Published in: The Applied Geometry and Engineering Graphics Vol. **70**, 43 – 50 (2002).

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Composition of Motions and Similarities - Two Geometric Aspects of Kinematics

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Summary. The paper gives an overview of four main results of the author adhering to the field of 3-dimensional Euclidean kinematics: At first we build up all motions with a two-parametric manifold of plane point paths as compositions of DARBOUX-motions with translation groups. Secondly these methods are extended to Euclidean similarities (equiform motions) by the use of scaling groups instead of the translation groups. These two-parametric equiform motions again have a two-parametric manifold of plane point paths – moreover, they contain all equiform DARBOUX-motions as one-parametric submotions. In the third section we outline an algorithm (based on plane equiform motions) to generate a series of overconstrained mechanisms with spherical 2R-joints.The last section leads us to robotics and the Stewart-Gough-platforms: By using equiform motions we are able to define a geometrically invarant "rigidity rate" of a platform at a given position.

Introduction

This paper shall give a short overview of some parts of the author's geometric work. It is focused on kinematics and the use of similarity concepts, including some examples and stating some results[1].

1. One-parametric spatial motions with many plane point paths

Within the group of Euclidean displacements B_6 of the Euclidean 3-space a one-parametric motion $\zeta(t)$ of a moving frame Σ with respect to a fixed frame Σ^* can be described[2] by

(1.1) $$\zeta(t):\ \vec{x}^*(\vec{x},t) = \vec{d}^*(t) + A(t)\,\vec{x}$$

with orthogonal 3×3-matrices $A(t)$. There exist nontrivial[3] motions, which move all points of Σ in planes of Σ^*. G. DARBOUX [1.1] listed them and was able to show that the general case has the following standard representation

(1.2) $$\zeta(t):\ \vec{x}^*_{Darboux}(\vec{x},t) = \begin{pmatrix} 0 \\ a\sin t \\ b\sin t + c(1-\cos t) \end{pmatrix} + \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x}$$

with $t \in [0, 2\pi]$ and real constants *a,b,c*. These motions are called *Darboux-motions.* They are compositions of spatial extensions of plane *Cardan*-motions

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[1] The references follow the sections of the paper.

[2] Points are marked by their position vectors $\vec{x} = (x,y,z)^t$ with respect to a Cartesian coordinate frame.

[3] The trivial examples are extensions of plane motions into the 3-space or pure translations with plane paths.

and harmonic oscillations orthogonal to its basic plane. The general point paths are ellipses.

In [1.2] and [1.3] I contributed to the question wether there are nontrivial motions with a two-parametric set of points with plane paths: The following two-parametric motion

(1.3) $\vec{x}^*(\vec{x},t,u) := \vec{x}^*_{Darboux}(\vec{x},t) + u\,\vec{e}^*$

with a fixed vector $\vec{e}^* = (d,e,f)^t$ and a second parameter $u \in \mathfrak{R}$ is the composition of the Darboux- motion (1.2) and the group of translations $\delta(u)$ with the direction $\vec{e}^*$. It has point-paths, which can be written as

$$(1.4)\quad \vec{x}^*(\vec{x},t,u) = \begin{pmatrix} 0 \\ 0 \\ c+z \end{pmatrix} + u \begin{pmatrix} d \\ e \\ f \end{pmatrix} + \begin{pmatrix} -y \\ x+a \\ b \end{pmatrix} \sin t + \begin{pmatrix} x \\ y \\ -c \end{pmatrix} \cos t.$$

General points move on cylinders of degree 2 given by the Darboux-path ellipse and generators parallel to $\vec{e}^*$. (1.4) makes clear, that all point paths are affinly equivalent to an arbitrary path on a nondegenerate cylinder of degree 2. Iff the last 3 vectors in (1.4) are linearly dependent, the path surface of the corresponding point will be a plane. These points of Σ are characterized by the equation

$$(1.5)\quad \det \begin{pmatrix} d & -y & x \\ e & x+a & y \\ f & b & -c \end{pmatrix} = 0.$$

