MÖBIUS MECHANISMS

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Abstract: The paper gives a construction of a new series of overconstrained spatial mechanisms with six systems connected via nine spherical 2R-joints. The mechanisms are designed by means of plane equiform motions. This new type of overconstrained mechanisms will be called *Möbius mechanisms.* By removing one of its joints, the so-called *reduced Möbius mechanisms* will be set up, still being overconstrained. A special example is being studied in detail: It admits interesting self-motions of different degrees of freedom. This is why it represents a new example of *kinematotropic mechanisms.*

1. Linear equiform motions with a globally fixed point

The equiform group of the Euclidean plane ε^* consists of equiform transformations: They do not change angles, but possibly lengths of geometric figures. Kinematics with respect to this group has first been studied by (Krause, 1910). We consider special one-parametric equiform motions ζ^*: We fix a point A^* and a line g^* not containing A^*. Then there exists a one-parametric equiform motion $\zeta^* = \zeta^*(t) := \varepsilon/\varepsilon^*$ of a moving plane ε with respect to the fixed plane ε^* such that a point $A \in \varepsilon$ coincides with the globally fixed point $A^* \in \varepsilon^*$ and a point $P \in \varepsilon$ moves on the line g^*. Then ζ^* moves all general points of ε on straight lines (see figure 1). Such equiform motions are called *linear equiform motions with a globally fixed point* A^* (cf. (Yaglom, 1968), p. 71).

Any line h^* (not through A^*) is the path of one and only one point $Q \in \varepsilon$ (construction with the help of equal angles - see figure 1).

We use Cartesian frames $\{A^*, x^*, y^*\}$ in ε^* and $\{A, x, y\}$ in ε, respectively. The motion ζ^* can be parametrized by

$$\begin{aligned} \zeta^*: \quad & (x,y) \longrightarrow (x^*,y^*) \qquad \text{with} \\ x^*(t,x,y) &:= (x\ \cos t\ -\ y \sin t)/\cos t \quad (-\pi/2 \le t \le \pi/2) \\ y^*(t,x,y) &:= (x\ \sin t\ +\ y \cos t)/\cos t. \end{aligned} \tag{1}$$

As a consequence, ζ^* can be generated by a rotation with the center A^* (angle t) followed by a scaling with the center A^* and the factor $1/\cos t$. The motion ζ^* has a rotational symmetry with respect to the center A^*. At the moment $t = 0$ we have: Observed from A^* the points of ε are the pedal points on their straight line paths.

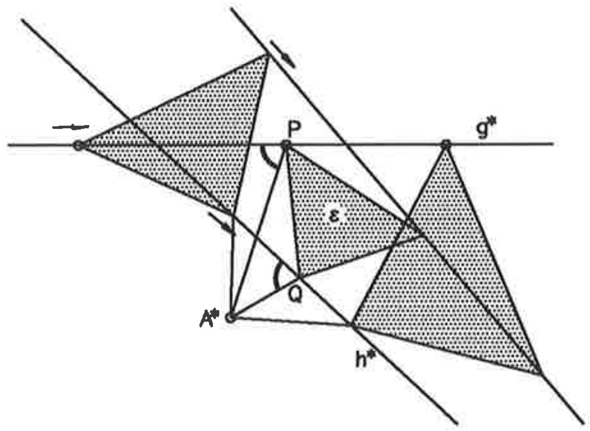

Figure 1. A linear equiform motion ζ^* with globally fixed point A^* and some point paths

2. A chain of six linear equiform motions

Further on we will use some considerations of the paper (Roeschel, 1996a). Taking a sphere Φ^* (center M^*) in a fixed Euclidean three-space E_3 we choose six different points $\{A_i^* \in \Phi^*;\ i = 1, ..., 6\}$. Let the plane τ_i^* be tangent to Φ^* at A_i^*. By $s_{i,j}^*$ we denote the intersection lines of the planes τ_i^* and τ_j^*; the plane containing $s_{i,j}^*$ and M^* is called $\sigma_{i,j}^*$ $(i, j = 1, ..., 6)$. Seen from A_i^* the pedal points on the lines $s_{i,j}^*$ are called $P_{i,j}^*$ $(j \neq i \ mod\ 2)$.

