norm on V^2. Suppose that $\|\cdot\|'$ is another norm on V^2. Since in every finite dimensional vector space any two norms are equivalent, there exist constants $c_0, c_1 > 0$ such that

$$c_0\|\binom{v}{w}\| \le \|\binom{v}{w}\|' \le c_1\|\binom{v}{w}\|, \quad \forall v, w \in V.$$

For a sequence $\mathbf{f} \in \ell(V^2)$, we define $\|\mathbf{f}\|'_\infty = \sup_{\alpha\in\mathbb{Z}} \|f(\alpha)\|'$. It follows immediately that $\|\mathbf{f}\|_\infty$ and $\|\mathbf{f}\|'_\infty$ are equivalent with the same constants $c_0, c_1 > 0$ from above:

$$c_0\|\mathbf{f}\|_\infty \le \|\mathbf{f}\|'_\infty \le c_1\|\mathbf{f}\|_\infty. \tag{5.3}$$

Therefore, a sequence $\mathbf{f}^n$ converges with respect to $\|\cdot\|_\infty$ if and only if it converges with respect to $\|\cdot\|'_\infty$. Furthermore, every inequality of the form $\|\mathbf{f}\|_\infty \le c\|\mathbf{g}\|_\infty$ (with $c > 0$) induces an inequality $\|\mathbf{f}\|'_\infty \le c'\|\mathbf{g}\|'_\infty$ (with $c' > 0$) and vice versa. The results in the following sections will either be concerned with convergence or with this type of inequalities. Therefore, they hold for any norm on V^2.

6. Convergence analysis. In this section we derive the following results concerning C^1 convergence of Hermite subdivision schemes:

- We define a "smoothness condition" for linear Hermite subdivision schemes which is sufficient for C^1 convergence.
- We introduce a "proximity condition", which bounds the difference between a nonlinear Hermite subdivision scheme and a linear one.
- We show that the nonlinear Hermite subdivision scheme is C^1 convergent if its linear counterpart is, and certain technical conditions are met.

We start by repeating the definition of convergence for point sequences and point-vector sequences. This needs the following notion: By $\mathcal{F}_n(\mathbf{g}^n)$ we denote the piecewise linear interpolant of the sequence $\mathbf{g}^n$ on the grid $2^{-n}\mathbb{Z}$. If $\mathbf{g}^n$ has more than one component, $\mathcal{F}_n(\mathbf{g}^n)$ is constructed componentwise.

DEFINITION 11 (Convergence of point and point-vector sequences). *Let V be a vector space. Then we define:*

(i) A point sequence $\mathbf{g}^n \in \ell(V^2)$ is said to be convergent if $\mathcal{F}_n(\mathbf{g}^n)$ converges uniformly on compact intervals to a continuous curve $\Psi \in C(\mathbb{R}, V^2)$.

(ii) A point-vector sequence $\left(\begin{smallmatrix}\mathbf{p}^n\\ \mathbf{v}^n\end{smallmatrix}\right) \in \ell(V^2)$ is said to be C^1 convergent if $\mathcal{F}_n(\mathbf{p}^n)$ converges uniformly on compact intervals to a continuously differentiable curve $\varphi \in C^1(\mathbb{R}, V)$ and $\mathcal{F}_n(\mathbf{v}^n)$ converges uniformly on compact intervals to its derivative φ'.

In the case of manifold-valued sequences, we require that (i) resp. (ii) is satisfied in a chart of the manifold.

In this paper, the sequences $\mathbf{g}^n$ and $\left(\begin{smallmatrix}\mathbf{p}^n\\ \mathbf{v}^n\end{smallmatrix}\right)$ are produced by subdivision. Due to the compact support of a subdivision operator U, the limit curve on compact intervals depends only on finitely many points of the initial data. In order to prove convergence and smoothness results for subdivision schemes it is therefore sufficient to consider finite input data. In particular we can assume that the input data is bounded.

For bounded input data Definition 11 is equivalent to the following:

(i) A point sequence $\mathbf{g}^n \in \ell^\infty(V^2)$ is convergent if there exists a continuous curve $\Psi \in C(\mathbb{R}, V^2)$ such that

$$\sup_{\alpha\in\mathbb{Z}} \|g^n(\alpha) - \Psi\Big(\frac{\alpha}{2^n}\Big)\| \to 0 \quad \text{as} \quad n \to \infty.$$

(ii) A point-vector sequence $\left(\begin{smallmatrix}\mathbf{p}^n\\ \mathbf{v}^n\end{smallmatrix}\right) \in \ell^\infty(V^2)$ is C^1 convergent if there exists a continuously differentiable curve $\varphi \in C^1(\mathbb{R}, V)$ such that

$$\sup_{\alpha\in\mathbb{Z}} \|p^n(\alpha) - \varphi\Big(\frac{\alpha}{2^n}\Big)\| \to 0 \quad \text{and} \quad \sup_{\alpha\in\mathbb{Z}} \|v^n(\alpha) - \varphi'\Big(\frac{\alpha}{2^n}\Big)\| \to 0 \quad \text{as} \quad n \to \infty.$$

REMARK 12. In Definition 11 we define C^1 *convergence* of a point-vector sequence $\left(\begin{smallmatrix}\mathbf{p}^n\\ \mathbf{v}^n\end{smallmatrix}\right)$. By this we mean that the limit curve φ enjoys *at least* C^1 smoothness. There exist linear Hermite schemes producing a point-vector sequence $\left(\begin{smallmatrix}\mathbf{p}^n\\ \mathbf{v}^n\end{smallmatrix}\right)$ and a C^2 limit curve φ (see [4]). We nevertheless use the terminology "C^1 convergence", since we do not prove anything about higher derivatives in this paper. $\diamondsuit$

REMARK 13. Recall from Section 2 that the Hermite scheme $D^n\mathbf{f}^n = U^n\mathbf{f}^0$ and the point scheme $\mathbf{g}^n = U^n\mathbf{g}^0$ only differ in the second component by a factor 2^n if $\mathbf{f}^0 = \mathbf{g}^0$. That is, for $\mathbf{f}^n = \left(\begin{smallmatrix}\mathbf{p}^n\\ \mathbf{v}^n\end{smallmatrix}\right)$ and $\mathbf{g}^n = \left(\begin{smallmatrix}\mathbf{p}^n\\ \mathbf{u}^n\end{smallmatrix}\right)$ we have $2^n\mathbf{u}^n = \mathbf{v}^n$.

If the Hermite scheme $\mathbf{f}^n$ is C^1 convergent with limit φ, then the point scheme $\mathbf{g}^n$ is convergent with limit $\Psi = \binom{\varphi}{0}$. In general, the converse is not true. Verification of the following, however, implies C^1 convergence of $\mathbf{f}^n$:

- the point sequence $\mathbf{p}^n$ converges to a continuously differentiable curve ψ_1,
- the point sequence $2^n\mathbf{u}^n$ converges to a continuous curve ψ_2 and
- $\psi_1' = \psi_2$.

In Sections 6.2 and 6.3 we will use this line of arguments to prove C^1 convergence of Hermite schemes. $\diamondsuit$

EXAMPLE 14. As already mentioned, the interpolatory linear scheme defined in Example 1 produces the piecewise cubic interpolant of given point-vector input data. Therefore the Hermite scheme is C^1 convergent in the sense of Definition 11 (and hence also the point scheme converges).

The scheme associated to the mask $\mathbf{E}$ we defined in Proposition 8 is interpolatory and adds midpoints between two consecutive points in every step. Therefore it produces a continuous limit which in general is not C^1. Hence the point scheme associated with it is convergent, but the associated Hermite scheme is not. $\diamondsuit$

6.1. Convergence results for derived schemes. This section deals solely with the convergence of linear point schemes. We treat the *Taylor scheme* $\mathbf{g}^n = (\partial_T S_\mathbf{A})^n \mathbf{g}^0$ as well as $\mathbf{g}^n = (\partial_{\Delta_0} S_\mathbf{A})^n \mathbf{g}^0$ and $\mathbf{g}^n = (\partial_{\Delta_1 T} S_\mathbf{A})^n \mathbf{g}^0$, where we use the notation for derived schemes introduced by Proposition 8. Results obtained here will be useful in Sections 6.2 and 6.3 for the C^1 analysis of Hermite schemes.

Define distance functions D_0, D_1 in $\ell(V^2)$ by

$$D_0(\mathbf{g}) = \|\Delta_0 \mathbf{g}\|_\infty \quad \text{and} \quad D_1(\mathbf{g}) = \|\Delta_1 \mathbf{g}\|_\infty. \tag{6.1}$$

Equation (5.2) implies that both $D_0(\mathbf{g})$ and $D_1(\mathbf{g})$ are finite if $\mathbf{g} \in \ell^\infty(V^2)$.

We already mentioned in Section 5 that a linear subdivision operator $S_\mathbf{A}$ has an operator norm. It is given by

$$\|S_\mathbf{A}\|_\infty = \sup\{\|S_\mathbf{A}\mathbf{g}\|_\infty, \text{ where } \|\mathbf{g}\|_\infty = 1\}.$$

PROPOSITION 15. *Let $S_\mathbf{A}$ satisfy the spectral condition (2.2) and let $\partial_T S_\mathbf{A}$ be its derived scheme w.r.t. the Taylor operator T. Then there exist constants c_0, c_1 such that for all $\mathbf{g} \in \ell^\infty(V^2)$*

$$\|\mathcal{F}_{n+1}(S_\mathbf{A}\mathbf{g}) - \mathcal{F}_n(\mathbf{g})\| \le c_0 D_0(\mathbf{g}),$$
$$\|\mathcal{F}_{n+1}(\partial_T S_\mathbf{A}\mathbf{g}) - \mathcal{F}_n(\mathbf{g})\| \le c_1 D_1(\mathbf{g}),$$

where we use the notation $\|\varphi\| = \sup_{t\in\mathbb{R}} \|\varphi(t)\|$ for a continuous curve $\varphi \in C(\mathbb{R}, V^2)$. Furthermore the constant c_0 (resp. c_1) only depends on $S_\mathbf{A}$ (resp. $\partial_T S_\mathbf{A}$) and neither on n nor on $\mathbf{g}$.

