Interpolation of Helical Patches by Kinematic Rational Bézier Patches

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Dedicated to Prof.Dr.H.Florian on the occasion of his 65th birthday

Abstract. We interpolate the helical motion σ by *rational axial motions of degree three and four* ζ. This will be done by interpolating one point path $\sigma(\vec{\mathbf{p}}^0)$ by $\zeta(\vec{\mathbf{p}}^0)$. Then all point paths of σ are interpolated in the same way if we use the method given here.

1. In the 3-dimemsional euclidean space E_3 we describe the points by cartesian coordinates $(x, y, z)^t$ and by homogeneous coordinates $(x_0, x_1, x_2, x_3)^t$ respectively. There is $\rho(1, x, y, z)^t = (x_0, x_1, x_2, x_3)^t := \vec{\mathbf{x}}$ with $\rho \neq 0 \in \Re$. Then E_3 is a subset of the 3-dimensional projective space $P_3(\Re)$

A euclidean one parameter motion ζ with a fixed axis is called *axial motion.* Examples are rotations and helical motions. This fixed axis shall be the $z-$axis of our cartesian frame $(x_1 = x_2 = 0)$. Then ζ is given by

$$\vec{\mathbf{x}}' = \begin{pmatrix} a(v) & 0 & 0 & 0 \\ 0 & b(v) & -c(v) & 0 \\ 0 & c(v) & b(v) & 0 \\ d(v) & 0 & 0 & a(v) \end{pmatrix} \vec{\mathbf{x}} := A_{44}(v)\vec{\mathbf{x}}, \tag{1}$$

with $v \in I \subset \Re,\quad a(v),\quad b(v),\quad c(v),\quad d(v) \in C^0(I),\quad a^2(v) = b^2(v) + c^2(v),$

$a(v) \neq 0 \quad \forall v \in I$. The point path $\zeta(\vec{\mathbf{p}}^0)$ of the point $\vec{\mathbf{p}}^0 := (1,1,0,0)^t$ is described by

$$\zeta(\vec{\mathbf{p}}^0) = \begin{pmatrix} a(v) \\ b(v) \\ c(v) \\ d(v) \end{pmatrix}. \tag{2}$$

It is situated on a cylinder of revolution with axis $x_1 = x_2 = 0$. Given the path (2) of one point $\vec{\mathbf{p}}^0$ the axial motion (1) is uniquely defined.

2. A helical motion σ with *pitch p and rotational angel* φ is given in (1) by $a(\varphi) := 1$, $b(\varphi) := \cos\varphi$, $c(\varphi) := \sin\varphi$, $d(\varphi) := p\varphi$, $\varphi \in [0, 2\psi]$. The helix $\sigma(\vec{\mathbf{p}}^0)$ is given by

$$\vec{\mathbf{X}}(\phi) = \begin{pmatrix} 1 \\ \cos\phi \\ \sin\phi \\ p\phi \end{pmatrix}. \tag{3}$$

At $\phi := 0$, $\phi := \psi$ and $\phi := 2\psi$ we get the points $\vec{\mathbf{p}}^0, \vec{\mathbf{p}}^{0.5}$ and $\vec{\mathbf{p}}^1$ and their tangents contain the points $\vec{\mathbf{t}}^0, \vec{\mathbf{t}}^{0.5}, \vec{\mathbf{t}}^1$ respectively. They are

$$\begin{array}{llll} \vec{\mathbf{p}}^0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & \vec{\mathbf{p}}^{0.5} = \begin{pmatrix} 1 \\ cos\psi \\ \sin\psi \\ p\psi \end{pmatrix}, & \vec{\mathbf{p}}^1 = \begin{pmatrix} 1 \\ \cos 2\psi \\ \sin 2\psi \\ 2p\psi \end{pmatrix} & \text{and} \\ \vec{\mathbf{t}}^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ p \end{pmatrix}, & \vec{\mathbf{t}}^{0.5} = \begin{pmatrix} 0 \\ -\sin\psi \\ \cos\psi \\ p \end{pmatrix}, & \vec{\mathbf{t}}^1 = \begin{pmatrix} 0 \\ -\sin 2\psi \\ \cos 2\psi \\ p \end{pmatrix}. & \end{array} \tag{4}$$

