On a particular class of cyclides in isotropic respectively pseudoisotropic space

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Dedicated to Prof.Dr.W.WUNDERLICH to his 75th birthday

G. DARBOUX [5] defines algebraic surfaces of the forth order as cyclides of the euclidean space $\mathbb{E}_3$, if they contain the absolute conic of this space as the double curve. Analog ously one can define those algebraic surfaces of forth order $F_4(d_1,d_2)$ as cyclides of the isotropic resp. pseudoisotropic space, if their double curve is decomposing into the two intersecting lines d_1, d_2.[1)]

1. In the real threedimensional projective space $\mathbb{P}_3(\mathbb{R})$ - on some occasions we will also use the complex extension- we adjoin homogeneous coordinates $(x_o:x_1:x_2:x_3) \neq (0:0:0:0)$ to the points of the space.

If one places the two intersecting and different double lines d_1 and d_2 into the lines $x_o = x_1 = 0$ and $x_o = x_2 = 0$, the equation of the surface $F_4(d_1,d_2)$ is

$$x_1^2 x_2^2 + x_o^2 f_2(x_o,x_1,x_2,x_3) + 2.x_o x_1 x_2 (Ax_1 + Bx_2 + Cx_3) = 0. \quad (1.1)$$

In this equation $f_2(x_o,x_1,x_2,x_3)$ means a real polynome of second degree and A, B, C are real constants (see A. LACKNER [16]). The planes $\delta_i(\alpha_i)$ $(i=1,2;\alpha \in \mathbb{R})$ containing the

1) In this paper we will only deal with the pseudoisotropic space; the isotropic space can be found by a complex projective transformation, wich transforms the real lines d_1 and d_2 into complex conjugate lines.

double lines d_i cut the surface $F_4(d_1,d_2)$ into d_i and conics $k_i(\alpha_i)$. If D is the intersection of d_1 and d_2, we denote principle axes the polars of D with respect to the conics $k_i(\alpha_i)$. With $f_2(x_o,x_1,x_2,x_3) = \sum_{i,j=0}^{3} b_{ij}x_ix_j \quad (b_{ij}=b_{ji})$ one can find the following equations for the principal axes of $k_i(\alpha_i)$

$$a_i(\alpha_i) \quad \begin{cases} x_i = \alpha_i . x_o \qquad (i = 1,2) \\ b_{o3}x_o + b_{13}x_1 + b_{23}x_2 + b_{33}x_3 + \alpha_i C x_2 = 0 \end{cases} \qquad (1.2)$$

These principle axes describe the surface of principle axes H

$$b_{o3}x_o^2 + b_{13}x_ox_1 + b_{23}x_ox_2 + b_{33}x_ox_3 + Cx_1x_2 = 0, \qquad (1.3)$$

In discussing the possibilities of decomposing of this surface H we find the following classification of types of these surfaces :

Type 1 : $b_{33}C \neq 0$ is a regular quadric and has the normal form

$$b_{33}x_ox_3 + Cx_1x_2 = 0 \quad (b_{o3}=b_{13}=b_{23}=0) \qquad (1.4)$$

Using (1.4) we get the following equation of the surface (1.1)

$$x_1^2x_2^2 + x_o^2(b_{oo}x_o^2 + 2b_{o1}x_ox_1 + 2b_{o2}x_ox_2 + b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2 + b_{33}x_3^2) + 2.x_ox_1x_2(Ax_1 + Bx_2 + Cx_3) = 0 \qquad (1.5)$$

Type 2 : $C = 0$, $b_{33} \neq 0$. H decomposes into the plane $x_o=0$ and another real plane, which contains neither d_1 nor d_2. Using the normal form

$$H \ldots \quad x_ox_3 = 0 \quad (b_{o3}=b_{13}=b_{23}=0,\ b_{33} \neq 0) \qquad (1.6)$$

the equation of the surface (1.1) gets the form

$$x_1^2x_2^2 + x_o^2(b_{oo}x_o^2 + 2b_{o1}x_ox_1 + 2b_{o2}x_ox_2 + b_{11}x_1^2 + + 2b_{12}x_1x_2 + b_{22}x_2^2 + b_{33}x_3^2) + 2x_ox_1x_2(Ax_1+Bx_2) = 0. \quad (1.7)$$

