NEW MODELS OF MOVEABLE POLYHEDRA

Dedicated to O. Univ.-Prof. Dr H. STACHEL on the occasion of his 60th birthday

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ABSTRACT
The first few sections of this paper recall methods and results of the series by Roeschel (1995) – (2001): The definition of overconstrained mechanisms, the set up of special plane equiform motions and their application to the making of overconstrained mechanisms. Subsequently we go for some novel mechanisms of this type: We will consider starting polyhedra with a *closed band structure*. Bands with and without twist will be studied in detail. Nice examples with one twist are gained from the Moebius strip. This will result in the generation of ocerconstrained polyhedral models which we call *Moebius-mechanisms*. A physical model without self-intersections is presented.
Key words: Kinematics, Overconstrained linkages.

1. OVERCONSTRAINED MECHANISMS

In the 3-dimesnional Euclidean space a displacement is determined by 6 parameters (3 from translations, 3 from rotations). Multibody mechanisms consist of a series of n rigid bodies, which are linked by r linkages. The number of independent parameters of the mobility of a linkage is called *the degree of freedom* f_i *of the linkage.* Without considering geometric details and possible dependencies the mechanism will have a *theoretical degree of freedom F*, which amounts to

$$(1) \qquad F = 6\times(n-1-r) + \sum_{i=1}^{r} f_i .$$

This formula is referred to as Gruebler's formula (see Beyer (1963) p.102. $F \le 0$ characterizes theoretically rigid mechansims. If all geometric properties and dependencies of the mechanisms are known, we are able to compute its actual degree of freedom Φ. If $\Phi \ge 1$ the mechanism will be moveable, regardless of $F \le 0$. Such mechanisms are called overconstrained. Many examples of such overconstrained mechanisms are known. The paper will add some new examples which are generated with the help of polyhedra – therefore we will speak of moveable models of polyhedra. As a special example we refer to the famous HEUREKA-Polyhedron and the proof of its moveability by Stachel (1991).

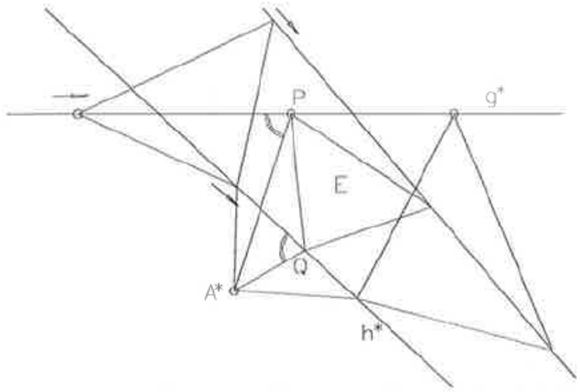

Figure 1: A linear equiform motion with globally fixed point $A*$ at 3 positions and 3 point paths

2. PLANE LINEAR EQUIFORM MOTIONS WITH A GLOBALLY FIXED POINT

In the Euclidean plane there are equiform motions $\varepsilon(t) := E / E^*$ (fixed plane E^*, moving plane E) with a globally fixed point $A^* \in E^*$ which move all points of E on straight lines (not through A*). We use Cartesian frames $\{A^*, x^*, y^*\}$ and $\{A, x, y\}$ in E^* and E, respectively. Then $\varepsilon(t)$ can be parametrized by

$$(2) \qquad \begin{aligned} (x, y) &\to \qquad (x^*, y^*) \\ x^*(t,x,y) &= (x\cos t - y\sin t)/\cos t \\ y^*(t,x,y) &= (x\sin t + y\cos t)/\cos t \end{aligned}$$

with $-\pi/2 \le t \le \pi/2$. Any straight line g^* (not through A^*) is the point path of exactly one point of the moving plane E (see Yaglom (1968) p. 71 and figure 1).