3. The rational motions of degree three are studied in [10]. In [9] these motions are represented in terms of the *Bernstein polynomials of degree three* $B_j^3(v) := \binom{3}{j}(1-v)^{3-j}v^j$. Here we investigate axial motions with axis $x_1 = x_2 = 0$ only. Such a *rational axial motion of degree n* ζ has a representation (1) with $a(v), b(v), c(v), d(v)$ being polynomials of degree n. The corresponding matrices $A_{44}(v)$ may be written as

$$A_{44}(v) = \sum_{j=0}^{3} C_j B_j^3(v) \tag{5}$$

with $4 \times 4-$ matrices C_j

$$C_j := \begin{pmatrix} a_j & 0 & 0 & 0 \\ 0 & b_j & -c_j & 0 \\ 0 & c_j & b_j & 0 \\ d_j & 0 & 0 & a_j \end{pmatrix} \tag{6}$$

and real constants a_j, b_j, c_j, d_j. Specializing the results from [9] we get

$$C_0 = E_{44}, \qquad C_1 = \frac{1}{3}\begin{pmatrix} 2\Delta+\Gamma & 0 & 0 & 0 \\ 0 & 2\Delta+\Gamma & -2\beta & 0 \\ 0 & 2\beta & 2\Delta+\Gamma & 0 \\ 3d_1 & 0 & 0 & 2\Delta+\Gamma \end{pmatrix},$$

$$C_2 = \frac{1}{3}\begin{pmatrix} \Delta^2+2\Delta\Gamma+\beta^2 & 0 & 0 & 0 \\ 0 & \Delta^2+2\Delta\Gamma-\beta^2 & -2\beta(\Delta+\Gamma) & 0 \\ 0 & 2\beta(\Delta+\Gamma) & \Delta^2+2\Delta\Gamma-\beta^2 & 0 \\ 3d_2 & 0 & 0 & \Delta^2+2\Delta\Gamma+\beta^2 \end{pmatrix}, \tag{7}$$

$$C_3 = \Delta\begin{pmatrix} (\Delta^2+\beta^2)\Gamma & 0 & 0 & \\ 0 & (\Delta^2-\beta^2)\Gamma & -2\beta\Delta\Gamma & 0 \\ 0 & 2\beta\Delta\Gamma & (\Delta^2-\beta^2)\Gamma & 0 \\ d_3 & 0 & 0 & (\Delta^2+\beta^2)\Gamma \end{pmatrix}.$$

There $d_1, d_2, d_3, \beta, \Gamma, \Delta$ are real constants. Then our *starting point* $\vec{\mathbf{p}}^0$ with coordinates $(1,1,0,0)^t$ will describe a rational path of degree three $\zeta(\vec{\mathbf{p}}^0)$. It is a *rational Bézier curve* with control points

$$\vec{\mathbf{b}}^j := C_j\vec{\mathbf{p}}^0 = \begin{pmatrix} a_j \\ b_j \\ c_j \\ d_j \end{pmatrix} \qquad j = 0\cdots 3. \tag{8}$$

The components a_j are the weights of $\vec{\mathbf{b}}^j$. Now the curve (2) has the representation

$$\zeta(\vec{\mathbf{p}}^0) := \vec{\mathbf{z}}(v) = \sum_{j=0}^{3} \vec{\mathbf{b}}^j B_j^3(v). \tag{9}$$

4. To interpolate the helix (3) by the rational cubic Bézier curve (9) we may require

$$\begin{aligned} \vec{\mathbf{z}}(0) &= \vec{\mathbf{p}}^0 \quad \text{with the same tangent,} \\ \vec{\mathbf{z}}(0.5) &= \vec{\mathbf{p}}^{0.5} \quad \text{and} \\ \vec{\mathbf{z}}(1) &= \vec{\mathbf{p}}^1 \quad \text{with the same tangent.} \end{aligned} \tag{10}$$

