# $GC^1$-Continuity of Integral Bézier-Patches

## Another Approach

Johann LANG, Hannes RASSI, Otto RÖSCHEL

**Abstract.** *The conditions for two $GC^1$-continous $(n,n)$-Bézier-Patches with common link curve are well known and give a 3-parametric variety of solutions. In order to gain more freedom of design we suggest to link an $(n,n)$- and an $(n+k,n)$-Patch. We give a constructive and algorithmic way to find all solutions in the general case.*

**1. The problem.** Given the shift operator representations[1] of integral $(n,m)$ and $(n+k,\bar{m})$ Bézier-Patches in an affine 3-space

$$\begin{aligned} \mathbf{x}(u,v) &= (1-u+uE)^n(1-v+vF)^m \quad \mathbf{b}_{0,0} \\ \mathbf{y}(u,v) &= (1-u+uE)^{n+k}(1-v+vF)^{\bar{m}} \quad \mathbf{c}_{0,0} \\ &\quad (u,v) \in [0,1]\times[0,1] \end{aligned} \tag{1}$$

with an arbitrary integer $k \geq 0$ and with the common curve

$$l \ \ldots v=0 \qquad \mathbf{x}(u,0) = \mathbf{y}(u,0) \ \text{ for all } u \in \Re.$$

Thus we have

$$(1-u+u)^k\,(1-u+uE)^n\mathbf{b}_{0,0} = (1-u+uE)^{n+k}\mathbf{c}_{0,0}.$$

The points of the 0-tread are gained by elevation of degree (see HOSCHEK, J./LASSER, D. [5], p. 131). *We*

[1] In the usual way we apply the operators $E, F$, such that $E^k\mathbf{b}_{0,0} := \mathbf{b}_{k,0}$ and $E^l\mathbf{b}_{0,0} := \mathbf{b}_{0,l}$.

*want to investigate, under which conditions the patch $\Psi$ is a $GC^1$-continuation of $\Phi$ along the border curve $l \ldots v = 0$ (i.e. $\Phi$ and $\Psi$ are tangent along $l$).* The partial derivative vectors to be regarded are

$$\begin{aligned} \mathbf{x}_u(u,0) &= n\,(1-u+uE)^{n-1}\,(E-1)\;\mathbf{b}_{0,0}, \\ \mathbf{x}_v(u,0) &= m\,(1-u+uE)^{n}\,(F-1)\;\mathbf{b}_{0,0}, \\ \mathbf{y}_v(u,0) &= \bar{m}\,(1-u+uE)^{n+k}\,(F-1)\;\mathbf{c}_{0,0}. \end{aligned} \tag{2}$$

For abbreviation we write (see fig. 1):

$$\begin{aligned} E^j(E-1)\mathbf{b}_{0,0} &= \mathbf{b}_{j+1,0}-\mathbf{b}_{j,0} &=:\; \mathbf{a}_j, \quad j=0,\ldots n-1 \\ E^j(F-1)\mathbf{b}_{0,0} &= \mathbf{b}_{j,1}-\mathbf{b}_{j,0} &=:\; \mathbf{b}_j, \quad j=0,\ldots n \\ E^j(F-1)\mathbf{c}_{0,0} &= \mathbf{c}_{j,1}-\mathbf{c}_{j,0} &=:\; \mathbf{c}_j, \quad j=0,\ldots n+k. \end{aligned} \tag{3}$$

Then we have $E^j\mathbf{a}_0 = \mathbf{a}_j,\; E^j\mathbf{b}_0 = \mathbf{b}_j,\; E^j\mathbf{c}_0 = \mathbf{c}_j$.