We start with six Euclidean congruent linear equiform motions $\zeta_i^*(t) := \varepsilon_i/\tau_i^*$ in the planes τ_i^* with globally fixed points A_i^* $(i = 1, ..., 6)$. Observed from the outside, the orientations of $\zeta_1^*(t), \zeta_3^*(t)$ and $\zeta_5^*(t)$ with increasing t are chosen clockwise, the others counterclockwise.

So for all $j \neq i \ mod\ 2$ the plane equiform motions $\zeta_j^*(t) := \varepsilon_j/\tau_j^*$ can be gained from $\zeta_i^*(t)$ by reflection with respect to the plane of symmetry $\sigma_{i,j}^*$. As a consequence, for all $t \in (-\pi/2, \pi/2)$ we can link the moving planes ε_j and ε_i $(j \neq i \ mod\ 2)$ via the point paths on the straight lines $s_{i,j}^*$. All (three) linked points of the moving plane ε_i determine a triangle. At the moment $t = 0$ the positions of these points are just the pedal points $P_{i,j}^*$. Figure 2 shows the situation for $i = 1$ - the orientation of the forward process of the motions being indicated by an arrow.

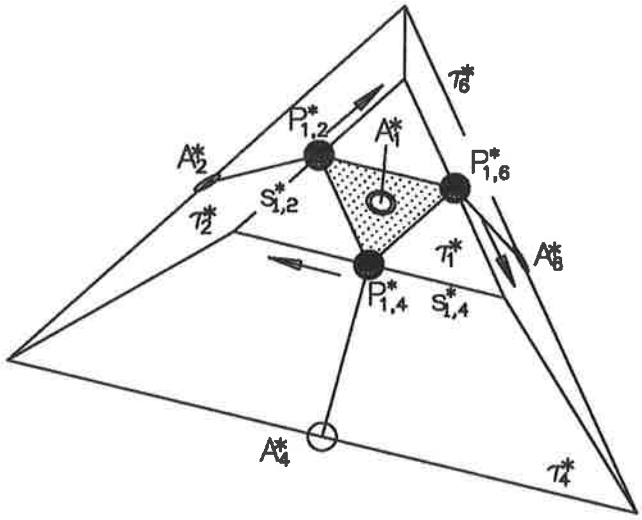

Figure 2. The tangent plane τ_1^* and its neighbours τ_2^*, τ_4^* and τ_6^*

This procedure in general gives a configuration of six triangles in the six tangent planes τ_i^* which are linked to each other. The configuration remains closed, if all plane linear equiform motions ζ_i^* with the globally fixed points A_i^* are performed with the same time parameter t.

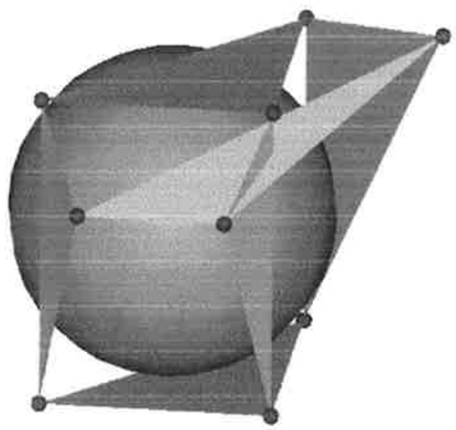

Figure 3. The closed chain of 6 triangles in the tangent planes of a sphere

As an example Figure 3 shows this situation at the moment $t = 0$. The triangles are parts of six planes tangent to the sphere. Our linked points are given by small spheres. As mentioned before we have: At the moment $t = 0$ the linked points are the pedal points on the straight lines $s^*_{i,j}$ $(i \neq j \; mod \, 2)$. Consequently, their construction is easy.