Then a short computation gives the result

$$\begin{aligned} &\beta = \sin\psi, \quad \Delta = \cos\psi, \quad \Gamma = 1, \\ &d_0 = 0, \quad d_1 = \tfrac{2p}{3}\sin\psi, \\ &d_2 = \tfrac{2p}{3}[\psi(1+\cos\psi) - \sin\psi], \quad d_3 = 2p\psi. \end{aligned} \tag{11}$$

Now all control points of the interpolant are fixed. They are

$$\vec{\mathbf{b}}^0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{b}}^1 = \tfrac{1}{3} \begin{pmatrix} 1 + 2\cos\psi \\ 1 + 2\cos\psi \\ 2\sin\psi \\ 2p\sin\psi \end{pmatrix}$$
$$\vec{\mathbf{b}}^2 = \tfrac{1}{3} \begin{pmatrix} 1 + 2\cos\psi \\ \cos 2\psi + 2\cos\psi \\ 2\sin\psi(1+\cos\psi) \\ 2p[\psi(1+2\cos\psi) - \sin\psi] \end{pmatrix} \quad \vec{\mathbf{b}}^3 = \begin{pmatrix} 1 \\ \cos 2\psi \\ \sin 2\psi \\ 2p\psi \end{pmatrix}. \tag{12}$$

5. The control points (12) determine the matrices C_j and by (5) the motion ζ. It has the rotational angel $2v\psi$. Points on $\sigma(\vec{\mathbf{p}}^0)$ with parameter φ and points on $\zeta(\vec{\mathbf{p}}^0)$ with parameter $v = \frac{\varphi}{2\psi}$ lie on a straight line parallel to $x_1 = x_2 = 0$.

Figure 1 shows a helix (dotted) and its cubic interpolant in an axonometric view (we have put $\psi := \frac{5\pi}{12}$). Both lie on the same cylinder of revolution.

Figure 1

If a rational Bézier curve of degree n with control points $\vec{\mathbf{c}}^0, \cdots \vec{\mathbf{c}}^n$ is moved by the helical motion σ we get a rectangular helical patch, which may be interpolated by a rational rectangular $n \times 3$ Bézier patch Φ. Its control points are

$$\vec{\mathbf{b}}^{ij} := C_j \vec{\mathbf{c}}^i \qquad i = 0 \cdots n, \quad j = 0 \cdots 3. \tag{13}$$

This patch is generated by our interpolating axial motion ζ (see [8]).

Figure 2 shows such an interpolating rational Bézier patch with its control points.

Figure 2

6. Now we are going to interpolate the helical motion by an axial motion of degree four. The rational motions of degree four are investigated in [Röschel '85]. For the subset of motions with a fixed direction a rational Bernstein representation is given in [9] . As mentioned above, the corresponding matrices $A_{44}(v)$ may be written as [1]

$$A_{44}(v) = \sum_{j=0}^{4} C_j B_j^4(v). \tag{14}$$

[1]The terms $B_j^4(v)$ are the Bernstein polynomials of degree four.

We specialize the results from [9] for axial motions and obtain

$$C_0 = E_{44}, \qquad C_1 = \begin{pmatrix} \Delta + \frac{B\alpha}{4} & 0 & 0 & 0 \\ 0 & \Delta + \frac{B\alpha}{4} & -\frac{B}{2} & 0 \\ 0 & \frac{B}{2} & \Delta + \frac{B\alpha}{4} & 0 \\ d_1 & 0 & 0 & \Delta + \frac{B\alpha}{4} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} \Delta^2 + \frac{\alpha B\Delta}{2} + \frac{B^2}{6}(\beta+1) & 0 & 0 & 0 \\ 0 & \Delta^2 + \frac{\alpha B\Delta}{2} + \frac{B^2}{6}(\beta-1) & -B(\Delta + \frac{\alpha B}{3}) & 0 \\ 0 & B(\Delta + \frac{\alpha B}{3}) & \Delta^2 + \frac{\alpha B\Delta}{2} + \frac{B^2}{6}(\beta-1) & 0 \\ d_2 & 0 & 0 & \Delta^2 + \frac{\alpha B\Delta}{2} + \frac{B^2}{6}(\beta+1) \end{pmatrix}, \tag{15}$$