Proof. Observe that

$$\|\mathcal{F}_n(\mathbf{g}) - \mathcal{F}_n(\mathbf{h})\| = \sup_{\alpha\in\mathbb{Z}} \|g(\alpha) - h(\alpha)\| = \|\mathbf{g} - \mathbf{h}\|_\infty,$$

for $n = 0, 1, \ldots$ and $\mathbf{g}, \mathbf{h} \in \ell^\infty(V^2)$. Therefore using Proposition 8 and $\mathcal{F}_n(\mathbf{g}) = \mathcal{F}_{n+1}(S_\mathbf{E}\mathbf{g})$, with $\mathbf{E}$ from Proposition 8,

$$\begin{aligned}\|\mathcal{F}_{n+1}(S_\mathbf{A}\mathbf{g}) - \mathcal{F}_n(\mathbf{g})\| &= \|\mathcal{F}_{n+1}(S_\mathbf{A}\mathbf{g}) - \mathcal{F}_{n+1}(S_\mathbf{E}\mathbf{g})\| \\ &= \|(S_\mathbf{A} - S_\mathbf{E})\mathbf{g}\|_\infty = \|S_\mathbf{G}\Delta_0\mathbf{g}\|_\infty \\ &\le \|S_\mathbf{G}\|_\infty \|\Delta_0\mathbf{g}\|_\infty = c_0 D_0(\mathbf{g}).\end{aligned}$$

A similar argument proves the result for $\partial_T S_\mathbf{A}$. □

PROPOSITION 16. *Let $S_\mathbf{A}$ satisfy the spectral condition (2.2) and let $\partial_{\Delta_0} S_\mathbf{A}$ be the derived scheme w.r.t. to the operator Δ_0. Then the following are equivalent:*

(i) The point scheme $\mathbf{g}^n = S_{\mathbf{A}}^n \mathbf{g}^0$ *is convergent with continuous limit curve* $\binom{\psi}{0}$ *for all input data* $\mathbf{g}^0$.
(ii) The point scheme $\mathbf{h}^n = (\partial_{\Delta_0} S_{\mathbf{A}})^n \mathbf{h}^0$ *converges to* 0 *for all input data* $\mathbf{h}^0 = \Delta_0 \mathbf{g}^0$.
(iii) There exists a positive constant c_0 *and* $\alpha \in (0, 1]$ *such that*

$$D_0(S_{\mathbf{A}}^n \mathbf{g}) \le c_0 2^{-\alpha n} D_0(\mathbf{g}) \quad \text{for } n = 0, 1, \dots \text{ and } \mathbf{g} \in \ell^\infty(V^2).$$

Proof. In fact this proposition follows from similar results in [1, 2, 11]. For the convenience of the reader we give a proof.

Recall that it is sufficent to prove convergence for bounded input data. We therefore assume that $\mathbf{g}^0 \in \ell^\infty(V^2)$.

(i) $\Longrightarrow$ (ii): Let $\mathbf{g}^n = \binom{\mathbf{p}^n}{\mathbf{u}^n}$. Convergence to $\binom{\psi}{0}$ implies that

$$\sup_{\alpha \in \mathbb{Z}} \|p^n(\alpha) - \psi\Big(\frac{\alpha}{2^n}\Big)\| \to 0 \quad \text{and} \quad \sup_{\alpha \in \mathbb{Z}} \|u^n(\alpha)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{6.2}$$

Furthermore $\mathbf{h}^n = (\partial_{\Delta_0} S_{\mathbf{A}})^n \mathbf{h}^0 = (\partial_{\Delta_0} S_{\mathbf{A}})^n \Delta_0 \mathbf{g}^0 = \Delta_0 S_{\mathbf{A}}^n \mathbf{g}^0 = \Delta_0 \mathbf{g}^n = \binom{\Delta \mathbf{p}^n}{\mathbf{u}^n}$, which immediately implies that the second component of $\mathbf{h}^n$ converges to 0. As to the first component, consider

$$\begin{aligned}\|(\Delta \mathbf{p}^n)(\alpha)\| &= \|p^n(\alpha+1) - p^n(\alpha)\| \\ &\le \Big\|p^n(\alpha+1) - \psi\Big(\frac{\alpha+1}{2^n}\Big)\Big\| + \Big\|\psi\Big(\frac{\alpha}{2^n}\Big) - p^n(\alpha)\Big\| + \Big\|\psi\Big(\frac{\alpha+1}{2^n}\Big) - \psi\Big(\frac{\alpha}{2^n}\Big)\Big\|.\end{aligned}$$

The continuity of ψ together with (6.2) implies that $\sup_{\alpha \in \mathbb{Z}} \|(\Delta \mathbf{p}^n)(\alpha)\|$ converges to 0 and hence $\mathbf{h}^n$ converges to 0.

(ii) $\Longrightarrow$ (iii): Suppose that $\mathbf{h}^n = S_{\mathbf{D}}^n \mathbf{h}^0$ converges to 0 with $\mathbf{h}_0 = \Delta_0 \mathbf{g}$. Then there exists a positive integer N such that

$$\|S_{\mathbf{D}}^N \Delta_0 \mathbf{g}\|_\infty \le \frac{1}{2} \|\Delta_0 \mathbf{g}\|_\infty.$$

Write a positive integer n as $n = mN + r$, where $m = \lfloor \frac{n}{N} \rfloor$ and $0 \le r < N$. Then we have

$$\begin{aligned}\|\Delta_0 S_{\mathbf{A}}^n \mathbf{g}\|_\infty &= \|S_{\mathbf{D}}^n \Delta_0 \mathbf{g}\|_\infty = \|S_{\mathbf{D}}^r S_{\mathbf{D}}^{mN} \Delta_0 \mathbf{g}\|_\infty \\ &\le 2^{-m} \|S_{\mathbf{D}}^r\|_\infty \|\Delta_0 \mathbf{g}\|_\infty = 2^{-\frac{n}{N}} 2^{\frac{r}{N}} \|S_{\mathbf{D}}^r\|_\infty \|\Delta_0 \mathbf{g}\|_\infty \\ &\le 2^{-n\alpha} \max_{0 \le r < N} 2^{\frac{r}{N}} \|S_{\mathbf{D}}^r\|_\infty \|\Delta_0 \mathbf{g}\|_\infty \\ &\le c_0 2^{-n\alpha} \|\Delta_0 \mathbf{g}\|_\infty,\end{aligned}$$

where $\alpha = \frac{1}{N}$ and $c_0 = \max_{0 \le r < N} 2^{\frac{r}{N}} \|S_{\mathbf{D}}^r\|_\infty$. This proves inequality (iii).

(iii) $\Longrightarrow$ (i): Let $\mathbf{g}^n = S_{\mathbf{A}}^n \mathbf{g}^0$. Since we want to prove convergence, we may assume that $\mathbf{g}^0$ is bounded. By Proposition 15 and assumption (i) we have

$$\|\mathcal{F}_{n+1}(S_{\mathbf{A}}^{n+1} \mathbf{g}^0) - \mathcal{F}_n(S_{\mathbf{A}}^n \mathbf{g}^0)\| \le c_0 D_0(S_{\mathbf{A}}^n \mathbf{g}^0) \le c_0 2^{-\alpha n} D_0(\mathbf{g}^0),$$

for all $n = 0, 1, \dots$. This shows that $\mathcal{F}_n(S_{\mathbf{A}}^n \mathbf{g}^0) = \mathcal{F}_n(\mathbf{g}^n)$ is a Cauchy sequence. The sequence $\mathbf{g}^n$ takes values in the finite dimensional vector space V^2 which implies that the space of continuous curves on V^2, equipped with the ∞-norm, is a Banach space. Therefore $\mathcal{F}_n(\mathbf{g}^n)$ converges to a continuous curve.

Let $\mathbf{g}^n = \binom{\mathbf{p}^n}{\mathbf{u}^n}$. Inequality (iii) implies that $\|\Delta\mathbf{p}^n\|_\infty, \|\mathbf{u}^n\|_\infty$ converge to 0. Therefore the second component of the limit curve equals 0. □

The following result can be proved analogously to Proposition 16:

PROPOSITION 17. *Let $S_\mathbf{D}$ satisfy the reproduction property (4.1) and let $S_\mathbf{F}$ be such that $\Delta_1 S_\mathbf{D} = S_\mathbf{F}\Delta_1$ ($S_\mathbf{F}$ exists by Proposition 8). Then the following are equivalent:*

(i) The point scheme $\mathbf{g}^n = S_\mathbf{D}^n\mathbf{g}^0$ is convergent with continuous limit curve $\binom{0}{\psi}$ for all input data $\mathbf{g}^0$.

(ii) The point scheme $\mathbf{h}^n = S_\mathbf{F}^n\mathbf{h}^0$ converges to 0 for all input data $\mathbf{h}^0 = \Delta_1\mathbf{g}^0$.

(iii) There exists a positive constant c_1 and $\alpha \in (0, 1]$ such that

$$D_1(S_\mathbf{D}^n\mathbf{g}) \le c_1 2^{-\alpha n} D_1(\mathbf{g}) \quad \text{for } n = 0, 1, \dots \text{ and } \mathbf{g} \in \ell^\infty(V^2).$$

Note that if $S_\mathbf{A}$ satisfies the spectral condition (2.2), then Proposition 17 can be applied to $S_\mathbf{D} = \partial_T S_\mathbf{A}$. The operator $S_\mathbf{F}$ is then given by $\partial_{\Delta_1 T} S_\mathbf{A}$.

6.2. C^1 results for linear Hermite schemes. The C^1 analysis of linear Hermite subdivision schemes is often transferred to the convergence analysis of the respective Taylor schemes, see [7, 8, 10, 25]. The main theorem of this section (Theorem 21) is similar in this regard, but differs somewhat in the proof. Instead of constructing the limit function explicitly, we invoke convergence and smoothness conditions which we later adapt to the manifold-valued case.