3. A CHAIN OF LINEAR EQUIFORM MOTIONS

This plane configuration is used as the starting element $A_0^* = A^*$, $\varepsilon_0(t) = E_0 = E / E_0^* = E^*$. Successive reflections with respect to planes $\sigma_i^*, i = 1, \ldots, k$ builds up a series of reflected plane equiform motions $\varepsilon_i(t) = E_i / E_i^*$ where each has straight line point paths and a globally fixed point $A_i^* \in E_i^*$, though they run in different planes E_i^* of the 3-space. Moreover, all are congruent even with respect to their parametrisation. Given that the fixed points A_i^* do not belong to the intersections $E_i^* \cap E_{i+1}^*$ or $E_{i-1}^* \cap E_i^*$ there exists exactly one point in E_i with its point path on $E_{i-1}^* \cap E_i^*$. This way, the procedure generates a chain of equiform plane motions, which are linked via common straight line paths.

In Roeschel (2001) closed chains of four linked motions of this type have been characterized: The configuration can be closed iff the four points $A_0 * := A*$ and $A_i *$ either belong to a circle, a straight line or are lying on a sphere κ. In the first two cases the corresponding planes $E_i *$ $(i = 1,2,3)$ are gained by succesive reflections of $E* := E_0 *$ with respect to the planes of symmetry of the pairs $A_{i-1}*, A_i *$. In the third case all planes $E_i *$ (including $E_0 *$) have to be tangent to the sphere κ. Figure 2 shows this situation. Any two linear equiform motions in the tangent planes $E_i *$ and $E_j *$ of a given sphere can be linked by a common point path, if the orientation fits. This way the linkage points generate polygons in the planes $E_i *$.[1] The arrows (figure 2) indicate the orientation of the point paths with decreasing t.

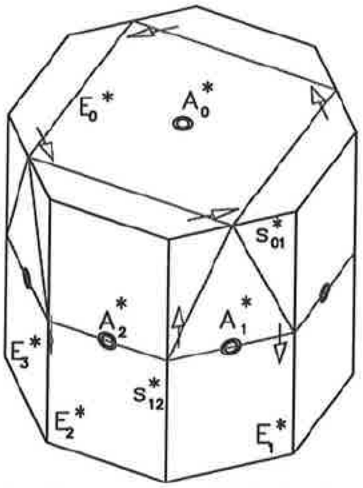

Figure 2: Equiform motion of linked polygons

3. A CHAIN OF LINKED SPACE MOTIONS

If we perform all these linked equiform plane motions, compose them with similarity transformations (scalings with factor $\cos t$) from a fixed point $O*$ cancelling out the common scaling factor and extend the outcome into the 3-space we get chains of linked bodies. As the angle between the planes of our equiform motions remains invariant under these scalings, each two neighbouring bodies $\Sigma_{i-1}*, \Sigma_i *$ can be linked via spherical 2R- joints[2]. The rigid bodies can be represented by orthogonal prisms: Their orthogonal intersections (basic polygons) are those from figure 2. The edges orthogonal to the basic plane are used as the 2 intersecting axes of rotation building up the spherical 2R-joint. Figure 3 displays that fact.

There is an interesting special case: It can happen that the planes $E_{i-1} *$, $E_i *$ coincide. Then the angle between the linked prisms will be 0. In these cases the 2R-joint will become a 1R-joint with its axis orthogonal to $E_{i-1} * = E_i *$.

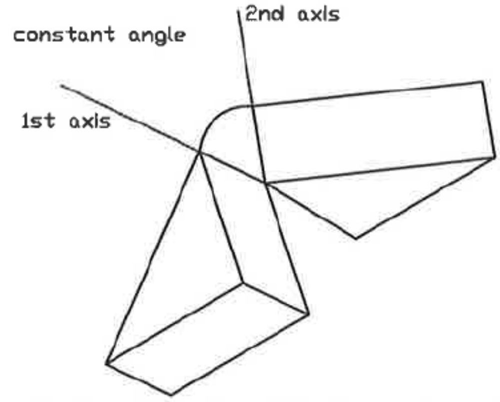

Figure 3: Two rigid prisms linked by a spherical 2R-

4. PRIMARY EXAMPLES

There is a great variety of mechanisms generated this way, which can be classified with respect to their topological structure. Most of the examples of Wohlhart (1993a) – (1998) and Verheyen (1989) can be generated this way. Figure 4 shows a topologic type of a sphere. It has the structure of a truncated cube and consists of 14 rigid parts linked by 24 spherical 2R-linkages. Its theoretical degree of freedom has the value $F = -18$.