This equation in general characterizes points on a cylinder of revolution, which contains the fixed direction of the given motions.[4] So we have found an example of motions with a two-parametric family of plane point paths. All these planes are parallel to the fixed direction $\vec{e}^*$. *The harder part of the work is to show, that (1.3) contains standard representations of all such motions. With the help of differential geometric methods* (see [1.2.]) we gain the following

Result 1.1: *All nontrivial Euclidean motions of the 3-space with a two-parametric manifold of plane point paths are gained by combining Darboux- motions with one-parametric translation groups with a fixed direction. These motions are two-parametric and have the standard representation* (1.3). *The point paths in general are situated on cylinders of degree two. Exactly the points of the cylinder of revolution* (1.5) *– or in the special case the points of the plane –move on planes.*

2. Darboux- motions within the group of Euclidean similarities

Adding scaling factors $\rho(t)$ to (1.1) we get a representation of the seven- parametric group A_7 of *similarities*[5] of the Euclidean 3-space. A. KARGER [2.1] used Lie-group methods in order to give a complete list of all types of equiform mo-

[4] In some cases this cylinder degenerates: Then it splits into a plane and the plane at infinity.

[5] This group is sometimes referred as *equiform group* of the Euclidean space.

tions, which move all points of the moving frame on plane paths. These motions are called *equiform Darboux- motions*. To work out examples of equiform motions with a two-parametric familiy of plane point paths we can generalize the procedure of section 1 (see [2.2]): We fix a point $A^* \in \Sigma^*$ (with position vector $\vec{a}^* := (d,e,f)^t$) and combine the group $\sigma(u)$ of scalings with centre A^* and scaling factor $u \in \Re$ with the Darboux- motion (1.2). We gain a two-parametric equiform motion $\sigma(u) \circ \zeta(t)$ with a standard representation

$$(2.1)\quad \vec{x}^*(\vec{x},t,u) = \vec{a}^*(1-u) + u\left[\begin{pmatrix} 0 \\ a\sin t \\ b\sin t + c(1-\cos t) \end{pmatrix} + \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\vec{x}\right].$$

From its generation it is clear, that the general point-paths of this two-parametric equiform motion are situated on cones of degree 2. They contain the Darboux-path ellipse and have their common vertex in the scaling centre A^*. As in section 1 (2.1) can be rewritten in terms of $\{1, u, u \sin t, u \cos t\}$. Any point path on a non-degenerate cone is affinly equivalent to all other paths. Exactly those points of Σ characterized by the equation

$$(2.2)\quad \det\begin{pmatrix} -d & -y & x \\ -e & x+a & y \\ z+c-f & b & -c \end{pmatrix} = 0$$

describe plane paths. (2.2) in general determines an algebraic surface of degree 3, which sometimes is referred to as MÜLLER's surface. For these points the affinity mentioned above is singular.

We can restrict a point-path on a nondegenerate cone to remain in a given plane. This specifies special one-parameter equiform motions within the two-parametric (2.1) one. In [2.2] I was able to show that these one-parametric equiform motions move all points on plane paths. So these motions are equiform Darboux-motions – all different types of A. KARGER [2.1] can be built up in this way. We have the

Result 2.1: *Combining Darboux- motions with one-parametric scaling groups with a fixed center $A^* \in \Sigma^*$ gives two-parametric equiform motions with the standard representation* (2.1). *The point paths in general are situated on cones of degree two. Exactly the points of the MÜLLER-surface* (2.2) *move on planes. All equiform Darboux-motions can be seen as special one-parametric submotions within the two-parametric equiform motion* (2.1).