Type 3 : $C \neq 0$, $b_{33} = 0$, $b_{o3}C \neq b_{13}b_{23}$. H is a cone of second order with the normal form

$$Cx_1x_2 = -b_{o3}x_o^2 \qquad (b_{13} = b_{23} = 0) \qquad (1.8)$$

and we get for the equation of the surface (1.1)

$$x_1^2x_2^2 + x_o^2(b_{oo}x_o^2 + 2b_{o1}x_ox_1 + 2b_{o2}x_ox_2 + 2b_{o3}x_ox_3 + + b_{11}x_1^2 + b_{22}x_2^2 + 2b_{12}x_1x_2) + 2x_ox_1x_2(Ax_1+Bx_2+Cx_3)=0 \quad (1.9)$$

Type 4 : $C \neq 0$, $b_{33} = 0$, $b_{o3}C = b_{13}b_{23}$. H decomposes into two planes with the normal form

$$x_1 = 0 \text{ und } x_2 = 0 \qquad (b_{13} = b_{23} = 0) \qquad (1.10)$$

The equation of the surface (1.1) gets the form

$$x_1^2x_2^2 + x_o^2(b_{oo}x_o^2 + 2b_{o1}x_ox_1 + 2b_{o2}x_ox_2 + b_{11}x_1^2 + + 2b_{12}x_1x_2 + b_{22}x_2^2) + 2x_ox_1x_2(Ax_1+Bx_2+Cx_3) = 0 \quad (1.11)$$

Type 5 : $C = b_{33} = 0$, $b_{13} \neq 0$. H consists of the plane $\omega = [d_1,d_2]$ and another plane containing one of the two double lines. The normal form of H is described by $x_ox_1 = 0$ and the equation of the surface (1.1) is

$$x_1^1x_2^2 + x_o^2(b_{oo}x_o^2+2b_{o1}x_ox_1+2b_{o2}x_ox_2+b_{11}x_1^2+ +2b_{12}x_1x_2+b_{22}x_2^2+2b_{13}x_1x_3) + 2x_ox_1x_2(Ax_1+Bx_2)=0 \quad (1.12)$$

Type 6 : $C = b_{33} = 0$, $b_{13}b_{23} \neq 0$. H consists of the plane ω and another plane containing D, but neither d_1 nor d_2. Using the normal form $x_1+x_2 = 0$ for H we get the following form out of the equation (1.1)

$$x_1^2x_2^2 + x_o^2(b_{oo}x_o^2+2b_{o1}x_ox_1+2b_{o2}x_ox_2+b_{11}x_1^2+2b_{12}x_1x_2+ +b_{22}x_2^2 +2b_{13}x_3(x_1+x_2)) + 2x_ox_1x_2(Ax_1+Bx_2) = 0 \quad (1.13)$$

Type 7 : $C=b_{33}=b_{13}=b_{23}=0$, $b_{o3}\neq 0$. H consists only of the plane ω counting twice. The equation of the surface (1.1) gets the form

$$x_1^2x_2^2 + x_o^2(b_{oo}x_o^2 + 2b_{o1}x_ox_1 + 2b_{o2}x_ox_2 + 2b_{o3}x_ox_3 + 2b_{12}x_1x_2 + \qquad (1.14)$$
$$+ b_{11}x_1^2 + b_{22}x_2^2) + 2x_ox_1x_2(Ax_1 + Bx_2) = 0$$

Type 8 : $C = b_{o3} = b_{13} = b_{23} = b_{33} = 0$. H does not exist; (1.1) describes then a cone of forth order.