Figure 5 shows a photo of a physical model of the topological type of a torus: It consists of 24 rigid parts linked by 42 spherical 2R-linkages and by 6 1R-linkages. Its theoretical degree of freedom is $F = -60$ (!). As these mechanisms at least provide one-parametric mobility, we have found a *series of highly overconstrained mechanisms* with a stun-

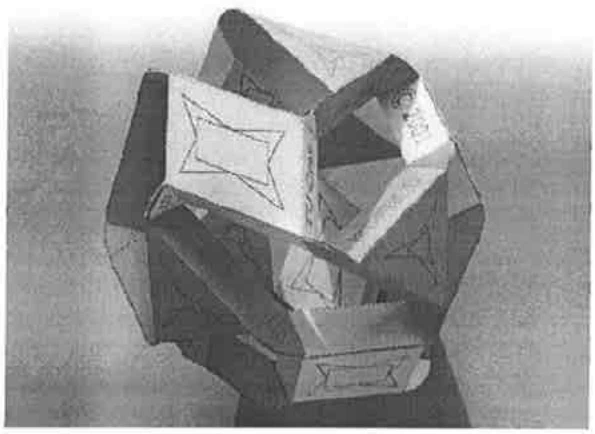

Figure 4: Overconstrained mechanism of the topologic type of a sphere

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[1] As all points are in the pedal points of their point paths (seen from $A_i *$) at the same moment it is easy to determine the polygons in planes $E_i *$.

[2] These considerations are a generalisation of a paper by Stachel (1991), who used this idea to prove the mobility of the so-called HEUREKA-Polyhedron.

ningly wide gap between F and Φ.

5. BAND-STRUCTURES

A viable option of geting a further class of overconstrained models of polyhedra is the study of a series of planes E_i * (i = 0, …, n) which build up a folded paper-strip with plane facets E_i *. The paper strip shall be closed – so we add a virtual plane $E_{n+1}* \equiv E_0*$. Then we proceed as indicated in section 2: We start with a globally fixed point $A_0* \in E_0*$ (not on any of the intersecting lines with the neighboring planes) and a given rotational orientation around A_0* in E_0*. This way we define a linear equiform motion $\varepsilon_0(t) = E_0 / E_0*$ with the globally fixed point A_0* in E_0*. In order to get good models of moveable polyhedra from this starting point we will take a second point $B_0* \neq A_0*$ in E_0* and define a linear equiform motion $\phi_0(t) = E_0 / E_0*$ with the globally fixed point B_0* which is linked to $\varepsilon_0(t)$ via the common point path on the perpendicular bisector of A_0*, B_0* (all in E_0*).

Then we succesively apply the reflections with respect to the planes of symmetry σ_i* between E_{i-1}* and E_i* containing $E_{i-1}* \cap E_i*$. The composition of all these reflections (i = 1, …, n+1) makes up a displacement δ which will transfrom $(E_0*, \varepsilon_0(t))$ into $(E_{n+1}*, \varepsilon_{n+1}(t))$ in $E_{n+1}* \equiv E_0*$.

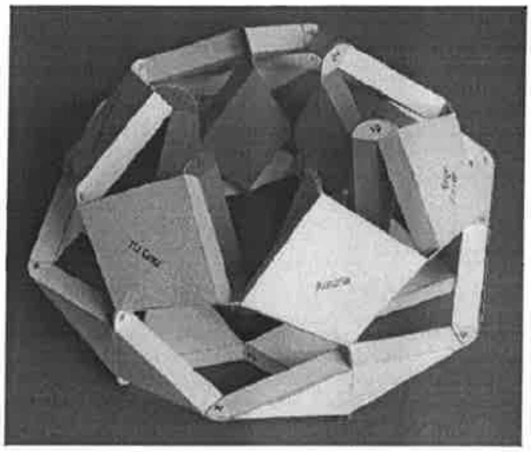

Figure 5: Overconstrained mechanism of the topological type of a torus

In order to get a closed loop of linked equiform motions we additionally must have: $\delta(\varepsilon_0(t)) = \varepsilon_{n+1}(t) = \varepsilon_0(t)$ and $\delta(\phi_0(t)) = \phi_0(t)$ for all t. As we proposed $A_0* \neq B_0*$, this implies that the restriction of δ to the plane E_0* has to be the identity mapping in E_0*.