3. Some remarkable overconstrained mechanisms

In the Euclidean plane there are equiform motions $\varepsilon(t) := E/E^*$ (fixed plane E^*, moving plane E) with a globally fixed point $A^* \in E^*$ which move all points of E on straight lines (not through A*). A standard representation of these motions is

$$(3.1)\quad \varepsilon(t):\ x^*(x,y,t) = \rho(t)(x\cos t - y\sin t);\quad y^*(x,y,t) = \rho(t)(x\sin t + y\cos t)$$

with the scaling factor $\rho(t)=1/\cos t$. Any straight line g^* (not through A^*) is the point path of exactly one point of the moving plane E. This plane configuration is successively reflected with respect to planes $o_i{}^*, i = 1,...,k$. This procedure results in a series of reflected plane equiform motions $\varepsilon_i(t) = E_i / E_i{}^*$ which all have straight line point paths and a globally fixed point $A_i{}^* \in E_i{}^*$, but run in different planes $E_i{}^*$ of the 3-space. Moreover, all are congruent even with respect to their parametrisation. Given that the fixed points $A_i{}^*$ do not belong to the intersections $E_i{}^* \cap E_{i+1}{}^*$ or $E_{i-1}{}^* \cap E_i{}^*$ there exists exactly one point in E_i with its point path on $E_{i-1}{}^* \cap E_i{}^*$. This way, the procedure generates a chain of equiform plane motions, which are linked via common straight line paths. In [3.7] I characterized closed chains of four linked motions of this type: The configuration can be closed iff the four points $A_0{}^* := A^*$ and $A_i{}^*$ either belong to a circle, a straight line or are lying on a sphere κ. In the first two cases the corresponding planes $E_i{}^*$ $(i = 1,2,3)$ are gained by succesive reflections of $E^* := E_0{}^*$ with respect to the planes of symmetry of the pairs $A_{i-1}{}^*, A_i{}^*$. In the third case all planes $E_i{}^*$ (including $E_0{}^*$) have to be tangent to the sphere κ.

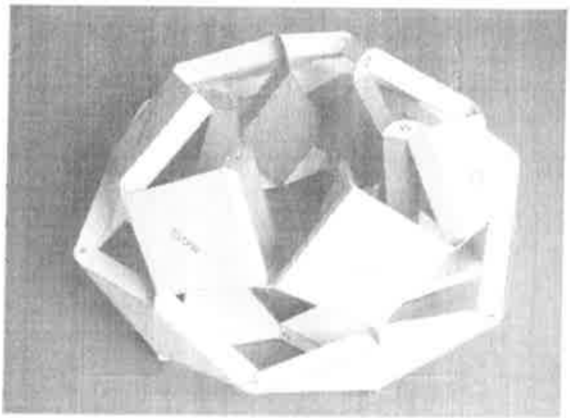

Figure 1

If we perform all these linked equiform plane motions, compose them with scalings from a fixed point O^* with factor $\cos t$ and extend the outcome into the 3-space we get chains of linked DARBOUX-motions (see [3.1]-[3.7]). Each two neighbouring bodies $\Sigma_{i-1}{}^*, \Sigma_i{}^*$ can be linked via spherical 2R- joints[6]. Of course we can build up two-dimensional chains of such arrangements consisting of $m * n$ rigid bodies (see [3.1]-[3.7]). There is a great variety of such mechanisms, which can be classified with respect to their topologic structure. Let us assume a rectangular array of rigid bodies with the topological structure of a torus: Here m and n have to be even numbers, the number of spherical 2R-joints then has to be $2 * m * n$. According to GRÜBLER's formula the theoretical degree of freedom of such a mechanism is $F = -2 * m * n - 6 < 0$. As these mechanisms at least provide a one-parametric mobility, we have found an interesting *series of overconstrained mechanisms*. Figure 1 shows the photo of an example of the case $m = 4$, $n = 6$ with the topological structure of a torus[7]. As 6 of the spherical 2R-joints are 1R-joints, its theoretical degree of freedom has the value $F = -60$, but it is still moveable due to our construction.