This classification is mainly motivated by the following

THEOREM 1 : All singularities of the surfaces (1.1) lie on the surface of principal axes or in the plane $\omega = [d_1, d_2]$.

Proof : The partial derivation of (1.1) with respect to x_3 is

$$\frac{dF}{dx_3} = b_{o3}x_o^2 + b_{13}x_ox_1 + b_{23}x_ox_2 + b_{33}x_ox_3 + Cx_1x_2. \quad (1.15)$$

Since (1.15) must be zero for singularities the theorem holds.

In [10] and [11] we investigated the surfaces of the types 1 - 4. A detailed examination of the other types is missing so far. It is the aim of this paper to inverstigate these types and to give a pseudoisotropic kinematical generation for some of these surfaces.

2. The surfaces of type 5 : These surfaces have the normal form (1.12); their surfaces of principal axes decompose into the plane ω ($x_o = 0$) and the plane $x_1 = 0$. The intersection of the plane $x_1 = 0$ with the surface (1.12) consists of the double line d_1 and the lines

$$x_1=0,\ b_{oo}x_o{}^2+2b_{o2}x_ox_2+b_{22}x_2{}^2 = 0. \qquad (2.1)$$

We now must discuss two cases seperately:

Case A : $b_{22} \neq 0$. Without any loss of generality we get $b_{o2} = 0$. The two lines (2.1) are the intersections of the planes $\varepsilon_1, \varepsilon_2$

$$x_2 = \pm \sqrt{-\frac{b_{oo}}{b_{22}}}\ x_o \qquad (2.3)$$

with $x_1=o$. Besides the lines already known each of the planes ε_1 and ε_2 cut the surface (1.13) in another line. Without any loss of generality we can demand that these lines shall lie on the surface of second order

$$x_ox_3 + \alpha x_1x_2 = 0 \qquad (\alpha \in \mathbb{R}). \qquad (2.4)$$

Hence the equation of the surface gets the form

$$x_ox_1(x_ox_3+\alpha x_1x_2) = -(\beta x_o{}^2+x_2{}^2)(\gamma x_o{}^2+2Bx_ox_1+x_1{}^2) \qquad (2.5)$$

($\alpha,\beta,\gamma,B \in \mathbb{R},\ \gamma \neq 0$). Thus we have nine types of surfaces with respect to the position and the reality of the lines lieing on the surfaces :

$$\begin{aligned}
&(a1)\quad x_ox_1(x_ox_3+\alpha x_1x_2) = (x_o^2 - x_2^2)(x_o^2 - (Bx_o+x_1)^2)\\
&(a2)\quad x_ox_1(x_ox_3+\alpha x_1x_2) = (x_o^2 - x_2^2)(x_o - x_1)^2\\
&(a3)\quad x_ox_1(x_ox_3+\alpha x_1x_2) = (x_o^2 - x_2^2)(x_o^2 + (Bx_o+x_1)^2)\\
&(a4)\quad x_ox_1(x_ox_3+\alpha x_1x_2) = (x_o^2 + x_2^2)(x_o^2 - (Bx_o+x_1)^2)\\
&(a5)\quad x_ox_1(x_ox_3+\alpha x_1x_2) = (x_o^2 + x_2^2)(x_o - x_1)^2\\
&(a6)\quad x_ox_1(x_ox_3+\alpha x_1x_2) = (x_o^2 + x_2^2)(x_o^2 + (Bx_o+x_1)^2)
\end{aligned} \qquad (2.6)$$

$$(a7)\quad x_ox_1(x_ox_3+\alpha x_1x_2) = x_2^2\,(x_o^2 - (Bx_o + x_1)^2)$$

$$(a8)\quad x_ox_1(x_ox_3+\alpha x_1x_2) = x_2^2\,(x_o - x_1)^2 \qquad (2.6)$$

$$(a9)\quad x_ox_1(x_ox_3+\alpha x_1x_2) = x_2^2\,(x_o^2 + (Bx_o + x_1)^2)$$

Case B: $b_{22}= 0$

1) $b_{o2}\neq 0$, $B \neq 0$

In this case (2.1) represents the two lines:

$$x_o=x_1=0 \quad \text{and} \quad x_1=0,\ b_{oo}x_o+ 2b_{o1}x_2 = 0 \qquad (2.7)$$