DEFINITION 18 (Convergence and smoothness conditions). *Consider a linear subdivision operator $S_\mathbf{A} : \ell(V^2) \to \ell(V^2)$. We use the following terminology:*

$S_\mathbf{A}$ satisfies a convergence condition, if there exists $\gamma_0 < 1$ and a positive constant c_0 such that

$$D_0(S_\mathbf{A}^n\mathbf{g}) \le c_0\gamma_0^n D_0(\mathbf{g}) \qquad \text{for all } \mathbf{g} \in \ell^\infty(V^2).$$

$S_\mathbf{A}$ satisfies a smoothness condition, if there exists $\gamma_1 < 1$ and a positive constant c_1 such that

$$D_1(2^n T S_\mathbf{A}^n\mathbf{g}) \le c_1\gamma_1^n D_1(T\mathbf{g}) \qquad \text{for all } \mathbf{g} \in \ell^\infty(V^2).$$

Define $\alpha = -\log_2(\gamma_0)$ and $\beta = -\log_2(\gamma_1)$. We are content to find contraction factors γ_i with $\frac{1}{2} \le \gamma_i$ for $i = 0, 1$, which is the optimum for linear functions. Therefore we w.l.o.g. assume that $\alpha, \beta \in (0, 1]$. We will always work with $\gamma_0 = 2^{-\alpha}$ and $\gamma_1 = 2^{-\beta}$.

Definition 18 is based on similar conditions in [29, 27, 32]. Note that the convergence condition is exactly our condition (iii) of Proposition 16. Furthermore, the smoothness condition is our condition (iii) of Proposition 17 for input data of the form $T\mathbf{g}$ applied to $\partial_T S_\mathbf{A}$. The following Lemma is a preparation for Theorem 21.

LEMMA 19. *Let $S_\mathbf{A}$ satisfy the spectral condition (2.2). If $S_\mathbf{A}$ satisfies the smoothness condition of Definition 18, then it also satisfies the convergence condition.*

Proof. Let $\mathbf{g}^0 = \binom{\mathbf{p}^0}{\mathbf{u}^0}$ be bounded input data. Define $\binom{\mathbf{p}^n}{\mathbf{u}^n} = S_\mathbf{A}^n\binom{\mathbf{p}^0}{\mathbf{u}^0}$ by iterated subdivision. By Proposition 8 the derived scheme $\partial_{\Delta_0} S_\mathbf{A}$ exists. We will prove that the point scheme $\mathbf{h}^n = (\partial_{\Delta_0} S_\mathbf{A})^n\mathbf{h}^0$ with $\mathbf{h}^0 = \Delta_0\mathbf{g}^0$ converges to 0. Then Proposition 16 implies the convergence condition.

In order to prove that $\mathbf{h}^n$ converges to 0, note that

$$\mathbf{h}^n = (\partial_{\Delta_0} S_\mathbf{A})^n\mathbf{h}^0 = (\partial_{\Delta_0} S_\mathbf{A})^n\Delta_0\mathbf{g}^0 = \Delta_0 S_\mathbf{A}^n\mathbf{g}^0 = \binom{\Delta\mathbf{p}^n}{\mathbf{u}^n}.$$

Therefore, we have to prove that

(i) $\sup_{\alpha\in\mathbb{Z}} \|(\Delta\mathbf{p}^n)(\alpha)\| \to 0$ and
(ii) $\sup_{\alpha\in\mathbb{Z}} \|u^n(\alpha)\| \to 0$.
This will follow from the smoothness condition:

By Proposition 8, the Taylor scheme $\partial_T S_{\mathbf{A}}$ exists. By Proposition 17, the smoothness condition implies that the point scheme $\mathbf{k}^n = (\partial_T S_{\mathbf{A}})^n \mathbf{k}^0$ converges to a curve $\binom{0}{\psi}$ for all input data $\mathbf{k}^0 = T\mathbf{g}^0$. Since $\mathbf{k}^n = 2^n T S_{\mathbf{A}}^n \mathbf{g}^0 = 2^n \binom{\Delta\mathbf{p}^n - \mathbf{u}^n}{\mathbf{u}^n}$ we have

$$\sup_{\alpha\in\mathbb{Z}} \|2^n(\Delta\mathbf{p}^n)(\alpha) - 2^n u^n(\alpha)\| \to 0,$$
$$\sup_{\alpha\in\mathbb{Z}} \|2^n u^n(\alpha) - \psi\big(\frac{\alpha}{2^n}\big)\| \to 0,$$

as $n \to \infty$. Therefore

$$\sup_{\alpha\in\mathbb{Z}} \|u^n(\alpha)\| \le \frac{1}{2^n} \sup_{\alpha\in\mathbb{Z}} \|2^n u^n(\alpha) - \psi\big(\frac{\alpha}{2^n}\big)\| + \frac{1}{2^n} \sup_{\alpha\in\mathbb{Z}} \|\psi\big(\frac{\alpha}{2^n}\big)\| \to 0,$$

as $n \to \infty$. This proves (ii). We use (ii) to prove (i):

$$\sup_{\alpha\in\mathbb{Z}} \|(\Delta\mathbf{p}^n)(\alpha)\| \le \frac{1}{2^n} \sup_{\alpha\in\mathbb{Z}} \|2^n(\Delta\mathbf{p}^n)(\alpha) - 2^n u^n(\alpha)\| + \sup_{\alpha\in\mathbb{Z}} \|u^n(\alpha)\| \to 0,$$

as $n \to \infty$. This proves that $\mathbf{h}^n$ converges to 0. Thus, by Proposition 16, $S_{\mathbf{A}}$ satisfies the convergence condition. □

As a preparation for the next theorem we state a well-known result, which will also be useful for the rest of the paper:

LEMMA 20. [29, Lemma 8] *Consider a sequence of polygons $\mathbf{p}^n$ such that $\mathcal{F}_n(\mathbf{p}^n)$ converges to φ as $n \to \infty$ and φ is continuous. If $\mathcal{F}_n(2^n\Delta\mathbf{p}^n)$ is a Cauchy sequence and $\|2^n\Delta^2\mathbf{p}^n\|_\infty \to 0$ as $n \to \infty$, then φ is continuously differentiable and $\mathcal{F}_n(2^n\Delta\mathbf{p}^n)$ converges to φ' as $n \to \infty$.*

THEOREM 21. *If $S_{\mathbf{A}}$ satisfies the spectral condition (2.2) and the smoothness condition of Definition 18, then the linear Hermite scheme $\mathbf{f}^n = D^{-n} S_{\mathbf{A}}^n \mathbf{f}^0$ is C^1 convergent for all input data $\mathbf{f}^0 \in \ell(V^2)$.*

Proof. As suggested in Remark 13, we will use the linear point scheme $\mathbf{g}^n = S_{\mathbf{A}}^n \mathbf{g}^0$ to prove C^1 convergence of the Hermite scheme $\mathbf{f}^n = D^{-n} S_{\mathbf{A}}^n \mathbf{f}^0$.

Let $\mathbf{f}^n = \binom{\mathbf{p}^n}{\mathbf{v}^n}$ and $\mathbf{g}^n = \binom{\mathbf{p}^n}{\mathbf{u}^n}$ with $2^n\mathbf{u}^n = \mathbf{v}^n$. By Definition 11, we have to prove that $\mathbf{p}^n$ converges to a C^1 curve φ and that $\mathbf{v}^n = 2^n\mathbf{u}^n$ converges to its derivative φ'.

By Proposition 8, the Taylor scheme $\partial_T S_{\mathbf{A}}$ exists. By Proposition 17, the point scheme $\mathbf{k}^n = (\partial_T S_{\mathbf{A}})^n \mathbf{k}^0$ converges to a continuous curve $\binom{0}{\psi}$ for all input data $\mathbf{k}^0 = T\mathbf{g}^0$. Since $\mathbf{k}^n = 2^n T S_{\mathbf{A}}^n \mathbf{g}^0 = 2^n T\binom{\mathbf{p}^n}{\mathbf{u}^n} = 2^n \binom{\Delta\mathbf{p}^n - \mathbf{u}^n}{\mathbf{u}^n}$, also $2^n \binom{\Delta\mathbf{p}^n - \mathbf{u}^n}{\mathbf{u}^n}$ converges to $\binom{0}{\psi}$. This implies that both $\mathbf{v}^n = 2^n\mathbf{u}^n$ and $2^n\Delta\mathbf{p}^n$ converge to ψ and that

$$\sup_{\alpha\in\mathbb{Z}} \|2^n(\Delta\mathbf{p}^n)(\alpha) - v^n(\alpha)\| \to 0 \quad \text{as } n \to \infty. \tag{6.3}$$

By Lemma 19, the convergence condition applies and by Proposition 16, the linear point scheme $\mathbf{g}^n = S_{\mathbf{A}}^n \mathbf{g}^0$ converges to a continuous curve $\binom{\varphi}{0}$. Hence, the sequence $\mathbf{p}^n$ converges to φ.