$$C_3 = \begin{pmatrix} \Delta^3 + \frac{\alpha B}{4}E + \frac{B^2\Delta}{2}(\beta+1) & 0 & 0 & 0 \\ 0 & \Delta^3 + \frac{\alpha B}{4}F + \frac{B^2\Delta}{2}(\beta-1) & -\frac{B}{2}(3\Delta^2 + 2\alpha B\Delta + \beta B^2) & 0 \\ 0 & \frac{B}{2}(3\Delta^2 + 2\alpha B\Delta + \beta B^2) & \Delta^3 + \frac{\alpha B}{4}F + \frac{B^2\Delta}{2}(\beta-1) & 0 \\ d_3 & 0 & 0 & \Delta^3 + \frac{\alpha B}{4}E + \frac{B^2\Delta}{2}(\beta+1) \end{pmatrix},$$

$$E := 3\Delta^2 + B^2, \quad F := 3\Delta^2 - B^2 \qquad \text{and}$$

$$C_4 = \begin{pmatrix} (\Delta^2 + B^2)(\Delta^2 + \alpha B\Delta + \beta B^2) & 0 & 0 & 0 \\ 0 & (\Delta^2 - B^2)(\Delta^2 + \alpha B\Delta + \beta B^2) & -2B\Delta(\Delta^2 + \alpha B\Delta + \beta B^2) & 0 \\ 0 & 2B\Delta(\Delta^2 + \alpha B\Delta + \beta B^2) & (\Delta^2 - B^2)(\Delta^2 + \alpha B\Delta + \beta B^2) & 0 \\ d_4 & 0 & 0 & (\Delta^2 + B^2)(\Delta^2 + \alpha B\Delta + \beta B^2) \end{pmatrix}.$$

There $d_1, d_2, d_3, d_4, B, \alpha, \beta, \Delta$ denote real constants. We investigate quartic rational Bézier curves now. Like we did in chapter 3 we put

$$\zeta(\vec{\mathbf{p}}^0) := \vec{\mathbf{z}}(v) = \sum_{j=0}^{4} \vec{\mathbf{b}}^j B_j^4(v) \tag{16}$$

and

$$\vec{\mathbf{b}}^j := C_j \vec{\mathbf{p}}^0 = \begin{pmatrix} a_j \\ b_j \\ c_j \\ d_j \end{pmatrix}. \tag{17}$$

To interpolate the helix (3) by our Bézier curve b (17) we may require

$$\begin{array}{ll} \vec{\mathbf{z}}(0) = \vec{\mathbf{p}}^0 & \text{with the same tangent,} \\ \vec{\mathbf{z}}(0.5) = \vec{\mathbf{p}}^{0.5} & \text{and} \\ \vec{\mathbf{z}}(1) = \vec{\mathbf{p}}^1 & \text{with the same tangent.} \end{array} \tag{18}$$

Hence the interpolant b lies on the same cylinder of revolution like the helix. Then a short computation gives the result

$$B = \sin\psi, \quad \Delta = \cos\psi, \quad \beta \sin^2\psi = \sin\psi(\sin\psi - \alpha\cos\psi), \tag{19}$$

$$d_0 = 0, \quad d_4 = 2p\psi \quad \text{and} \quad p\psi(a_0 + 4a_1 + 6a_2 + 4a_3 + a_4 = (d_0 + 4d_1 + 6d_2 + 4d_3 + d_4). \tag{20}$$