The second line can be transformed by a projectiv transformation into the x_3-axis ($b_{oo}= 0$). Then the plane $x_2=0$ cuts the surface in the double line d_2, in the x_3-axis and in another line:

$$b_{o1}x_o + b_{11}x_1 + b_{13}x_3 = 0 \qquad (2.8)$$

Without any loss of generality the line (2.8) can be transformed into the x_1-axis ($b_{o1} = b_{11} = 0$).

If we cut the surface(1.3) with the planes $x_2 = ux_o$, then the poles of the intersecting lines which refer to the double line d_2 lie in the plane $x_1 = 0$ and describe the curve

$$b_{12}x_2x_o + b_{13}x_ox_3 + Bx_2^2 = 0. \qquad (2.9)$$

After transforming (2.9) into the normal form

$$b_{13}x_ox_3 + Bx_2^2 = 0 \qquad (2.10)$$

we get with

$$\alpha = \frac{b_{o2} + 4AB^2}{2B^2},\quad D = \frac{b_{o2}}{2B^2},\ b_{13} = \frac{1}{2} \qquad (2.11)$$

the following normal form of these types of surfaces

$$x_o x_1(\alpha x_1 x_2 + x_o x_3) = x_2(x_1+2Bx_o)(Dx_o x_1-2Bx_o{}^2-x_1 x_2).(2.12)$$

Thus we have four projective different types of surfaces : One will get these types from (2.12) with $\alpha = 0$, resp. $\alpha \neq 0$ and $\alpha \neq D$, resp. $\alpha = D$ ($A = 0$ resp. $A \neq 0$).

2) $b_{o2} \neq 0$, $B = 0$

Analog ously to the case 1 we can get $b_{oo}=b_{o1}=b_{11}=b_{12}=0$. With the aid of this and $b_{13}=\frac{1}{2}$ we get the normal form of this type of surfaces

$$x_1{}^2 x_2{}^2+x_o{}^2(x_1 x_3+2b_{o2}x_o x_2)+2Ax_o x_1{}^2 x_2 = 0 \ (b_{o2}\neq 0). \quad (2.13)$$

This normal form describes two different types of surfaces which can be characterised in the following way :

We consider the conics $k_1(\alpha_1)$

$$\alpha_1{}^2 x_2{}^2+\alpha_1 x_o x_3+2b_{o2}x_o x_2+2A\alpha_1{}^2 x_o x_2 = 0, \ x_1=\alpha_1 x_o. \quad (2.14)$$

Along the line $x_2=x_3=0$ these conics have the tangents $t(\alpha_1)$

$$x_1=\alpha_1 x_o, \quad x_2(b_{o2}+2A\alpha_1{}^2) + \frac{\alpha_1}{2} x_3 = 0 \quad , \quad (2.15)$$

which describe a ruled surface R

$$2x_2(b_{o2}x_o{}^2+2Ax_1{}^2)+x_o x_1 x_3 = 0. \quad (2.16)$$

For $A \neq 0$ R is a ruled surface of third order; for $A = 0$ R decomposes into the plane $x_o=0$ and a surface of second order containing d_1 and d_2.