By Proposition 8, the derived scheme $\partial_{\Delta_1 T} S_{\mathbf{A}}$ exists. By Proposition 17, the linear point scheme $\mathbf{h}^n = (\partial_{\Delta_1 T} S_{\mathbf{A}})^n \mathbf{h}^0$ converges to 0 for all input data $\mathbf{h}^0 = \Delta_1 \mathbf{k}^0 =$

$\Delta_1 T\mathbf{g}^0$. Since

$$\mathbf{h}^n = (\partial_{\Delta_1 T} S_{\mathbf{A}})^n \mathbf{h}^0 = (\partial_{\Delta_1 T} S_{\mathbf{A}})^n \Delta_1 T\mathbf{g}^0 = 2^n \Delta_1 T S_{\mathbf{A}}^n \mathbf{g}^0 = 2^n \Delta_1 T \begin{pmatrix} \mathbf{p}^n \\ \mathbf{u}^n \end{pmatrix} = 2^n \begin{pmatrix} \Delta\mathbf{p}^n - \mathbf{u}^n \\ \Delta\mathbf{u}^n \end{pmatrix},$$

both $2^n(\Delta\mathbf{p}^n - \mathbf{u}^n)$ and $2^n\Delta\mathbf{u}^n$ converge to 0. Therefore

$$\sup_{\alpha\in\mathbb{Z}} \|(\Delta\mathbf{v}^n)(\alpha)\| = \sup_{\alpha\in\mathbb{Z}} \|2^n(\Delta\mathbf{u}^n)(\alpha)\| \to 0 \quad \text{as } n \to \infty.$$

This together with (6.3) and $\|\Delta\|_\infty = 2$ indicates that

$$\begin{aligned}\sup_{\alpha\in\mathbb{Z}} \|2^n(\Delta^2\mathbf{p}^n)(\alpha)\| &\le \sup_{\alpha\in\mathbb{Z}} \|2^n(\Delta^2\mathbf{p}^n)(\alpha) - (\Delta\mathbf{v}^n)(\alpha)\| + \sup_{\alpha\in\mathbb{Z}} \|(\Delta\mathbf{v}^n)(\alpha)\| \\ &\le \sup_{\alpha\in\mathbb{Z}} 2\|2^n(\Delta\mathbf{p}^n)(\alpha) - v^n(\alpha)\| + \sup_{\alpha\in\mathbb{Z}} \|(\Delta\mathbf{v}^n)(\alpha)\| \to 0,\end{aligned}$$

as $n \to \infty$. To summarise, we showed that

(i) $\mathbf{p}^n$ converges to a continuous function φ,
(ii) $2^n\Delta\mathbf{p}^n$ as well as $\mathbf{v}^n$ converge to a continuous function ψ and
(iii) $2^n\Delta^2\mathbf{p}^n$ converges to 0.

Lemma 20 implies that φ is continuously differentiable and that $\varphi' = \psi$, which gives the result. □

Corollary 22. *If $S_{\mathbf{A}}$ satisfies the spectral condition (2.2) and there exists a positive integer N such that the derived scheme $\partial_{\Delta_1 T} S_{\mathbf{A}}$ satisfies $\|\partial_{\Delta_1 T} S_{\mathbf{A}}^N\|_\infty < 1$, then the linear Hermite scheme $\mathbf{f}^n = D^{-n} S_{\mathbf{A}}^n \mathbf{f}^0$ is C^1 convergent for all input data $\mathbf{f}^0 \in \ell(V^2)$.*

Proof. A similar argument as in the proof of Prop. 16 shows that $\|\partial_{\Delta_1 T} S_{\mathbf{A}}^N\|_\infty < 1$ implies the smoothness condition. The rest then follows from Theorem 21. □

Remark 23. We would like to comment on the independence of the chosen norm, continuing the discussion of Section 5. Certainly, if $\|\partial_{\Delta_1 T} S_{\mathbf{A}}^N\|_\infty < 1$ holds for $\|\cdot\|_\infty$, then in general it does not hold for an equivalent norm $\|\cdot\|'_\infty$. So in this case, the choice of norm is relevant. Nevertheless, the result is independent of the chosen norm in the following sense: If we can prove that $\|\partial_{\Delta_1 T} S_{\mathbf{A}}^N\|_\infty < 1$ for *any* norm $\|\cdot\|_\infty$, then the linear Hermite scheme is C^1 convergent with respect to this norm, and therefore, it is C^1 convergent with respect to every other norm. ♢

6.3. C^1 convergence from proximity. In this section we are going to derive C^1 convergence for nonlinear Hermite subdivision schemes, which are *close enough* to linear ones. The comparison of a subdivision operator $U_{\mathbf{A}}$ on TM to a linear subdivision operator $S_{\mathbf{A}}$ only makes sense in a chart or in an embedding of M. This paper uses charts. Therefore, from now on, all results are to be understood in a chart of M. Hence we w.l.o.g. assume that $TM \subset V^2$. Furthermore, since the particular mask the subdivision operator is not important, we write S instead of $S_{\mathbf{A}}$ and U instead of $U_{\mathbf{A}}$.

We say that a subdivision operator U on TM and a linear subdivision operator S satisfy the *proximity condition* if there exists a constant c_0 such that

$$\|\big(U - S\big)\mathbf{g}\|_\infty \le c_0 D_0(\mathbf{g})^2. \tag{6.4}$$

This is analogous to the proximity condition defined in [29].

We start with two theorems similar to [29, Theorem 2 and 5] and [32, Theorem 2.4]. By $\ell^\epsilon(V^2)$ we denote the set of sequences $\mathbf{g} : \mathbb{Z} \to V^2$ which satisfy $D_0(\mathbf{g}) < \epsilon$.

THEOREM 24. *Let U be a subdivision operator on TM and let S be a linear subdivision operator. Suppose that S and U satisfy the proximity condition (6.4) for all $\mathbf{g} \in \ell^\epsilon(TM)$ and that S satisfies the convergence condition*

$$D_0(S^n\mathbf{g}) \leq c_0 2^{-\alpha n} D_0(\mathbf{g}). \tag{6.5}$$

Then there exists a positive integer m such that for every $\beta \in \left(0, \alpha - \frac{\log_2(c)}{m}\right)$ there exists $\epsilon' \in (0, \epsilon)$ such that $\bar{U} = U^m$ satisfies

$$D_0(\bar{U}^n\mathbf{g}) \leq 2^{-\beta m n} D_0(\mathbf{g}), \tag{6.6}$$

for all $\mathbf{g} \in \ell^{\epsilon'}(TM)$.

Proof. Choose m such that $\mu = \alpha - \frac{\log_2(c_0)}{m} > 0$. Then $\bar{S} = S^m$ satisfies

$$D_0(\bar{S}\mathbf{g}) \leq 2^{-\mu m} D_0(\mathbf{g}), \tag{6.7}$$

for all $\mathbf{g} \in \ell^\epsilon(TM)$. Furthermore, it is proved in [27, Lemma 3] that the iterates $\bar{U}, \bar{S}$ satisfy a proximity condition as well. Hence

$$\|(\bar{U} - \bar{S})\mathbf{g}\|_\infty \leq \bar{c} D_0(\mathbf{g})^2.$$

Choose $\beta \in (0, \mu)$ and $\epsilon' \in (0, \epsilon)$ such that ϵ' satisfies $2\bar{c}\epsilon' + 2^{-\mu m} < 2^{-\beta m}$. Note that we can choose such an ϵ' since $2^{-\beta m} - 2^{-\mu m} > 0$. Then for $\mathbf{g} \in \ell^{\epsilon'}(TM)$ we have

$$\begin{aligned} D_0(\bar{U}^n\mathbf{g}) &\leq \|(\Delta_0\bar{U}^n - \Delta_0\bar{S}\bar{U}^{n-1})\mathbf{g}\|_\infty + \|\Delta_0\bar{S}\bar{U}^{n-1}\mathbf{g}\|_\infty \\ &\leq 2\|(\bar{U} - \bar{S})\bar{U}^{n-1}\mathbf{g}\|_\infty + D_0(\bar{S}\bar{U}^{n-1}\mathbf{g}) \\ &\leq 2\bar{c} D_0(\bar{U}^{n-1}\mathbf{g})^2 + 2^{-\mu m} D_0(\bar{U}^{n-1}\mathbf{g}). \end{aligned}$$

Using this recursion, we will prove by induction that (6.6) holds. For $n = 1$ we have

$$D_0(\bar{U}\mathbf{g}) \leq 2\bar{c} D_0(\mathbf{g})^2 + 2^{-\mu m} D_0(\mathbf{g}) < (2\bar{c}\epsilon' + 2^{-\mu m}) D_0(\mathbf{g}) < 2^{-\beta m} D_0(\mathbf{g}).$$

Assume that (6.6) holds for $1, \ldots, n-1$. Then

$$\begin{aligned} D_0(\bar{U}^n\mathbf{g}) &\leq 2\bar{c} D_0(\bar{U}^{n-1}\mathbf{g})^2 + 2^{-\mu m} D_0(\bar{U}^{n-1}\mathbf{g}) \\ &\leq 2\bar{c} 2^{-2\beta m(n-1)} D_0(\mathbf{g})^2 + 2^{-\mu m} 2^{-\beta m(n-1)} D_0(\mathbf{g}) \\ &\leq \left(2\bar{c}\epsilon' + 2^{-\mu m}\right) 2^{-\beta m(n-1)} D_0(\mathbf{g}) \\ &\leq 2^{-\beta m n} D_0(\mathbf{g}), \end{aligned}$$

which concludes the induction. ∎

THEOREM 25. *Let U be a subdivision operator on TM and let S be a linear subdivision operator. Suppose that S and U satisfy the proximity condition (6.4) for all $\mathbf{g} \in \ell^\epsilon(TM)$ and that S satisfies the smoothness condition*

$$D_1(2^n T S^n \mathbf{g}) \leq c_1 2^{-\alpha_1 n} D_1(T\mathbf{g}). \tag{6.8}$$

Then there exists a positive integer m such that for every $\beta_1 \in \left(0, \alpha_1 - \frac{\log_2(c_1)}{m}\right)$ there exist $\epsilon' \in (0, \epsilon)$ and a linear polynomial P such that $\bar{U} = U^m$ satisfies

$$D_1(2^{mn} T \bar{U}^n \mathbf{g}) \leq 2^{-\beta_1 m n}\Big(D_1(T\mathbf{g}) + P(n) D_0(\mathbf{g})\Big), \tag{6.9}$$

for $\mathbf{g} \in \ell^{\epsilon'}(TM)$.

Proof. The linear subdivision operator S satisfies the smoothness condition (6.8) and hence by Lemma 19 it also satisfies the convergence condition (6.5). Hence there exist $c_0 > 0$ and $\alpha_0 \in (0, 1]$ such that

$$D_0(S_A^n \mathbf{g}) \leq c_0 2^{-\alpha_0 n} D_0(\mathbf{g})$$

and for all $\mathbf{g} \in \ell^{\epsilon}(TM)$.