It seems to be impossible to gain a configuration without self-intersections of the six triangles. Our configuration can be interpreted as an immersion of a part of a Möbius strip into the sphere (see figure 4). As this is impossible without self-intersection, the same is supposed to hold for our configuration as well.

The same procedure can be performed for any even number $2n \geq 6$. This would generate more complicated configurations consisting of $n-$gons in the planes tangent to our sphere. As the number of intersections increases with rising n, we will not consider these generalisations in this paper.

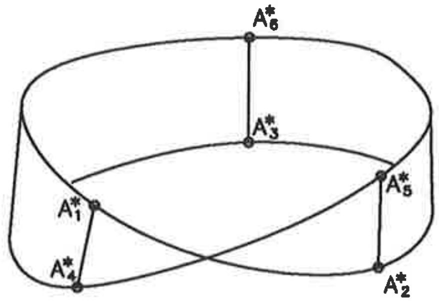

Figure 4. Möbius strip: Systems given by points A^*_i, linked systems indicated by connections of points

3. Möbius mechanisms

Our six linear equiform motions $\zeta^*_i = \varepsilon_i/\tau^*_i$ are congruent (even with respect to the time parameter t). The common scaling factor is $1/\cos t$. If we perform all linear equiform motions $\zeta^*_i(t)$ and an additional scaling with respect to the center of the sphere and the factor $\cos t$, we gain a series of *spatial one-parameter motions which turn out to be Euclidean*, due to the scaling factor having been chosen appropriately. These motions shall be called $\eta^*_i(t)$ of systems ϕ_i with respect to the fixed space. All these motions are pairwise congruent (including their parametrisation!). Each of them has a fixed direction ($a^*_i := [M^*, A^*_i]$). As shown in (Roeschel, 1996a), $\eta^*_i(t)$ is an axial DARBOUX-motion with the fixed axis a^*_i. Its parametrisation is given by (see (Bottema and Roth, 1979) pp. 301)

$$\begin{array}{rl} \eta^* : & (x, y, z) \longrightarrow (x^*, y^*, z^*) \qquad \text{with} \\ & x^*(t, x, y, z) := x \; \cos t \; - \; y \; \sin t \\ & y^*(t, x, y, z) := x \; \sin t \; + \; y \; \cos t \\ & z^*(t, x, y, z) := \cos t \; + \; z \; . \end{array} \tag{2}$$

The *relative motion* of two different systems has been studied in (Roeschel, 1996b): In general it is a *rational one-parameter motion of degree 4.*

Scaling does not change the angles between the moving planes ϕ_i and ϕ_j. Therefore spherical 2R-joints at the linked points do not affect the one-parameter motions. We have arrived at a mechnism shown in figure 5: It consists of six systems linked via nine spherical 2R-joints. Its general degree of freedom is supposed to be - according to the number of systems and linkages -

$$f = 5 \times 6 - 9 \times 4 = -6. \tag{3}$$

Our generation, however, guarantees one-parametric movability at least. Given that it has the topolgy of a Möbius strip (see figure 4), we suggest to call this new family of overconstrained mechansims *Möbius mechanisms.* In order to avoid undesired self-intersections we take some offsets of systems and build some kind of *bridge* (see figure 5).

Figure 5. A Möbius mechanism

4. Reduced Möbius mechanisms

Removing one of the spherical 2R-joints, we can get a *reduced Möbius mechanism.* The moving systems remain well-defined, if one vertex of the triangles is omitted. The corresponding triangle degenerates into a 2–gon. Together with the corresponding revolutes (belongig to one of our 2R-joints) it determines a system by itself being displayed as a rectangle. In the figures 6 and 7 we have two of them.

This reduced Möbius mechanism consists of six systems linked via eight spherical 2R-joints. As its theoretical degree of freedom would be $f =$

$5 \times 6 - 8 \times 4 = -2$ this new mechanism keeps on being overconstrained. It possibly allows more motions than that of our construction.