3) $b_{o2} = 0$

Using similar methods as with the cases 1 and 2 we can find the projective transformation

$$(x_o:x_1:x_2:x_3) \longrightarrow (x_o:x_1:x_2+Ax_o:2b_{13}x_3+2b_{12}x_2+ +2b_{o1}x_o+(b_{11}-A^2)x_1-2AB(Ax_o+2x_2)), \quad (2.17)$$

which transforms the general equation (1.12) into the normal form (with $b_{oo} = 1$)

$$x_1^{\ 2}x_2^{\ 2} + x_o^{\ 2}(x_o^{\ 2} + x_1x_3) + 2Bx_ox_1x_2^{\ 2} = 0. \quad (2.18)$$

One can see that (2.18) describes two different types of surfaces ($B \neq 0$ and $B = 0$). We get a STEINER surface for $B = 0$ (see W. WUNDERLICH [29] and [30]), whereby in our case two of the three possible double lines of the STEINER surface become one double line in d_1 ($x_o=x_1= 0$).
Hence in the case B we have eight projective different types of surfaces and we can formulate

THEOREM 2: The surfaces $F_4(d_1,d_2)$ of type 5 consist of 17 projective different types. Their normal forms are given by the formulas (2.6), (2.12), (2.13) resp. (2.18).

3. A kinematic generation : We now choose the double lines d_1, d_2 to represent the absolute lines of a pseudoistropic space $\bar{I}_3^{(1)}$ (see H. SACHS [23] and K.STRUBECKER [25], [26]) and apply the usual affin coordinates $x_o:x_1:x_2:x_3 = 1:x:y:z$ ($x_o \neq 0$). Then

$$\varphi'(u)=\begin{bmatrix} a_{11}(u) & 0 & 0 \\ 0 & a_{22}(u) & 0 \\ a_{31}(u) & a_{32}(u) & a_{33}(u) \end{bmatrix}\varphi + \begin{bmatrix} d_1(u) \\ d_2(u) \\ d_3(u) \end{bmatrix} \quad (3.1)$$

($a_{ij}(u)$, $d_i(u) \in C^o(\mathbb{R})$) describes the transformation group H_8 of similarities in the space $\bar{I}_3^{(1)}$. It will be shown that the surfaces $F_4(d_1,d_2)$ of type 5 can be generated by pseudo-isotropic oneparameter motions, whereby one conic $k_1(\alpha_1)$ is moved in such a way, that the paths of the points of $k_1(\alpha_1)$ describe the conics $k_2(\alpha_2)$.

Case A :

We determine the coefficients in (3.1) with the following conditions :

(1) The paths of the points of the y - axis shall be those generators of the surface of second order M

$$z' = \alpha x' y' \qquad (\alpha \in \mathbb{R}) \tag{3.2}$$

which intersects the double line d_2. With this condition we get

$$a_{22} = 1,\ a_{32} = \alpha d_1,\ d_2 = d_3 = 0. \tag{3.3}$$

(2) The path of the point (0,0,1) shall be the hyperbola

$$y'=0,\ z'=\frac{\beta(x'+1)^2+\gamma(x'+1)+1-\beta-\gamma}{x'+1} \tag{3.4}$$

$(\beta,\gamma \in \mathbb{R})$.Hence we have the oneparameter motion

$$\mathfrak{y}'(u)=\begin{bmatrix} a_{11}(u) & 0 & 0 \\ 0 & 1 & 0 \\ a_{31}(u) & \alpha u & \frac{\beta(u+1)^2+\gamma(u+1)+1-\beta-\gamma}{u+1} \end{bmatrix}\mathfrak{y} + \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} \tag{3.5}$$

$(u \in (-\infty,+\infty),\ a_{11}(u),\ a_{31}(u)$ arbitrary $\in C^0(\mathbb{R}))$.