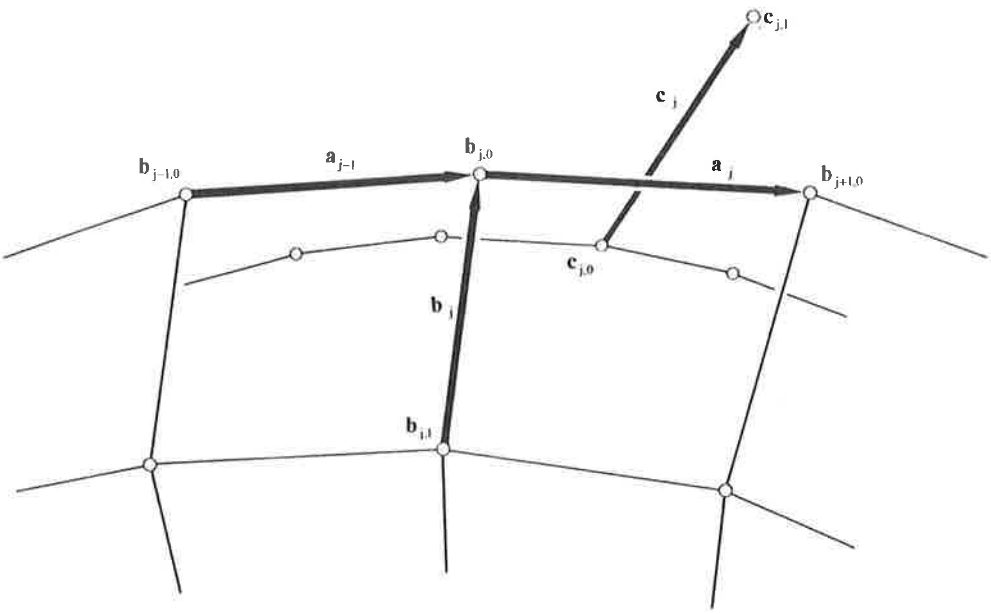


Fig. 1

The vectors $\mathbf{c}_j$ $(j=0,\ldots,n+k)$ determine the 1-thread of the control net $(\mathbf{c}_{ij})$ (see fig. 1): $\mathbf{c}_{j1} = \mathbf{c}_{j0} + \mathbf{c}_j$ for $j = 0,\ldots n+k$.

The characteristic condition is

$$\det[\mathbf{x}_u(u,0), \mathbf{x}_v(u,0), \mathbf{y}_u(u,0)] = 0,$$

which means:

$$\begin{aligned} \det[(1-u+uE)^{n-1} \quad & \mathbf{a}_0, \\ (1-u+uE)^{n} \quad & \mathbf{b}_0, \\ (1-u+uE)^{n+k} \quad & \mathbf{c}_0] \quad = \quad 0. \end{aligned} \tag{4}$$

Equation (4) is a **polynomial in the variable $u$ of degree** $3n+k-1$, which has to vanish for all $u \in \Re$. Comparison of coefficients yields $3n+k$ conditions for the $3(n+k+1)$ coefficients of the unknown vectors $\mathbf{c}_0, \ldots \mathbf{c}_{n+k}$. This is a **system of linear homogenous equations.** Thus we have:

**Theorem 1.** *In the general case the problem of finding vectors* $\mathbf{c}_j$ $(j = 0, \ldots n+k)$ *of the control net of a* $GC^1$*-continuation surface* $\Psi$ *yields a* $2k+3$*-dimensional variety as solution.*

In fact the system of equations found above may in special cases be dependent; as a consequence the dimension of the solution variety still increases. For small integers $n$ it may be possible to write down the conditions, under which that may happen.

We notice, that the problem in the case $k = 0$ still leads to a 3-dimensional solution, which is well known. (c.f. FARIN, G. [3], HOSCHEK, J./LASSER, D. [6], WASSUM, P. [15]; for further references see [6]). For the special cases $n = 3$, $k = 2$ see HOSAKA, M./KIMURA, F. [4], [5].

Degree elevation of the solution $k = 0$ does not change the variety of solution. But for some applications a **greater variety of solutions** may be of interest even in the general case. This is why we suggest to put $k > 0$.

**2. A fundamental system of solutions.** For practical use it is important to know a fundamental system of solutions for the general case. **Hence other solutions can be constructed by linear combination.**

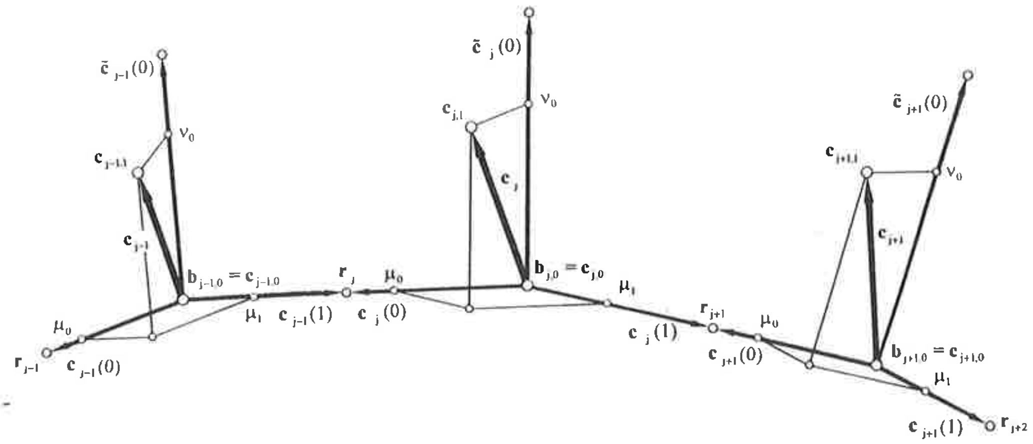


**Fig. *2***

Figure 2 illustrates that for the case $k = 0$.