For $i = 0, 1$ choose m_i such that $\mu_i = \alpha_i - \frac{\log_2(c_i)}{m_i} > 0$ and set $m = \max\{m_0, m_1\}$. Decrease α_1 until $0 < \alpha_1 < 2\alpha_0$ and increase c_1 until $2^m c_0^2 < c_1$. Note that these modifications do not affect the correctness of (6.8).

Then $\bar{S} = S^m$ satisfies

$$\begin{aligned} D_0(\bar{S}\mathbf{g}) &\leq 2^{-\mu_0 m} D_0(\mathbf{g}) \\ D_1(T\bar{S}\mathbf{g}) &\leq 2^{(-\mu_1 - 1)m} D_1(T\mathbf{g}), \end{aligned}$$

for $\mathbf{g} \in \ell^{\epsilon}(TM)$. As in the proof of Theorem 24, the proximity condition for the iterates $\bar{S} = S^m, \bar{U} = U^m$ is satisfied:

$$\|(\bar{U} - \bar{S})\mathbf{g}\|_\infty \leq \bar{c} D_0(\mathbf{g})^2.$$

Furthermore, by Theorem 24, for $\beta_0 = \left(\alpha_0 - \frac{\log_2(c_0)}{m}\right) - \eta$ with $\eta \in \left(0, \frac{2\alpha_0 - \alpha_1}{2}\right)$ there exists $\epsilon' \in (0, \epsilon)$ such that

$$D_0(\bar{U}^n \mathbf{g}) \leq 2^{-\beta_0 mn} D_0(\mathbf{g}), \tag{6.10}$$

for $\mathbf{g} \in \ell^{\epsilon'}(TM)$. Note that the modifications of c_1 and α_1 now imply that $1 + \mu_1 - 2\beta_0 < 0$.

We have for all $\mathbf{g} \in \ell^{\epsilon'}(TM)$

$$\begin{aligned} D_1(T\bar{U}^n \mathbf{g}) &= \|\Delta_1 T \bar{U}^n \mathbf{g}\|_\infty \leq \|\Delta_1 T \bar{S} \bar{U}^{n-1} \mathbf{g}\|_\infty + \|\Delta_1 T (\bar{S} - \bar{U}) U^{n-1} \mathbf{g}\|_\infty \\ &\leq 2^{(-\mu_1 - 1)m} D_1(T\bar{U}^{n-1}\mathbf{g}) + 4\bar{c} D_0(\bar{U}^{n-1}\mathbf{g})^2. \end{aligned}$$

Choose β_1 such that $\beta_1 \in (0, \mu_1)$. Then iteration of this recursion gives

$$\begin{aligned} D_1(T\bar{U}^n \mathbf{g}) &\leq 2^{(-\mu_1 - 1)mn} D_1(T\mathbf{g}) + 4\bar{c} \sum_{i=0}^{n-1} 2^{m(-\mu_1 - 1)(n-1-i)} D_0(\bar{U}^i \mathbf{g})^2 \\ &= 2^{(-\mu_1 - 1)mn} \Big(D_1(T\mathbf{g}) + 4\bar{c} 2^{m(1+\mu_1)} \sum_{i=0}^{n-1} 2^{m(1+\mu_1)i} D_0(\bar{U}^i \mathbf{g})^2 \Big) \\ &\overset{(6.10)}{\leq} 2^{(-\beta_1 - 1)mn} \Big(D_1(T\mathbf{g}) + 4\bar{c} 2^{m(1+\mu_1)} \sum_{i=0}^{n-1} 2^{m(1+\mu_1)i + 2(-\beta_0 mi)} D_0(\mathbf{g})^2 \Big) \\ &= 2^{(-\beta_1 - 1)mn} \Big(D_1(T\mathbf{g}) + 4\bar{c} 2^{m(1+\mu_1)} \sum_{i=0}^{n-1} 2^{(1+\mu_1 - 2\beta_0)mi} D_0(\mathbf{g})^2 \Big) \\ &\leq 2^{(-\beta_1 - 1)mn} \Big(D_1(T\mathbf{g}) + 4\bar{c} 2^{m(1+\mu_1)} \epsilon' n D_0(\mathbf{g}) \Big). \end{aligned}$$

Defining $P(n) = 4\bar{c} 2^{m(1+\mu_1)} \epsilon' n$ gives the result. □

THEOREM 26. *Let U be a subdivision operator on TM and let S be a linear subdivision operator which satisfies the spectral condition (2.2). Suppose that S satisfies the smoothness condition of Definition 18 and that S and U satisfy the proximity condition (6.4) for all input from $\ell^{\epsilon}(TM)$ for some $\epsilon > 0$. Then there exists $\epsilon' > 0$ such that the Hermite scheme $\mathbf{f}^n = D^{-n} U^n \mathbf{f}^0$ is C^1 convergent for all input $\mathbf{f}^0 \in \ell^{\epsilon'}(TM)$.*

Proof. As suggested in Remark 13, we use the point scheme $\mathbf{g}^n = U^n\mathbf{g}^0$ to prove C^1 convergence of the Hermite scheme $\mathbf{f}^n = D^{-n}U^n\mathbf{f}^0$. Let $\left(\begin{smallmatrix}\mathbf{p}^n\\ \mathbf{v}^n\end{smallmatrix}\right) = \mathbf{f}^n$ and $\left(\begin{smallmatrix}\mathbf{p}^n\\ \mathbf{u}^n\end{smallmatrix}\right) = \mathbf{g}^n$ with $2^n\mathbf{u}^n = \mathbf{v}^n$. We want to show that $\mathbf{p}^n$ converges to a continuously differentiable curve φ and $\mathbf{v}^n = 2^n\mathbf{u}^n$ converges to its derivative φ'.

Since S satisfies the smoothness condition for all $\mathbf{f}^0 \in \ell^\epsilon(TM)$, Theorems 24 and 25 imply that there exist a positive integer m such that $\bar{U} = U^m$ satisfies both the convergence (6.6) and the smoothness condition (6.9).

We start by considering the case $m = 1$. Then by (6.6) and (6.9) there exists an $\epsilon' \in (0, \epsilon)$ such that U satisfies

$$\begin{aligned} D_0(U^n\mathbf{f}^0) &\le 2^{-\beta_0 n} D_0(\mathbf{f}^0),\\ D_1(2^n TU^n\mathbf{f}^0) &\le 2^{-\beta_1 n}\Big(D_1(T\mathbf{f}^0) + P(n)D_0(\mathbf{f}^0)\Big), \end{aligned} \tag{6.11}$$

for all $\mathbf{f}^0 \in \ell^{\epsilon'}(TM)$ and β_0, β_1 as in Theorems 24, 25.

We prove C^1 convergence of $\mathbf{f}^n$ for all input data $\mathbf{f}^0 \in \ell^{\epsilon'}(TM)$. By arguments given at the beginning of this section, we may assume that $\mathbf{f}^0$ is bounded.

For simplicity, we denote the input data by $\mathbf{f}$. We first prove that the sequence $\mathbf{p}^n$ converges to a continuous curve:

$$\begin{aligned} \|\mathcal{F}_{n+1}(U^{n+1}\mathbf{f}) - \mathcal{F}_n(U^n\mathbf{f})\| &\le \|\mathcal{F}_{n+1}(U^{n+1}\mathbf{f}) - \mathcal{F}_{n+1}(SU^n\mathbf{f})\|\\ &\quad + \|\mathcal{F}_{n+1}(SU^n\mathbf{f}) - \mathcal{F}_n(U^n\mathbf{f})\|\\ &\le \|(U-S)(U^n\mathbf{f})\|_\infty + \|\mathcal{F}_{n+1}(SU^n\mathbf{f}) - \mathcal{F}_n(U^n\mathbf{f})\|\\ &\le c_0 D_0(U^n\mathbf{f})^2 + c_1 D_0(U^n\mathbf{f})\\ &\le c_0 2^{-2\beta_0 n} D_0(\mathbf{f})^2 + c_1 2^{-\beta_0 n} D_0(\mathbf{f}). \end{aligned} \tag{6.12}$$

Inequality (6.12) uses the proximity condition (6.4) for S and U and Proposition 15. We have proved that the point scheme $\mathbf{g}^n = U^n\mathbf{f}$ is convergent. In particular, $\mathbf{p}^n$ converges to a continuous curve φ.

We continue by proving that $\mathbf{v}^n$ converges to a continuous function. By Proposition 8 the Taylor scheme $\partial_T S$ exists. We compute

$$\begin{aligned} &\|\mathcal{F}_{n+1}(2^{n+1}TU^{n+1}\mathbf{f}) - \mathcal{F}_n(2^nTU^n\mathbf{f})\| \le\\ &\|\mathcal{F}_{n+1}(2^{n+1}TU^{n+1}\mathbf{f}) - \mathcal{F}_{n+1}(2^{n+1}TSU^n\mathbf{f})\| + \|\mathcal{F}_{n+1}(2^{n+1}TSU^n\mathbf{f}) - \mathcal{F}_n(2^nTU^n\mathbf{f})\|\\ &\le \|2^{n+1}T(U-S)U^n\mathbf{f}\|_\infty + \|\mathcal{F}_{n+1}(\partial_T S(2^nTU^n\mathbf{f})) - \mathcal{F}_n(2^nTU^n\mathbf{f})\|\\ &\le 2^{n+1}3c_0 D_0(U^n\mathbf{f})^2 + c_1\|2^n\Delta_1 TU^n\mathbf{f}\|_\infty\\ &= 2^{n+1}3c_0 D_0(U^n\mathbf{f})^2 + c_1 D_1(2^nTU^n\mathbf{f})\\ &\le 2^{n+1-2\beta_0 n}3c_0 D_0(\mathbf{f})^2 + c_1 2^{-\beta_1 n}\Big(D_1(T\mathbf{f}) + P(n)D_0(\mathbf{f})\Big)\\ &\le 2^{(1-2\beta_0)n}6c_0 D_0(\mathbf{f})^2 + c_1 2^{-\beta_1 n}\Big(D_1(T\mathbf{f}) + P(n)D_0(\mathbf{f})\Big) =: b_n. \end{aligned}$$

In this computation we used $\|T\|_\infty = 3$, the proximity condition (6.4) for S and U and Proposition 15. By performing modifications as in the proof of Theorem 25, we can achieve that $1 + \mu_1 - 2\beta_0 < 0$ for some $\mu_1 > 0$. This implies that $1 - 2\beta_0 < 0$. Therefore $\sum b_n < \infty$, which shows that $2^nTU^n\mathbf{f}$ is convergent. Since $2^nTU^n\mathbf{f} = 2^nT\mathbf{g}^n = 2^nT\left(\begin{smallmatrix}\mathbf{p}^n\\ \mathbf{u}^n\end{smallmatrix}\right) = 2^n\left(\begin{smallmatrix}\Delta\mathbf{p}^n - \mathbf{u}^n\\ \mathbf{u}^n\end{smallmatrix}\right)$, we have proved that $2^n\mathbf{u}^n = \mathbf{v}^n$ converges to a continuous curve ψ.