We get the control points of the interpolant as follows [2]

$$\vec{\mathbf{b}}^0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{b}}^1 = \begin{pmatrix} \cos\psi + \frac{\alpha}{4}\sin\psi \\ \cos\psi + \frac{\alpha}{4}\sin\psi \\ \frac{\sin\psi}{2} \\ \frac{p}{2}\sin\psi \end{pmatrix} \quad \vec{\mathbf{b}}^2 = \begin{pmatrix} \cos^2\psi + \frac{\sin\psi}{3}(\sin\psi + \alpha\cos\psi) \\ cos\psi(\cos\psi + \frac{\alpha}{3}\sin\psi) \\ \sin\psi(\cos\psi + \frac{\alpha}{3}\sin\psi) \\ \frac{p\psi}{3}(1 + 2\cos^2\psi + \alpha\sin\psi\cos\psi) \end{pmatrix}$$

$$\vec{\mathbf{b}}^3 = \begin{pmatrix} \cos\psi + \frac{\alpha}{4}\sin\psi \\ \cos^3\psi + \frac{\alpha}{4}\sin\psi(\cos^2\psi - \sin^2\psi) \\ \frac{sin\psi}{2}(1 + 2\cos^2\psi + \alpha\sin\psi\cos\psi) \\ \frac{p}{2}(4\psi\cos\psi - \sin\psi + \alpha\psi\sin\psi) \end{pmatrix} \quad \vec{\mathbf{b}}^4 = \begin{pmatrix} 1 \\ \cos 2\psi \\ \sin 2\psi \\ 2p\psi \end{pmatrix}. \tag{21}$$

The parameter α may be choosen by the following proposition: The tangent of b at $v = \frac{1}{2}$ contains the point $\vec{\mathbf{t}} = \frac{1}{8}[\vec{\mathbf{b}}^0 + 3(\vec{\mathbf{b}}^1 + \vec{\mathbf{b}}^2) + \vec{\mathbf{b}}^3]$. If the three points $\vec{\mathbf{p}}^{0.5}$, $\vec{\mathbf{t}}^{0.5}$ and $\vec{\mathbf{t}}$ lie on one straight line, the tangents of the Bézier curve b at $v = \frac{1}{2}$ and the helix at $\phi = \psi$ coincide. This gives the condition

$$\sin\psi[d_0+3(d_1+d_2)+d_3] = p\{(\psi\sin\psi+\cos\psi)[a_0+3(a_1+a_2)+a_3]-[b_0+3(b_1+b_2)+b_3]\}. \tag{22}$$

By the use of (21) we are able to compute α and get the tenuous result

$$\alpha = \frac{2\psi(1 + 2\cos\psi) - 2\sin\psi(2 + \cos\psi)}{\sin\psi(\sin\psi - \psi)}. \tag{23}$$

Using this constant all free parameters of our Bézier interpolant are determined and so the axial motion too. As mentioned above,we use this motion to generate kinematic rational Bézier patches again, which interpolate given helical surface patches.

There is no use to draw a figure showing a helix and its quartic interpolant, because the eye cannot perceive any difference between the two curves. Figure 3 shows a rational 3×4 rational Bézier patch which is generated by the algorithm given above.

Figure 3

Remark: There are further possibilities to interpolate such a helical motion by an axial one. The authors heave tried to give another quartic interpolation, where the interpolant b and the helix (3) have the same tangents and osculating planes at $v = 0$ and $v = 1$. But we recognized that in this case the difference between helix and interpolant is greater than that of chapter 6.