When we move the parabola

$$(0,v,\delta v^2+\varepsilon) \qquad \delta,\varepsilon \in \mathbb{R},\ v \in (-\infty,+\infty) \tag{3.6}$$

in the motion (3.5) we get a surface with the algebraic equation

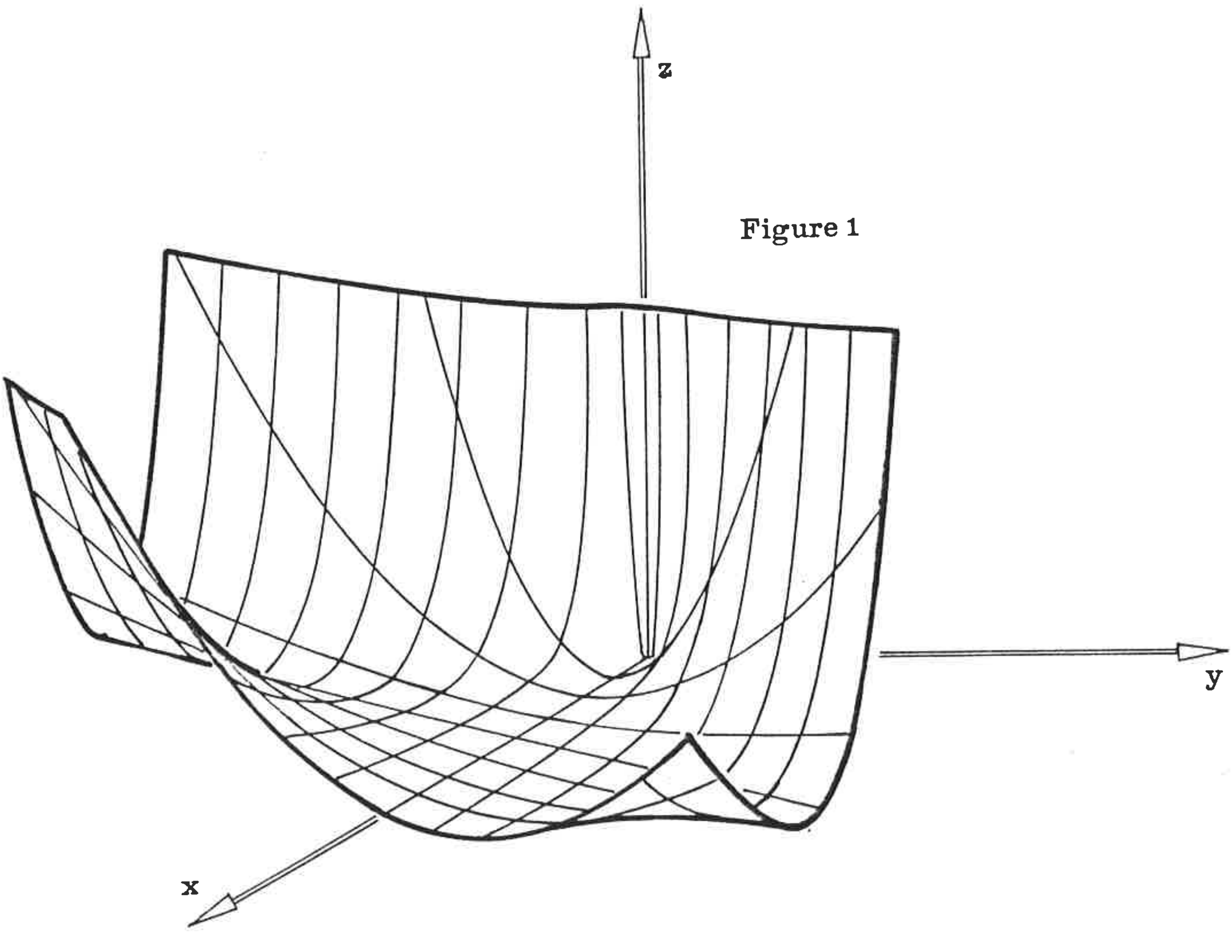

Figure 1

$$(x'+1)(z'-\alpha x'y')=(\delta y'^2+\varepsilon)[\beta(x'+1)^2+\gamma(x'+1)+1-\beta-\gamma]. \quad (3.7)$$

The pseudoisotropic similarity

$$x = x'+1,\ y=y',\ z=z'+\alpha y' \quad (3.8)$$

transforms (3.7) into

$$x(z-\alpha xy)=\varepsilon\delta(y^2+\frac{\varepsilon}{\delta})(x^2+x\frac{\gamma}{\beta}+\frac{1-\beta-\gamma}{\beta}). \quad (3.9)$$

By a suitable choice of coefficients this equation corresponds with (2.5). So it is possible to generate all surfaces (2.6) in this way.