The overcoinstrained polyhedral model built up by this procedure then will have $2\times(n+1)$ rigid prisms (the basic polygons in general are triangles in the planes E_i*) which are linked by $2\times(n+1)$ spherical 2R-joints and n+1 1R-joints. They stem from the linkage of $\varepsilon_i(t)$ and $\phi_i(t)$. The theoretical degree of freedom of these overconstrained mechansims therefore will take on the value

(3) $F = 6\times(-n-2)+5\times(n+1) = -n-7.$

The last considerations make clear that we have to look for closed paper strips with plane facets E_i* (i = 0, …, n) with the additional property that the composition δ of the reflections with respect to σ_i* (i = 1, …, n+1) induces the identity in E_0*.

At this point we have to split our considerations into two parts depending on the number of twists of the paper strip:

6. CLOSED BAND-STRUCTURES WITH AN EVEN NUMBER OF TWISTS

Let us restrict ourselves to the case of no twist. Then our paper strip has the topological structure of a cylinder. Models of this type are well-known. As an example figure 6 displays a physical model gained by the facets of an orthogonal quadratic prisma (n = 3 in formula (3), see Roeschel (2001)). Other even twist numbers can be treated similarily.

7. CLOSED BAND-STRUCTURES WITH AN ODD NUMBER OF TWISTS

Let us restrict ourselves to the case of one single twist.

Figure 6: Overconstrained model of cylindrical type

Other odd twist numbers can be treated in a similar way and shall not be studied in this paper. Then our paper strip has the topological structure of a Moebius strip. Models of this topological structure – but based on a sphere – have been called *Moebius mechanisms* and have been studied in Roeschel (2000). As a Moebius strip cannot be imdedded into a sphere without self-intersections its physical models either must have bridges or some parts must be cancelled (*reduced Moebius mechanisms*).

The procedure presented in this paper is capable of generating Moebius mechanisms without these distortions: Let us have a closer look to the situation displayed in figure 7: It shows a typical model of a Moebius strip. As the Moebius strip is a one-sided surface we have *to go around twice* in order to close the loop.

Let now (see the schematic sketch of figure 8 for $k = 5$) $E_i *$ be the strarting facets for $i = 0, \ldots, k$ and $E_{k+i+1} * := E_i *$ for $i = 0, \ldots, k$. This way the last k+1 plane facet coincides with the first one and we have n=2k+2. The planes of symmetry of $E_{i-1} *$ and $E_i *$ are again denoted by $\sigma_i *$ ($i = 0, \ldots, k$-1). The successive composition of the reflections with respect to the planes $\sigma_i *$ ($i = 0, \ldots k$) will transform any starting point $A_0 * \in E_0 *$ into points $A_i * \in E_i *$ and finally a point $A_{k+1} * \in E_{k+1} * = E_0 *$. This composition will be called γ. But *we have to go into the second loop*: Now $A_{k+1} *$ is used as the starting point, which now is transformed into further points $A_{k+i+1} * \in E_{k+i+1} * = E_i *$. Our closure condition of section 5 here can be written as: $\delta := \gamma \circ \gamma$ has to induce the identity in the plane $E_0 *$. This fact will call *identity property*.

Remarks: 1) This last condition makes clear, that not all

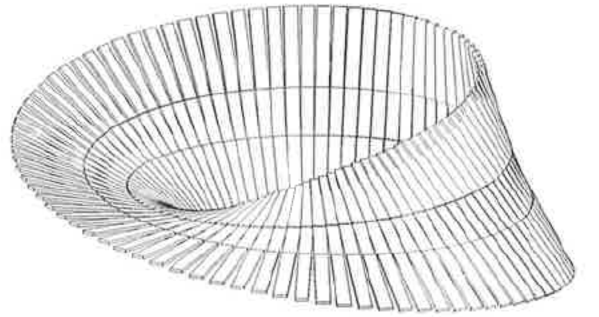

Figure 7: A Moebius strip

physical models of Moebius strips with plane facets can be used as a basic configuration of our proccedure (the one displayed in figure 8 does not have this property!). We have to look for Moebius strips with plane facets which additionally fulfill this *identity-property*.