The rest of the proof is analogous to the proof of Theorem 21, that is, inequality (6.11) implies that

- $2^n\Delta\mathbf{p}^n - 2^n\mathbf{u}^n$ converges to 0 and hence $2^n\Delta\mathbf{p}^n$ converges to ψ,
- $2^n\Delta\mathbf{u}^n$ converges to 0 and hence $2^n\Delta^2\mathbf{p}^n$ converges to 0.

Therefore, we have proved that

(i) $\mathbf{p}^n$ converges to a continuous curve φ,
(ii) $\mathbf{v}^n$ and $2^n\Delta\mathbf{p}^n$ converge to a continuous curve ψ,
(iii) $2^n\Delta^2\mathbf{p}^n$ converges to 0.

It follows from Lemma 20 that φ is C^1 with $\varphi' = \psi$.

For general m, the proof is analogous. We only have to replace U by U^m, 2 by 2^m, etc. □

7. Verification of proximity conditions. In the following, we verify that the proximity condition (6.4) holds between a linear subdivision operator and its nonlinear analogues we constructed in Section 3. In particular, this will imply the following result:

THEOREM 27. *Let M be a Riemannian manifold, Lie group or symmetric space (resp. a submanifold of Euclidean space $\mathbb{R}^n$). Let $S_\mathbf{A}$ be a linear subdivision operator that satisfies the spectral condition (2.2), let $\partial_{\Delta_1 T} S_\mathbf{A}$ be its derived scheme w.r.t. to $\Delta_1 T$ and let $U_\mathbf{A}$ be the Log-exp analogue (resp. the projection analogue) of $S_\mathbf{A}$ on TM. If there exists a positive integer N such that $\|(\partial_{\Delta_1 T} S_\mathbf{A})^N\|_\infty < 1$, then the Hermite scheme $\mathbf{f}^n = D^{-n} U_\mathbf{A}^n \mathbf{f}^0$ is C^1 convergent for dense enough input data $\mathbf{f}^0$.*

Proof. A similar argument as in the proof of Prop. 16 shows that $\|(\partial_{\Delta_1 T} S_\mathbf{A})^N\|_\infty < 1$ implies the smoothness condition of Definition 18. In Sections 7.1 and 7.2 we will show that the proximity condition holds between $S_\mathbf{A}$ and $U_\mathbf{A}$. Theorem 26 then implies that the Hermite scheme $\mathbf{f}^n$ is C^1 convergent for dense enough input data. □

Note that the result of the theorem is independent of the norm $\|\cdot\|_\infty$. This follows from arguments in Remark 23.

EXAMPLE 28. We consider the linear subdivision operator $S_\mathbf{A}$ defined in Example 1. In [25] it is shown that mask $\mathbf{F}$ of $\partial_{\Delta_1 T} S_\mathbf{A}$ is given by

$$F(0) = \begin{pmatrix} 1 & -\frac{1}{4} \\ \frac{3}{2} & -\frac{1}{4} \end{pmatrix} \quad \text{and} \quad F(1) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{5}{4} \end{pmatrix}$$

and that $\|\partial_{\Delta_1 T} S_\mathbf{A}\|_\infty < 1$. This implies that the Log-exp analogue (resp. the projection analogue) on any Riemannian manifold, Lie group or symmetric space (resp. submanifold of Euclidean space) of $S_\mathbf{A}$ is C^1 convergent for dense enough input data. In particular, this includes Example 6. ♢

7.1. Proximity for the Log-exp analogue. Let M denote a Riemannian manifold, Lie group or symmetric space and $\exp_p$ its exponential map. Using Taylor expansion, in a chart we have

$$\begin{aligned} p \oplus v &= \exp_p(v) = p + v + O(\|v\|^2), \\ q \ominus p &= \exp_p^{-1}(q) = q - p + O(\|q-p\|^2), \end{aligned} \tag{7.1}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$, for $n = \dim M$.

Let $S_\mathbf{A}$ be a linear subdivision operator with mask $\mathbf{A} = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right)$ that satisfies the spectral condition (2.2) and let $U_\mathbf{A}$ be its Log-exp analogue constructed in Section 3.1. As mentioned in Section 6, we can restrict the analysis to finite input data $\left(\begin{smallmatrix} \mathbf{p} \\ \mathbf{v} \end{smallmatrix}\right)$. We aim to prove that the proximity condition (6.4) is satisfied in a chart of M:

$$\|(S_\mathbf{A} - U_\mathbf{A})\left(\begin{smallmatrix} \mathbf{p} \\ \mathbf{v} \end{smallmatrix}\right)\|_\infty \le c_0 \|\left(\begin{smallmatrix} \Delta\mathbf{p} \\ \mathbf{v} \end{smallmatrix}\right)\|_\infty^2, \tag{7.2}$$

for some constant c_0. In order to prove this, we introduce some notation.

Notation. Let $\mathbf{q} = \{p(\alpha) + v(\alpha) : \alpha \in \mathbb{Z}\}$ and $\mathbf{r} = \{p(\alpha) \oplus v(\alpha) : \alpha \in \mathbb{Z}\}$. Recall that in Section 3.1 we introduced $S_{\tilde{\mathbf{A}}}$, which is constructed from $S_{\mathbf{A}}$ by transforming point-vector $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)$ to point-point data $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right)$:

$$S_{\tilde{\mathbf{A}}}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}(\alpha) = \begin{pmatrix}\sum_{\beta\in\mathbb{Z}} \tilde{a}(\alpha-2\beta)p(\beta) + \tilde{b}(\alpha-2\beta)q(\beta)\\ \sum_{\beta\in\mathbb{Z}} \tilde{c}(\alpha-2\beta)p(\beta) + \tilde{d}(\alpha-2\beta)q(\beta)\end{pmatrix},$$

with mask given by $\tilde{A}(\alpha) = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) A(\alpha) \left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)$ for $\alpha \in \mathbb{Z}$. Furthermore, we introduced the sequences $\mathbf{s}_1, \mathbf{r}_1$ by

$$\begin{pmatrix}s_1(\alpha)\\ r_1(\alpha)\end{pmatrix} = \begin{pmatrix}m_0(\alpha) \oplus \sum_{\beta\in\mathbb{Z}} \tilde{a}(\alpha-2\beta)\,(p(\beta) \ominus m_0(\alpha)) + \tilde{b}(\alpha-2\beta)\,(r(\beta) \ominus m_0(\alpha))\\ m_1(\alpha) \oplus \sum_{\beta\in\mathbb{Z}} \tilde{c}(\alpha-2\beta)\,(p(\beta) \ominus m_1(\alpha)) + \tilde{d}(\alpha-2\beta)\,(r(\beta) \ominus m_1(\alpha))\end{pmatrix}$$

in order to define $U_{\mathbf{A}}$ by

$$U_{\mathbf{A}}\begin{pmatrix}\mathbf{p}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{s}_1\\ \mathbf{r}_1 \ominus \mathbf{s}_1\end{pmatrix}.$$

Let $\left(\begin{smallmatrix}\mathbf{p^1}\\ \mathbf{v^1}\end{smallmatrix}\right) = S_{\mathbf{A}}\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)$. It is easy to see that $S_{\tilde{A}}\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) S_{\mathbf{A}}\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)$, which shows that the first component is also $\mathbf{p^1}$. This justifies the definition $\left(\begin{smallmatrix}\mathbf{p^1}\\ \tilde{\mathbf{q}}^1\end{smallmatrix}\right) = S_{\tilde{A}}\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right)$ with $\mathbf{v^1} = \tilde{\mathbf{q}}^\mathbf{1} - \mathbf{p^1}$. Furthermore, we set and $\left(\begin{smallmatrix}\tilde{\mathbf{p}}^\mathbf{1}\\ \tilde{\mathbf{r}}^\mathbf{1}\end{smallmatrix}\right) = S_{\tilde{A}}\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{r}\end{smallmatrix}\right)$.

What we want to prove. Using the notation we just introduced, the proximity condition (7.2) reads:

$$\begin{aligned}\|\mathbf{p^1} - \mathbf{s_1}\|_\infty &\le c_0 \left\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\right\|_\infty^2,\\ \|\mathbf{v^1} - \mathbf{r_1} \ominus \mathbf{s_1}\|_\infty &\le c_0 \left\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\right\|_\infty^2.\end{aligned}$$

In order to prove this, we show that every term in the following inequalities is less or equal to $c\left\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\right\|_\infty^2$:

$$\begin{aligned}\|\mathbf{p^1} - \mathbf{s_1}\|_\infty \le& \|\mathbf{p^1} - \tilde{\mathbf{p}}^\mathbf{1}\|_\infty + \|\tilde{\mathbf{p}}^\mathbf{1} - \mathbf{s_1}\|_\infty,\\ \|\mathbf{v^1} - \mathbf{r_1} \ominus \mathbf{s_1}\|_\infty \le& \|\tilde{\mathbf{q}}^\mathbf{1} - \tilde{\mathbf{r}}^\mathbf{1}\|_\infty + \|\tilde{\mathbf{p}}^\mathbf{1} - \mathbf{p^1}\|_\infty + \|\tilde{\mathbf{r}}^\mathbf{1} - \mathbf{r_1}\|_\infty\\ &+ \|\mathbf{s_1} - \tilde{\mathbf{p}}^\mathbf{1}\|_\infty + \|\mathbf{r_1} - \mathbf{s_1} - \mathbf{r_1} \ominus \mathbf{s_1}\|_\infty.\end{aligned}$$

A closer look at the above terms shows that in fact we only have to analyse three of them:

(i) $\|\mathbf{p^1} - \tilde{\mathbf{p}}^\mathbf{1}\|_\infty$. The term $\|\tilde{\mathbf{q}}^\mathbf{1} - \tilde{\mathbf{r}}^\mathbf{1}\|_\infty$ can be estimated analogously.
(ii) $\|\tilde{\mathbf{p}}^\mathbf{1} - \mathbf{s_1}\|_\infty$. The term $\|\tilde{\mathbf{r}}^\mathbf{1} - \mathbf{r_1}\|$ can be estimated analogously.
(iii) $\|\mathbf{r_1} - \mathbf{s_1} - \mathbf{r_1} \ominus \mathbf{s_1}\|_\infty$.