Remark : In [11] we have shown that the further conditions $a_{11}(u) = a_{33}(u)$ and $a_{31}(u)$ define oneparameter motions (3.5) being CLIFFORD translations in an indefinit elliptic resp. indefinit quasielliptic space with the absolute M (3.2). In this way the surfaces (3.9) can be generated by translating the conic $k_1(\alpha_1)$ along the hyperbola (3.4).

Figure 1 shows the surface (2.6) (a8) with $\alpha = 0.25$ in an axonometric mapping (for $0 \leqq x \leqq 2,\ -2 \leqq y,z \leqq 2$)

Figure 1

Case B : b_1

The coefficients of (3.1) are determined by the following conditions :

(1) The paths of the point (0,0,0) shall be the x - axis and the path of (0,0,1) shall be the straigt line

$$\mathcal{X}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \lambda . \begin{bmatrix} 1 \\ 0 \\ \frac{1}{1+2B} \end{bmatrix} \qquad (B \in \mathbb{R},\ \lambda \in (-\infty,+\infty)). \quad (3.10)$$

These two conditions give us $d_2 = d_3 = 0$ and $a_{33} = 1 + \frac{d_1}{1+2B}$ in (3.1).

(2) The path of the point (0,1,0) shall be the hyperbola h

$$y' = 1, \quad z' = \frac{(x+1+2B)[(x+1)(D-1)-2B]-\alpha(u+1)^2}{x+1} - \beta \quad (3.11)$$

with $\beta := (1+2B)(D-2B-1) + \alpha$. Then the oneparameter motion gets the form

$$\mathcal{X}'(u) = \begin{bmatrix} a_{11}(u) & 0 & 0 \\ 0 & 1 & 0 \\ a_{13}(u) & \frac{(u+1+2B)[(u+1)(D-1)-2B]-\alpha(u+1)^2}{u+1} - \beta & 1+\frac{u}{1+2B} \end{bmatrix} \mathcal{X} + \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}. \quad (3.12)$$

We consider now the motion of the parabola

$$(x,y,z) = (0,v,(1+2B)(1-v)v) \qquad (v \in (-\infty,+\infty)) \quad (3.13)$$

under (3.12). It describes a surface with the equation

$$\begin{aligned} &(x'+1)\{z'+\alpha(x'+1)y'+[(1+2B)(D-1-2B)+\alpha]y'\} = \\ &= y'(x'+1+2B)[D(x'+1)-2B-(x'+1)y']. \end{aligned} \quad (3.14)$$

Using the transformation

$$(x',y',z') \longrightarrow (x=x'+1, y=y', z=z'+\beta y') \quad (3.15)$$

we get from (3.14) the normal form (2.12) of these types of surfaces.

b_2 : Putting B = 0 in (3.10) and using the hyperbola h

$$y'=1,\quad z'=-\frac{(x'+1)^2(2A+1)+(x'+1)C+2b_{o2}}{x'+1} \qquad (3.16)$$

(b_{o2}, A, C $\in \mathbb{R}$) as the path of the point (0,1,0) the oneparameter motion is given by

$$\varphi'(u)=\begin{bmatrix} a_{11}(u) & 0 & 0 \\ 0 & 1 & 0 \\ a_{31}(u) & -\dfrac{(u+1)^2(2A+1)+C(u+1)+2b_{o2}}{u+1} & u+1 \end{bmatrix}\varphi + \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}. \qquad (3.17)$$