[2] There we have substituted β as given in (19)

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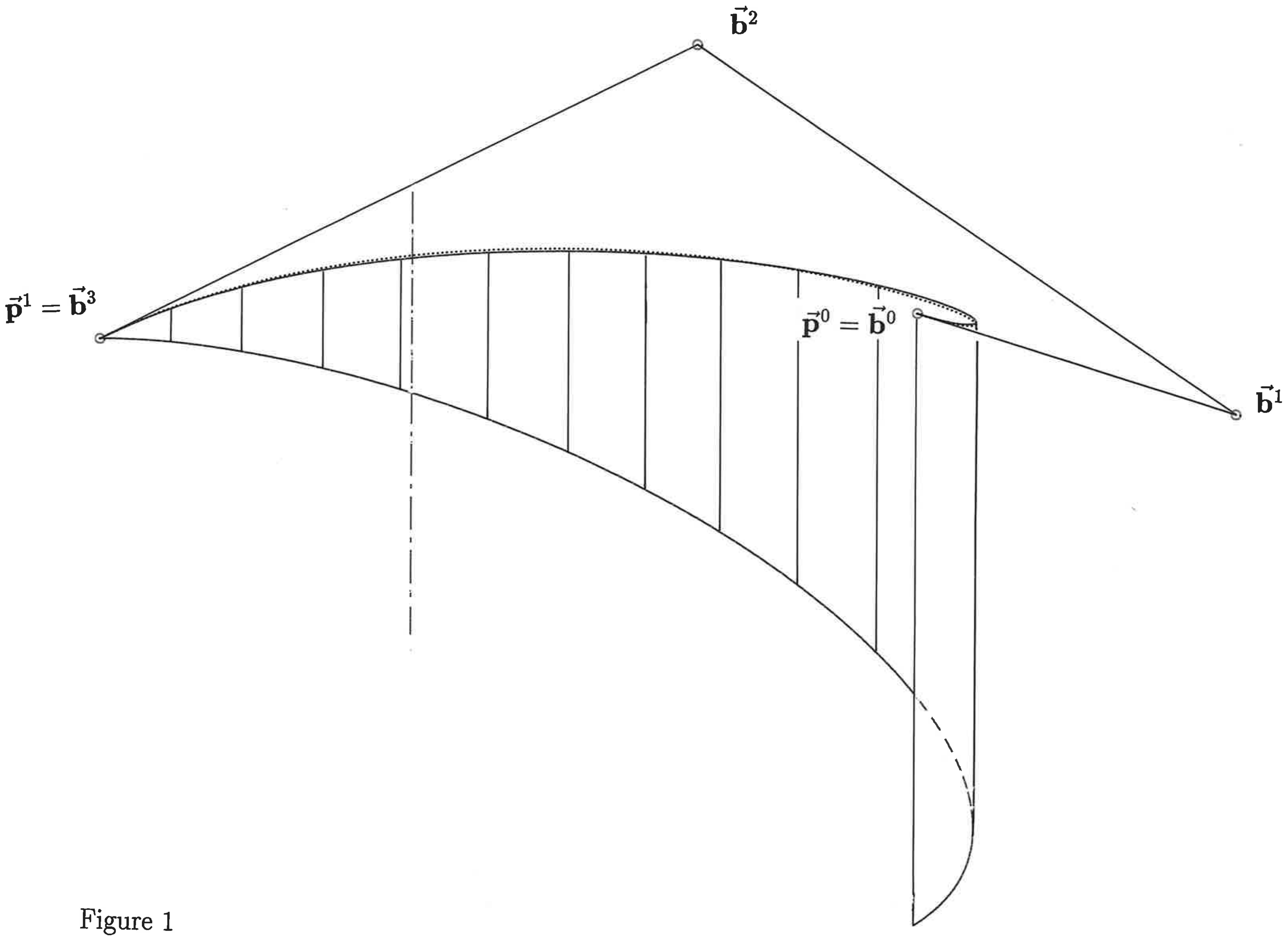
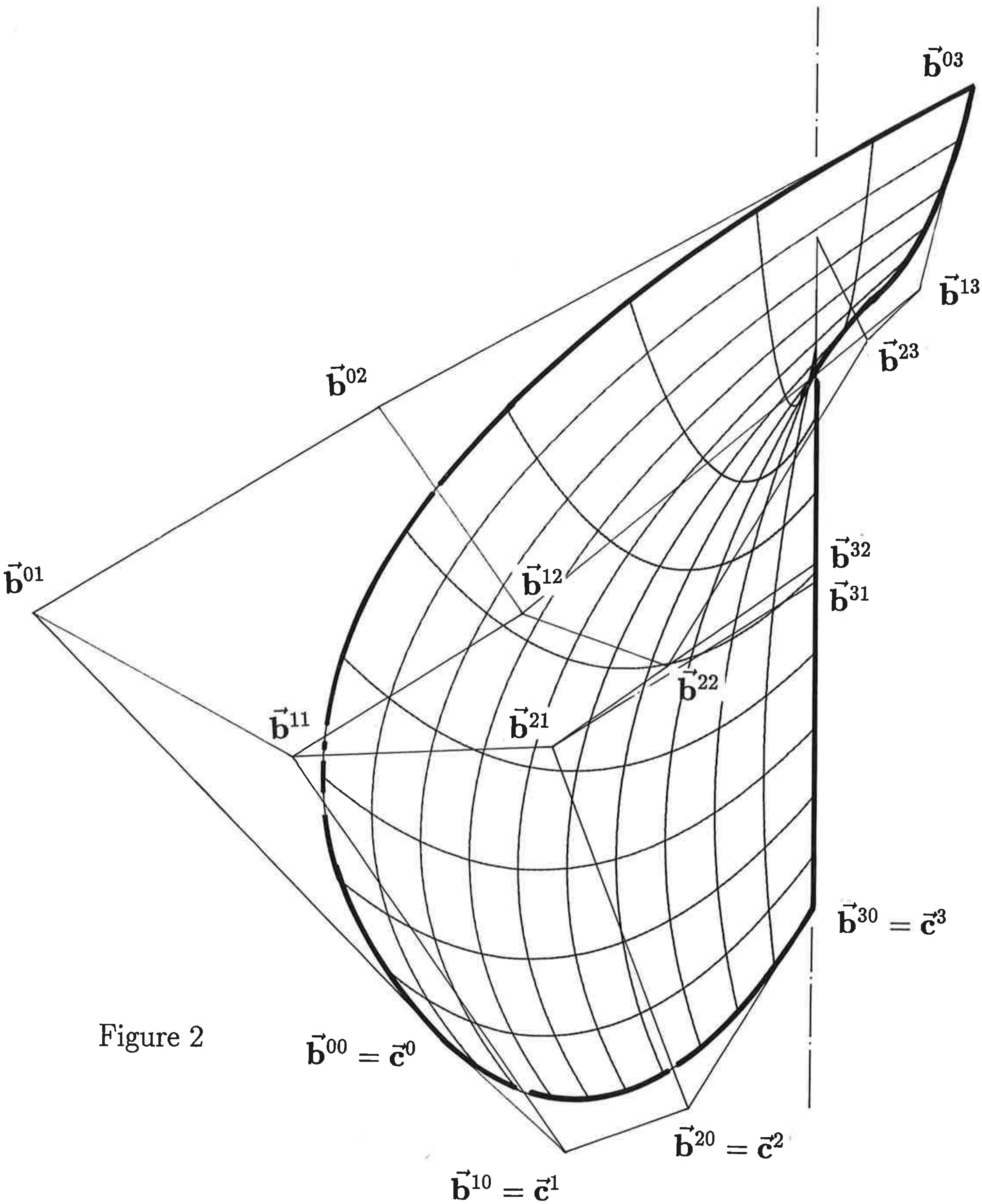