In order to realize a $GC^1$-continuation as described, the user still has to choose 3 constant real numbers $\mu_0, \mu_1, \nu_0$, such that for any $j \in \{0, \ldots n\}$ the vector $\mathbf{c}_j$ is composed by

$$\mathbf{c}_j = \mu_0 \; \mathbf{c}_j(0) + \mu_1 \; \mathbf{c}_j(1) + \nu_0 \; \tilde{\mathbf{c}}_j(0) \qquad (22)$$

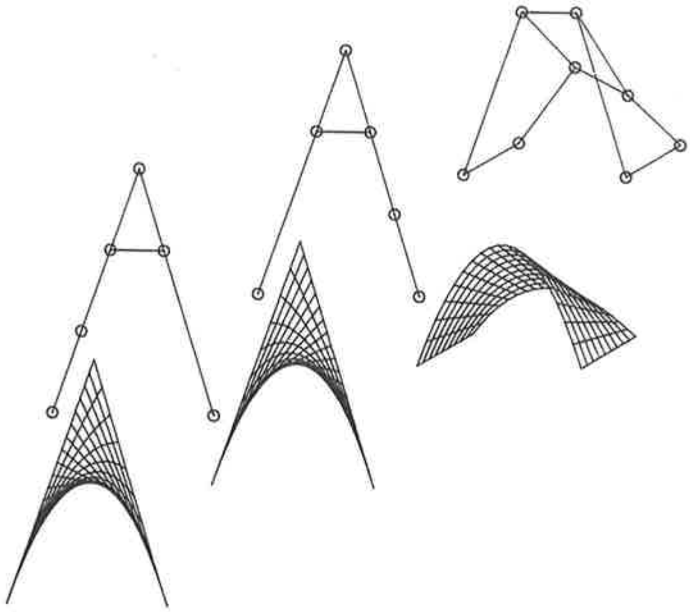

**Fig.** *3*

Figure 3 illustrates the fundamental solutions $\mu_0 = 1.5$, $\mu_1 = \nu_0 = 0$ and $\mu_0 = 0$, $\mu_1 = -1.5$, $\nu_0 = 0$. They are singular ones. Only the choice of $\nu_0 \neq 0$ would give nonsingular solutions.

With the help of the recursion formulas (13), (13), (15), and (16) we now treat with case $k = 1$, which already yields dimension 5 for the solution of the $GC^1$ continuation problem.

We apply (13), (14), (15), and (16) to the vectors $\mathbf{c}_j(0,0)$ and $\mathbf{c}_j(0,0)$, respectively:

$$\begin{aligned} \mathbf{c}_j(\lambda,1) &= \tfrac{n+1-j}{n+2}\ \mathbf{c}_j(\lambda,0) \qquad \lambda = 0,1, \\ \mathbf{c}_j(\lambda,1) &= \tfrac{j}{n+2}\ \mathbf{c}_{j-1}(\lambda-1,0) \quad \lambda = 1,2. \end{aligned} \tag{23}$$

$$\begin{aligned} \tilde{\mathbf{c}}_j(0,1) &= \tfrac{n+1-j}{n+1}\ \tilde{\mathbf{c}}_j(0,0) \\ \tilde{\mathbf{c}}_j(1,1) &= \tfrac{j}{n+1}\ \tilde{\mathbf{c}}_{j-1}(0,0). \end{aligned} \tag{24}$$

For $n = 3$ we get:

$$
\begin{array}{llll}
\mathbf{c}_j(0,1) = \frac{4-j}{5} & \mathbf{c}_j(0,0) & j = 0,1,2,3 & \\
\mathbf{c}_j(1,1) = \frac{4-j}{5} & \mathbf{c}_j(1,0) & j = 0,1,2,3 & \\
\mathbf{c}_j(1,1) = \frac{j}{5} & \mathbf{c}_{j-1}(0,0) & j = 1,2,3,4 & \\
\mathbf{c}_j(2,1) = \frac{j}{5} & \mathbf{c}_{j-1}(1,0) & j = 1,2,3,4 & (25) \\
 & & & \\
\tilde{\mathbf{c}}_j(0,1) = \frac{4-j}{4} & \tilde{\mathbf{c}}_j(0,0) & j = 0,1,2,3 & \\
\tilde{\mathbf{c}}_j(1,1) = \frac{j}{4} & \tilde{\mathbf{c}}_{j-1}(0,0) & j = 0,1,2,3 &
\end{array}
$$