Estimation of (i). The estimation of this term reduces to estimating $\|\mathbf{q} - \mathbf{r}\|_\infty$. At every index $\alpha \in \mathbb{Z}$ of the sequences $\mathbf{q}$ and $\mathbf{r}$ we have

$$q - r = p + v - p \oplus v \overset{(7.1)}{=} p + v - (p + v + O(\|v\|^2)) = O(\|v\|^2),$$

and therefore $\|\mathbf{q} - \mathbf{r}\|_\infty \le c\|\mathbf{v}\|_\infty^2$.

Estimation of (ii). In order to estimate $\|\tilde{\mathbf{p}}^{\mathbf{1}} - \mathbf{s_1}\|_\infty$, we introduce some notation. For $\tilde{a}(\alpha - 2\beta)$ write $\tilde{a}_\beta$ and similar with other indices of the form $\alpha - 2\beta$. Furthermore, we generally suppress the index α. Define $u_\beta := p(\beta) \ominus m_0$ and $t_\beta := r(\beta) \ominus m_0$. Note that $p(\beta) = \exp_{m_0}(u_\beta)$ and $r(\beta) = \exp_{m_0}(t_\beta)$. We rewrite $\|\tilde{p}^1 - s_1\|$ (the index α is suppressed) as follows:

$$\begin{aligned}
&\|m_0 + \textstyle\sum_\beta \tilde{a}_\beta(p(\beta) - m_0) + \tilde{b}_\beta(r(\beta) - m_0) \\
&\qquad\qquad -m_0 \oplus \Big(\textstyle\sum_\beta \tilde{a}_\beta(p(\beta) \ominus m_0) + \tilde{b}_\beta(r(\beta) \ominus m_0)\Big)\| \\
&= \|m_0 + \textstyle\sum_\beta \tilde{a}_\beta(\exp_{m_0}(u_\beta) - m_0) + \tilde{b}_\beta(\exp_{m_0}(t_\beta) - m_0) \\
&\qquad\qquad - \exp_{m_0}\Big(\textstyle\sum_\beta \tilde{a}_\beta u_\beta + \tilde{b}_\beta t_\beta\Big)\| \\
&= \|\textstyle\sum_\beta \tilde{a}_\beta u_\beta + \tilde{b}_\beta t_\beta + O(\|u_\beta\|^2) + O(\|t_\beta\|^2) \\
&\qquad\qquad -\Big(\textstyle\sum_\beta \tilde{a}_\beta u_\beta + \tilde{b}_\beta t_\beta + O(\|\textstyle\sum_\beta \tilde{a}_\beta u_\beta + \tilde{b}_\beta t_\beta\|^2)\Big)\| \\
&\leq O(\sup_\beta \|u_\beta\|^2) + O(\sup_\beta \|u_\beta\| \sup_\beta \|t_\beta\|) + O(\sup_\beta \|t_\beta\|^2).
\end{aligned}$$

Define $w := \max\{\sup_\beta \|u_\beta\|, \sup_\beta \|t_\beta\|\}$. Then we have the estimate

$$\|\tilde{p}^1 - s_1\| \leq c_0 w^2. \tag{7.3}$$

By Lemma 29 (see below), $w \leq c_0\|\Delta\mathbf{p}\|_\infty + c_1\|\mathbf{v}\|_\infty \leq c_0\|\binom{\Delta\mathbf{p}}{\mathbf{v}}\|_\infty$, which gives the right estimate.

Estimation of (iii). We use notation as introduced for the estimation of (ii). The Taylor expansion (7.1) at every index α implies

$$\|r_1 - s_1 - r_1 \ominus s_1\| \leq c_0\|r_1 - p_1\|^2.$$

Define $U_\beta := p(\beta) \ominus m_1, T_\beta := r(\beta) \ominus m_1$ and $W := \max\{\sup_\beta \|U_\beta\|, \sup_\beta \|T_\beta\|\}$. Then we have

$$\begin{aligned}
\|r_1 - s_1\| =& \|\exp_{m_1}\Big(\sum_\beta \tilde{c}_\beta U_\beta + \tilde{d}_\beta T_\beta\Big) - \exp_{m_0}\Big(\sum_\beta \tilde{a}_\beta u_\beta + \tilde{b}_\beta t_\beta\Big)\| \\
\leq& \|m_1 - m_0\| + O(\sup_\beta \|U_\beta\|) + O(\sup_\beta \|T_\beta\|) + O(\sup_\beta \|u_\beta\|) + O(\sup_\beta \|t_\beta\|) \\
\leq& \|m_1 - m_0\| + c_0 W + c_1 w \\
\leq& c_0\|\Delta\mathbf{p}\|_\infty + c_1\|\mathbf{v}\|_\infty,
\end{aligned}$$

where w is taken from above and the last inequality follows from Lemma 29 below. This shows that $\|\mathbf{r_1} - \mathbf{s_1} - \mathbf{r_1} \ominus \mathbf{s_1}\|_\infty \leq c_0\|\binom{\Delta\mathbf{p}}{\mathbf{v}}\|_\infty^2$.

LEMMA 29. *In a chart of M consider finite inital data $\binom{\mathbf{p}}{\mathbf{v}}$ and $\mathbf{r} = \{p(\alpha) \oplus v(\alpha) : \alpha \in \mathbb{Z}\}$. Let $\mathbf{m_0} = \mathbf{p}$ or $\mathbf{m_0} = \{\mathrm{mean}(p(\alpha+1), p(\alpha)) : \alpha \in \mathbb{Z}\}$ and $\mathbf{m_1} = \mathbf{r}$ or $\mathbf{m_1} = \{\mathrm{mean}(r(\alpha+1), r(\alpha)) : \alpha \in \mathbb{Z}\}$. Then there exist constants $c_0, \ldots, c_8 > 0$ such that*

$$\begin{aligned}
\|p(\beta) \ominus m_0(\alpha)\| &\leq c_0\|\Delta\mathbf{p}\|_\infty, \\
\|r(\beta) \ominus m_0(\alpha)\| &\leq c_1\|\Delta\mathbf{p}\|_\infty + c_2\|\mathbf{v}\|_\infty, \\
\|p(\beta) \ominus m_1(\alpha)\| &\leq c_3\|\Delta\mathbf{p}\|_\infty + c_4\|\mathbf{v}\|_\infty, \\
\|r(\beta) \ominus m_1(\alpha)\| &\leq c_5\|\Delta\mathbf{p}\|_\infty + c_6\|\mathbf{v}\|_\infty, \\
\|m_1(\alpha) - m_0(\alpha)\| &\leq c_7\|\Delta\mathbf{p}\|_\infty + c_8\|\mathbf{v}\|_\infty,
\end{aligned}$$

for all $\alpha, \beta \in \mathbb{Z}$.

Proof. In a sufficiently bounded neighbourhood any smooth function is Lipschitz, hence $\|F(x) - F(x_0)\| \le c\|x - x_0\|$. Encode the finite input data $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)$ in a vector $x = \left(\begin{smallmatrix} p(i_0) & \dots & p(i_n) \\ v(i_0) & \dots & v(i_n)\end{smallmatrix}\right)$ and define $x_0 = \left(\begin{smallmatrix} p(i_0) & \dots & p(i_0) \\ 0 & \dots & 0\end{smallmatrix}\right)$. In order to prove the second inequality, set $F(x) = r(\beta) \ominus m_0(\alpha)$, where the indices α and β appear among $i_0, \dots, i_n$. Evaluation at x_0 gives $F(x_0) = 0$ and hence by the Lipschitz condition

$$\|r(\beta) \ominus m_0(\alpha)\| = \|F(x)\| \le c\|x - x_0\| \le c_1\|\Delta\mathbf{p}\| + c_2\|\mathbf{v}\|.$$

The other inequalities are proved analogously. □

7.2. Proximity for the projection analogue. Let M be a submanifold of Euclidean space $\mathbb{R}^n$, π a projection map onto M and $S_{\mathbf{A}}$ be a linear subdivision operator with mask $\mathbf{A} = \left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{smallmatrix}\right)$ that satisfies the spectral condition (2.2). We want to prove that the proximity condition (7.2) holds between $S_{\mathbf{A}}$ and its projection analogue $U_{\mathbf{A}}$, which is defined by $U_{\mathbf{A}}\mathbf{f}(\alpha) = d\pi(S_{\mathbf{A}}\mathbf{f}(\alpha))$.