[6] These considerations are a generalisation of a paper by H. STACHEL [3.8], who used this idea to prove the mobility of the so-called HEUREKA-Polyhedron.
[7] Further examples can be seen on the web-page www.cis.tu.-graz./ig.

4. The rigidity-rate of a Stewart-Gough-platform in a given position

In robotics so-called Stewart-Gough-platforms (SGP) are widely used. An SGP is a robot consisting of a fixed and a moving platform plane $\varepsilon \subset \Sigma^*$ and $\varphi \subset \Sigma$. Both of these planes are linked by telescopic legs. Each leg is connected to the fixed plane and to the moving plane by spherical linkages (see figure 2). Let us assume that – at the present position - each one of the 6 leg lengths is being kept constant. Given that the Euclidean displacements in 3-space form a 6-parameter group, the 6 leg length conditions will – in general – prevent the mechanism from moving. There are, however, positions where the mechanical system surprisingly turns out to be *'shaky'* or even *'movable'*. Such positions of the SGP are called *'singular'*. If an SGP happens to be singular at each position of the robot, we call the platform *'architecturally singular'*. Such SGPs are useless for technical applications. A. KARGER [4.1] characterised such cases. In cooperation with S. MICK (see [4.3] and [4.4]) I tried to work out the close connection to projective geometry. The result may elucidate the structure of architecturally singular SGPs in a very geometric way:

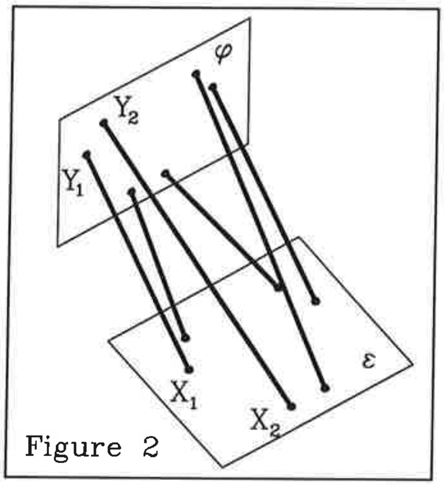

Figure 2

Result 4.1: *The architecturally singular SGPs are exactly those whose pairs of anchor points* $(X_i, Y_i) \in \varepsilon \times \varphi$ *(i=1, ..., 6) are at least 4-fold conjugate with respect to a linear manifold of correlations* $\varepsilon \rightarrow \varphi$.

Let us now switch over to SGPs which are not architecturally singular. Singular positions will only show up sporadically, but they have to be strictly avoided[8]. Moreover it is of prime importance to rate how far the present position is from a singular one. This question is indeed fairly tricky as the word *'far'* refers to some sort of measurement which still has to be introduced in a geometrically sound (i.e. 'invariant') way. This is the moment when our **similarity motions** again take centre stage in our considerations [4.2]: Within the group A_7 of similarities the 6 leg length conditions will no longer fix the platform's position – in general the robot can still perform a one-parametric equiform motion. Its infinitesimal transformation at the given position provides a possibility to rate the local deviation from the Euclidean displacement group. This directly leads into investigations in the tangent space of the corresponding Lie-groups. The action of the adjoint group in this tangent space allows to define (singular) Non-Euclidean distances and angles. In order to get a geometrically sensible rating we used some appropriate Non-Euclidean angle as indicator of the rigidity of an SGP at the given position. We sum up in

[8] If the robot takes on a singular position the forces exerted to the linkages tend to grow infinitely large and may cause considerable damage to the manipulator.

Result 4.2: *A given position of a Stewart-Gough-platform can be assessed by a so-called "rigidity rate". This rate is defined as the Non-Euclidean angle of the tangent vector of the viewed equiform motion with respect to the tangent space of the Euclidean displacement group.*

Conclusion

I have tried to give an overview of some results in the Euclidean space, which either was based on compositions of one-parametric motions or on the use of Euclidean similarities. These methods enabled us to see some kinematic problems more clearly and to develop strategies of solving them in a geometric way.

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