Figure 1



$\vec{b}^{03}$
$\vec{b}^{13}$
$\vec{b}^{23}$
$\vec{b}^{02}$
$\vec{b}^{32}$
$\vec{b}^{01}$
$\vec{b}^{12}$
$\vec{b}^{31}$
$\vec{b}^{22}$
$\vec{b}^{11}$
$\vec{b}^{21}$
$\vec{b}^{30} = \vec{c}^3$
$\vec{b}^{00} = \vec{c}^0$
$\vec{b}^{20} = \vec{c}^2$
$\vec{b}^{10} = \vec{c}^1$

Figure 2

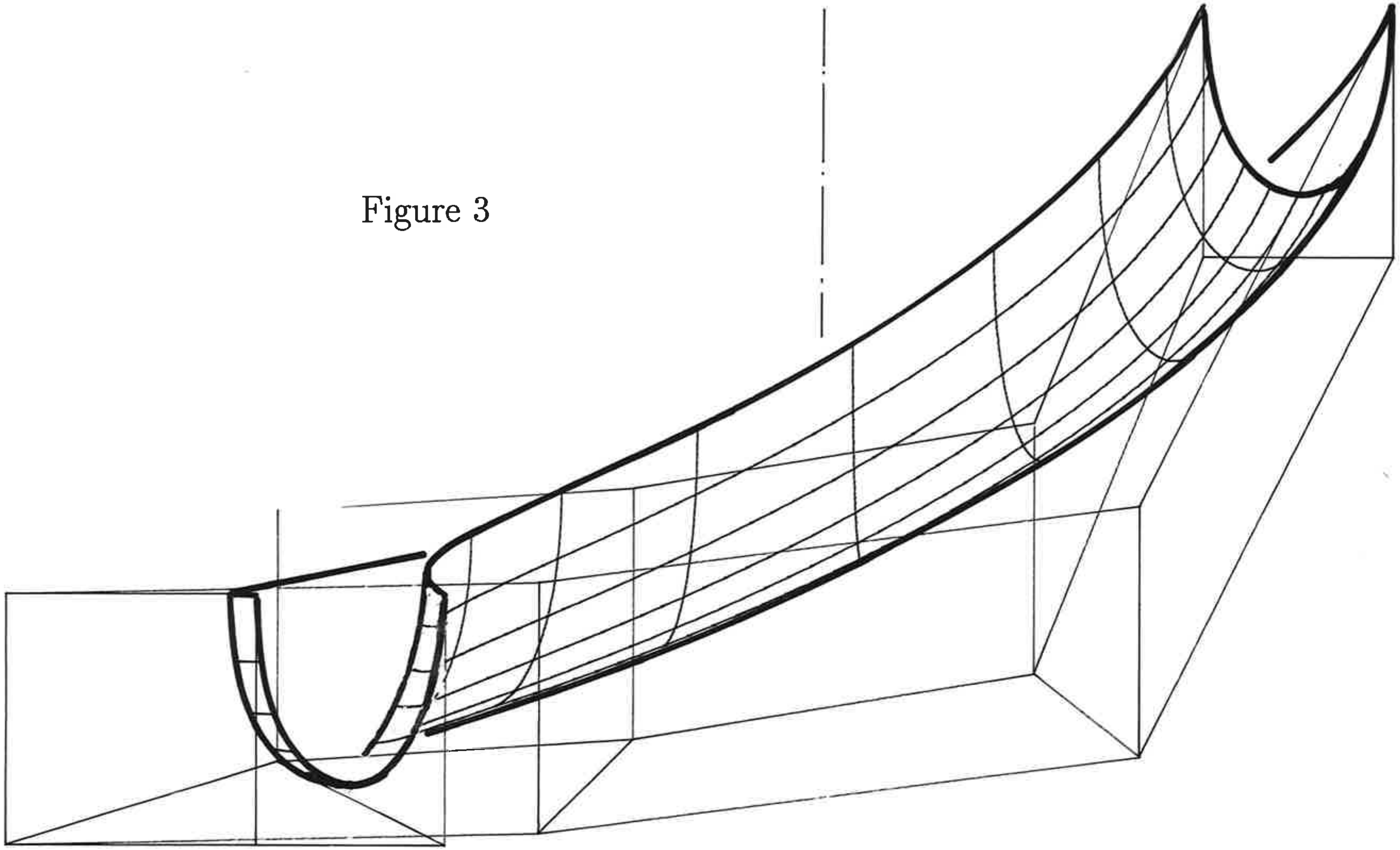


Figure 3