For $\alpha \in \mathbb{Z}$ and finite input data $\mathbf{f} = \left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)$, define sequences $\mathbf{p^1}$ and $\mathbf{v^1}$ by

$$\begin{pmatrix} p^1(\alpha) \\ v^1(\alpha)\end{pmatrix} = S_{\mathbf{A}}\begin{pmatrix}\mathbf{p}\\ \mathbf{v}\end{pmatrix}(\alpha) = \begin{pmatrix} \sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)p(\beta) + b(\alpha-2\beta)v(\beta) \\ \sum_{\beta\in\mathbb{Z}} c(\alpha-2\beta)p(\beta) + d(\alpha-2\beta)v(\beta)\end{pmatrix}.$$

Hence in order to prove the proximity condition, we have to show the following inequalities

$$\sup_{\alpha\in\mathbb{Z}} \|p^1(\alpha) - \pi(p^1(\alpha))\| \le c_0\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\|_\infty^2, \tag{7.4}$$

$$\sup_{\alpha\in\mathbb{Z}} \|v^1(\alpha) - d_{p^1(\alpha)}\pi(v^1(\alpha))\| \le c_0\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\|_\infty^2, \tag{7.5}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

Preparation. Before we prove (7.4) and (7.5), we rewrite the terms $p^1, \pi(p^1), v^1$ and $d_{p^1}\pi(v^1)$ at an index α. The main ingredient is the following Taylor expansion of the projection π:

$$\pi(p+v) = \pi(p) + d_p\pi(v) + O(\|v\|^2), \tag{7.6}$$

where p is a point in M and v is a tangent vector. This implies

$$\begin{aligned} p(\beta) &= \pi(p(\beta)) = \pi(p(\alpha)) + d_{p(\alpha)}\pi(p(\beta) - p(\alpha)) + O(\|p(\beta) - p(\alpha)\|^2),\\ v(\beta) &= d_{p(\beta)}\pi(v(\beta)),\end{aligned}$$

for $\alpha, \beta \in \mathbb{Z}$. Therefore

$$\begin{aligned} p^1(\alpha) &= \sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)p(\beta) + b(\alpha-2\beta)v(\beta) \\ &= \sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)\Big(\pi(p(\alpha)) + d_{p(\alpha)}\pi(p(\beta)-p(\alpha)) + O(\|p(\beta)-p(\alpha)\|^2)\Big) \\ &\qquad + b(\alpha-2\beta)d_{p(\beta)}\pi(v(\beta)).\end{aligned} \tag{7.7}$$

Using $\sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)=1$, which is the reproduction of constants property (2.3) and (7.6) we have

(7.8)
$$\begin{aligned}
\pi(p^1(\alpha)) &= \pi\Big(p(\alpha)+\sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)(p(\beta)-p(\alpha))+b(\alpha-2\beta)v(\beta)\Big)\\
&= \pi(p(\alpha))+\sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)d_{p(\alpha)}\pi(p(\beta)-p(\alpha))+b(\alpha-2\beta)d_{p(\alpha)}\pi(v(\beta))\\
&\quad + O(\sup_\beta \|p(\beta)-p(\alpha)\|^2)+O(\sup_\beta \|p(\beta)-p(\alpha)\|\sup_\beta\|v(\beta)\|)+O(\sup_\beta\|v(\beta)\|^2)\\
&= \sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)\Big(\pi(p(\alpha))+d_{p(\alpha)}\pi(p(\beta)-p(\alpha))\Big)+b(\alpha-2\beta)d_{p(\alpha)}\pi(v(\beta))\\
&\quad + c_1\|\Delta\mathbf{p}\|_\infty^2+c_2\|\Delta\mathbf{p}\|_\infty\|\mathbf{v}\|_\infty+c_3\|\mathbf{v}\|_\infty^2.
\end{aligned}$$

Note that $\sup\|p(\beta)-p(\alpha)\|=O(\|\Delta\mathbf{p}\|_\infty)$ since p is finite. Using $\sum_\beta c(\alpha-2\beta)=0$, which is the reproduction of constants property (2.3), we have

$$\begin{aligned}
v^1(\alpha) &= \sum_{\beta\in\mathbb{Z}} c(\alpha-2\beta)p(\beta)+d(\alpha-2\beta)v(\beta)\\
&= \sum_{\beta\in\mathbb{Z}} c(\alpha-2\beta)\Big(p(\beta)-p^1(\alpha)\Big)+d(\alpha-2\beta)d_{p(\beta)}\pi(v(\beta)).
\end{aligned}\tag{7.9}$$

Furthermore

$$d_{p^1(\alpha)}\pi(v^1(\alpha))=\sum_{\beta\in\mathbb{Z}} c(\alpha-2\beta)d_{p^1(\alpha)}\Big(p(\beta)-p^1(\alpha)\Big)+d(\alpha-2\beta)d_{p^1(\alpha)}\pi(v(\beta)).\tag{7.10}$$

Proving the proximity condition. The first part of proximity (7.4) can be estimated using (7.7) and (7.8):

$$\begin{aligned}
\|p^1(\alpha)-\pi(p^1(\alpha))\| \le & c_0\sup_\beta\|d_{p(\beta)}\pi(v(\beta))-d_{p(\alpha)}\pi(v(\beta))\|+c_1\|\Delta\mathbf{p}\|_\infty^2\\
& + c_2\|\Delta\mathbf{p}\|_\infty\|\mathbf{v}\|_\infty+c_3\|\mathbf{v}\|_\infty^2.
\end{aligned}$$

Similarly, the second part of the proximity condition (7.5) can be estimated using (7.9) and (7.10):

$$\begin{aligned}
\|v^1(\alpha)-d_{p^1(\alpha)}\pi(v^1(\alpha))\| \le & c_0\sup_\beta\|(p(\beta)-p^1(\alpha))-d_{p^1(\alpha)}\pi(p(\beta)-p^1(\alpha))\|\\
& + c_1\sup_\beta\|d_{p(\beta)}\pi(v(\beta))-d_{p^1(\alpha)}\pi(v(\beta))\|.
\end{aligned}$$

Therefore, in order to prove the proximity condition, we have to show that the following terms are bounded by quadratic terms in $\|\Delta\mathbf{p}\|_\infty$ and $\|\mathbf{v}\|_\infty$:

(i) $\sup_\beta\|d_{p(\beta)}\pi(v(\beta))-d_{p(\alpha)}\pi(v(\beta))\|$,
(ii) $\sup_\beta\|d_{p(\beta)}\pi(v(\beta))-d_{p^1(\alpha)}\pi(v(\beta))\|$,
(iii) $\sup_\beta\|(p(\beta)-p^1(\alpha))-d_{p^1(\alpha)}\pi(p(\beta)-p^1(\alpha))\|$.

This is done below.

Proof of (i)-(iii). For fixed y consider the map $\varphi : x \mapsto d_x\pi(y)$. Using Taylor expansion one can conclude that $\varphi(z) - \varphi(x) = \varphi(x + (z - x)) - \varphi(x) = d_x\varphi(z - x) + O(\|z - x\|^2)$. Hence

$$d_z\pi(y) - d_x\pi(y) = d_x^2\pi(y, z - x) + O(\|z - x\|^2).$$

With $z = p(\beta), y = v(\beta)$ and $x = p(\alpha)$, we have

(7.11)
$$\begin{aligned}\|d_{p(\beta)}\pi(v(\beta)) - d_{p(\alpha)}\pi(v(\beta))\| &\le \|d^2_{p(\alpha)}\pi(v(\beta), p(\beta) - p(\alpha))\| + O(\|p(\beta) - p(\alpha)\|^2)\\ &\le c_0\|v(\beta)\|\|p(\beta) - p(\alpha)\| + O(\|p(\beta) - p(\alpha)\|^2).\end{aligned}$$

The last inequality follows from the fact that in a sufficiently small neighbourhood there exists a uniform constant c_0 such that

$$\|d^2_{p(\alpha)}\pi(v(\beta), p(\beta) - p(\alpha))\| \le c_0\|v(\beta)\|\|p(\beta) - p(\alpha)\|.$$

Taking the supremum over inequality (7.11) proves (i). In particular this proves (7.4). The second inequality (ii) can be proved analogously. Note that $\|p(\beta) - p^1(\alpha)\| \le c_0\|\Delta\mathbf{p}\|_\infty + c_1\|\mathbf{v}\|_\infty$. For part (iii) we have

$$\begin{aligned}&\|p(\beta) - p^1(\alpha) - d_{p^1(\alpha)}\pi(p(\beta) - p^1(\alpha))\|\\ &\le \|\pi(p(\beta)) - \pi(p^1(\alpha)) - d_{p^1(\alpha)}\pi(p(\beta) - p^1(\alpha))\| + \|\pi(p^1(\alpha)) - p^1(\alpha)\|\\ &\le \|\pi(p^1(\alpha) + (p(\beta) - p^1(\alpha))) - \pi(p^1(\alpha)) - d_{p^1(\alpha)}\pi(p(\beta) - p^1(\alpha))\| + \|\pi(p^1(\alpha)) - p^1(\alpha)\|\\ &\le O(\|p(\beta) - p^1(\alpha)\|^2) + \|\pi(p^1(\alpha)) - p^1(\alpha)\|,\end{aligned}$$

where the last inequality follows from (7.6). The bounds on (iii) follow from (7.4). This proves (7.5) and concludes the proof of Theorem 27.

Conclusion. We have studied two natural nonlinear analogues of Hermite subdivision schemes: One is an intrinsic construction on manifolds equipped with an exponential mapping, the other one applies to embedded manifolds (i.e., surfaces) and uses projections. Furthermore, a full C^1 analysis for Hermite schemes was presented which allows to deduce C^1 convergence from the verification of a proximity condition. This is used to prove the result stated in Theorem 27: Both nonlinear analogues of a linear Hermite scheme are C^1 convergent, if the derived scheme is appropriately bounded and the input data is dense enough.

Generalisations. We would like to comment on generalisations of the results presented in this paper:

It would be natural to consider sequences with more than 2 components, with the $(k + 1)$-st component representing k-th derivatives. Judging form the behaviour of point-subdivision in manifolds we expect that the case $k = 2$ is similar to the case $k = 1$, setting aside the technicalities of the jet bundle, but there is a big difference between $k \le 2$ and $k > 2$, see [19, 32].

In this paper we are concerned with the analysis of stationary Hermite schemes, i.e. schemes, whose mask $\mathbf{A}$ (except for the dilation matrix D) is independent of the subdivision level. Recent results [3] provide a factorisation of non-stationary Hermite subdivision operators generalising the Taylor factorisation. We believe that this allows to extend our results to the non-stationary setting. Non-stationary masks are a topic of future research.

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