The following figure 6 shows a special example starting with the six planes of a cube. There $\tau_1^*, ..., \tau_4^*$ are the faces of a *belt*, τ_5^* and τ_6^* being the other faces. Given that τ_5^* and τ_6^* are parallel planes, their 2R-joint would be at infinity. If we omit this joint we get the reduced Möbius mechanism shown in figure 6.

It remarkably admits a great variety of self-motions: Some of them turn out to offer two degrees of movability:

a) Two independent rotations with respect to two different
axes of the system
b) A one-parametric motion with parallel (or antiparallel)
arms combined with an independent rotation (two possibilities)
c) A two-parametric translatoric motion with spherical paths

All three do not contain the one-parametric motion, which has been worked out by our construction. There exist further one-parametric self-motions of the mechanism, which have not been listed above. This is why the investigation of this interesting type of mechanism is by no way finished until now. This mechanism belongs to the family of linkages offering various

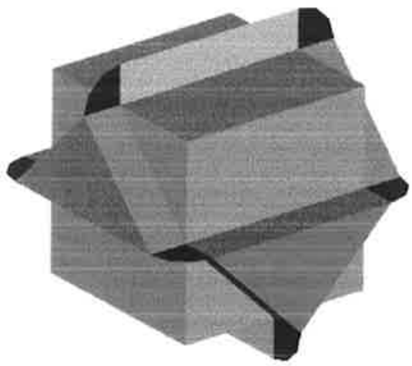

Figure 6. A reduced Möbius mechanism starting in the planes of a cube ($t = 0$)

self-motions of different degrees of freedom, which are essentially diverse. Such mechanisms are called *kinematotropic linkages*, which have recently been studied in (Wohlhart, 1996).

Figure 7 shows some positions of the one-parameter motion which we were starting with. At $t = \pm\pi/2$ some edges of the mechanism coincide. So our mechanisms provides movability without self-intersections for $t \in (-\pi/2, \pi/2)$.

5. Conclusions

Following the ideas of a previous paper (Roeschel, 1996a) we were able to design a new series of overconstrained mechanisms. They consist of six systems linked via nine spherical 2R-joints. As the generation of these mechanisms used an immersion of a Möbius strip into a sphere, they were called *Möbius mechanisms*. Removing one of the 2R-joints we could avoid undesired self-intersections gaining the so-called *reduced Möbius mechanisms*. These new mechanisms are related to the famous *Heureka-polyhedron* and the *Turning Tower* studied in various papers (e.g. (Stachel, 1991), (Stachel, 1992), (Wohlhart, 1993a), (Wohlhart, 1993b), (Wohlhart, 1995) and (Wohlhart, 1998)). Different to those mechanisms our new series consists of merely six moving systems.

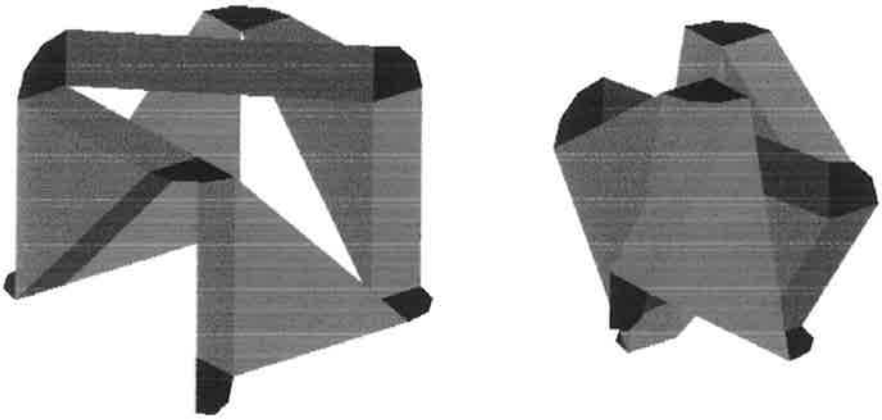

Figure 7. The reduced Möbius mechanism of figure 6 at $t = \pi/4$ and $t = 4\pi/9$

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