2) Having found such a model with this identity property, we are able to proceed according to section 5. We will be able to establish a physical model of an overconstrained polyhedron with the basic structure of a Moebius strip.

Figure 9 displays a Moebius strip with 6 plane factes ($k = 5$) which fulfills our identity property.[3] This interesting model has an axial symmetry (the axis is displayed in the figure).

[3] Its generation is not trivial: We have to start with 5 plane facets and have to determine the position of the 6th facet appropriately to our indentity property.

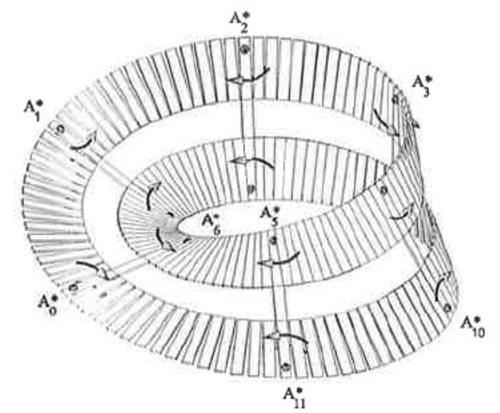

Figure 8: Moebius strip and possible basic points

There a starting point $A_0 *$ is transformed according to our procedure. The resulting polygons are displayed, too.

Figure 10 shows the resulting physical model of the moving model consisting of 12 rigid prismas linked by 12 spherical 2R-joints and by 6 1R-joints. According to Gruebler's formula (1) its theoretical degree takes the value $F = -12$.

CONCLUSIONS

In the first sections of paper we recalled methods and results. We presented the use of plane equiform motions for the generation of a series of overconstrained mechanisms which are based on equiform considerations in the facets of a polyhedron. In the present paper we considered starting polyhedra with a *closed-band-structure*. Bands with one

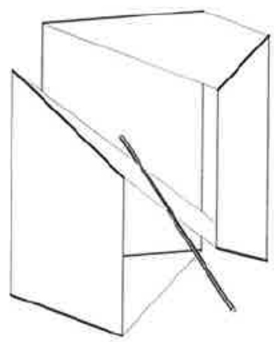

Figure 9: Moebius strip with plane facets with an axis of symmetry (with identity-property)

and without twist are studied in detail. Nice examples with one twist are gained from the Moebius-strip. Accorind to our procedure we have built up a physical model of the Moebius type without selfintersections.

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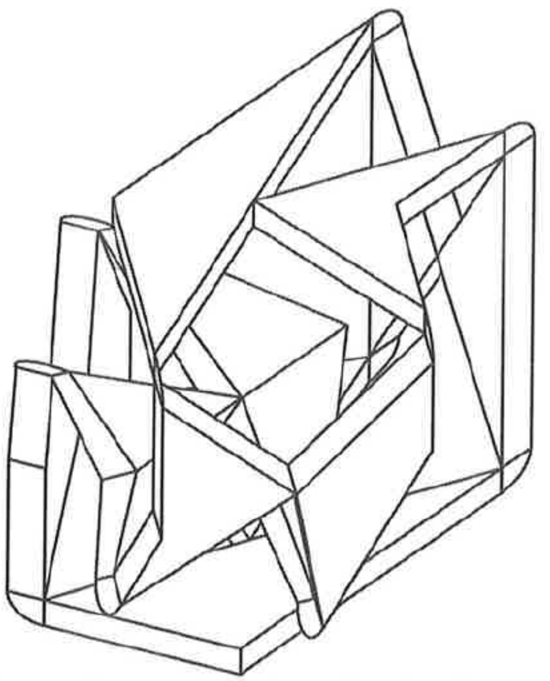

Figure 10: A physical model of a Moebius mechansim based on figure 9

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