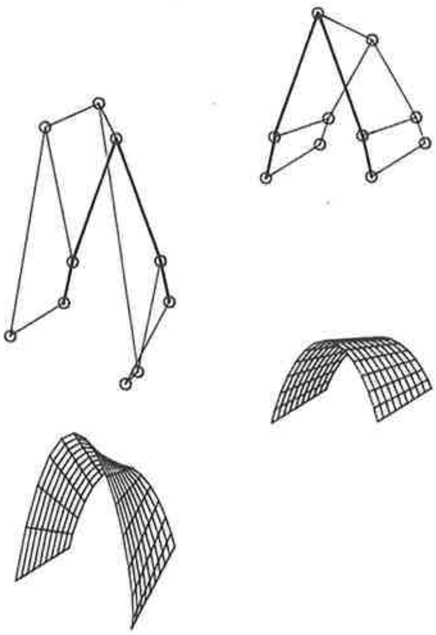

**Fig. 4**

Fig. 4 shows an example of a pair of Bézier-Patches, one being a $GC^1$-continuation of the other one in the case $k = 1$. There we put $\mu_0 = 0$, $\mu_1 = 3$, $\mu_2 = 0.3$, $\nu_0 = = \nu_1 = 1$ according to (18).

**References:**


[1] DEGEN, W. L. F., Explicit continuity conditions

for adjacent Bézier surface patches. Computer Aided Geometric Design 7 (1990) 181-189.

[2] DU, W. H./SCHMITT, F. J. M., On the $G^1$ Continuity of piecewise Bézier surfaces: a review with new results. Computer Aided design 22 (1990) 556-573.

[3] FARIN, G., A Construction for Visual $C^1$ Continuity of Polynomial Surface Patches. Computer Graphics and Image Processing 20 (1982) 272-282.

[4] HOSAKA, M./ KIMURA, F., Synthesis Methods of Curves and Surfaces in Interactive CAD, Proceedings: Interactive Techniques in Computer aided design, Bologna (1978), 151 - 156.

[5] HOSAKA, M./ KIMURA, F., A Theory and Methods for three dimensional free-form shape construction. Journal of Information Processing **3** (1980), 140 - 156.

[6] HOSCHEK, J./LASSER, D., Grundlagen der geometrischen Datenverarbeitung. Teubner 1992.

[7] KAHMANN, J., Krümmungsübergänge zusammengesetzter Kurven und Flächen. Diss. Braunschweig 1982.

[8] KAHMANN, J., Continuity of Curvature between Adjacent Bézier Patches. in Barnhill, R.E.; Böhm, W. (ed.): Surfaces in Computer Aided Geometric Design. North Holland (1983) 65-75.

[9] LIU, D., A geometric condition for smoothness between adjacent rectangular Bézier patches. Acta Math. Sinica 9 (1986)432-442.

[10] LIU, D./HOSCHEK, J., $GC^1$ continuity conditions between adjacent rectangular and triangular Bézier surface patches. Computer Aided Design 21 (1989) 194-200.

[11] POTTMANN, H., Projectively invariant classes of geometric continuity for CAGD. Computer Aided Geometric Design **6** (1989) 307 – 321. (1989) 194-200.

[12] RASSI, H., Geometrische $C^1$-Fortsetzung von Bézierflächen. Diploma Thesis at Graz University of Technology, 1995, Graz, Austria.

[13] ROSE DE, T., Necessary and sufficient conditions for tagent plane continuity of Bézier surfaces. Computer Aided Geometric Design 7 (1990) 165-179.

[14] VERON, M./RIS, G./MUSSE, J., Continuity of biparametric surface patches. Computer Aided Design 8 (1976) 267-273.

[15] WASSUM, P., Bedingungen und Konstruktionen zur geometrischen Stetigkeit und Anwendungen auf approximative Basistransformationen, Dissertation, Darmstadt 1991.

Authors' adress: Graz University of Technology, Institute for Geometry, Kopernikusgasse 24, A - 8010 Graz, Austria.