The parabola

$$(x,y,z) = (0,v,v(1-v)) \qquad (v \in (-\infty,+\infty)) \qquad (3.18)$$

describes under the motion (3.17) an algebraic surface with the equation

$$z'(x'+1)+y'[2A(x'+1)^2+C(x'+1)+2b_{o2}] + y'^2(x'+1)^2=0. \qquad (3.19)$$

Using the transformation

$$(x',y',z') \longrightarrow (x=x'+1,\ y=y',\ z=z'+y'(2b_{o2}+2A+1)) \qquad (3.20)$$

(3.19) gets the normal form (2.13).

b_3 : In this case the oneparameter motion is determined by the paths of the following three points :

$$\begin{aligned} &(1)\ (0,0,-1) \text{ shall be moved on } y'=0,\ z'=\frac{1}{x'+1}, \\ &(2)\ (0,1,-2(1+B)) \text{ on } y'=1,\ z'=\frac{1+2B(x'+1)+(x'+1)^2}{x'+1} \text{ and} \\ &(3)\ (0,2,-5+8B) \text{ on } y'=2,\ z'=\frac{1+8B(x'+1)+4(x'+1)^2}{x'+1}. \end{aligned} \qquad (3.21)$$

The motion (3.1) can be described by

$$\varphi'(u) = \begin{bmatrix} a_{11}(u) & 0 & 0 \\ 0 & 1 & 0 \\ a_{31}(u) & 0 & \frac{1+2B+u}{1+2B} \end{bmatrix} \varphi + \begin{bmatrix} u \\ 0 \\ \frac{u}{1+2B} + \frac{u}{u+1} \end{bmatrix}. \quad (3.22)$$

Now we move the parabola

$$(x,y,z) = (1,v,-1-v^2(1+2B)) \qquad (v \in (-\infty,+\infty)) \quad (3.23)$$

and get the following equation of the surface of their point-paths

$$z'(x'+1)+(x'+1)^2y'^2+(x'+1)y'^2 2B + 1 = 0. \qquad (3.24)$$

By the transformation

$$(x',y',z') \longrightarrow (x=x'+1,y=y',z=z') \qquad (3.25)$$

(3.24) is transformed into the normal form (2.18). So we have the following

THEOREM 3 : All surfaces $F_4(d_1,d_2)$ of type 5 can be generated by pseudoisotropic oneparameter motions. These motions move all points of a conic $k_1(\alpha_1)$ on the conics $k_2(\alpha_2)$.

Remark : In [11] we have shown that oneparameter motions with the structure of (3.12) resp. (3.17) can be extended to the whole space by putting $a_{11}(u) = a_{33}(u)$ and $a_{31}(u) = 0$. Then (3.12) and (3.17) represent CLIFFORD translations in an indefinite quasielliptic space; the conic $k_1(\alpha_1)$ is moved along one of the conics $k_2(\alpha_2)$ by translations. By putting $a_{11}(u) = a_{33}(u)$ and $a_{31}(u) = 0$ (3.22) represents nonisotropic CLIFFORD translations in a suitable chosen flag-space $I_3^{(2)}$ (see H. SACHS [23]).

4. Surfaces $F_4(d_1,d_2)$ of type 6. For these surfaces the surface of principle axes H is the plane $x_1+x_2 = 0$ (see (1.13)). If we look for a generation similar that of type 5 we have to move the conics $k_1(\alpha_1)$ along the conics $k_2(\alpha_2)$. The principle axis $a_1(\alpha_1)$ (1.2) has to be moved in the plane $x_1+x_2 = 0$, but (for our generation) also in a plane through d_2 $(x_o=x_2=0)$. This is impossible and so we are able to formulate

THEOREM 4 : Surfaces $F_4(d_1,d_2)$ of type 6 do not admit a pseudoisotropic generation, which moves the points of a conic $k_1(\alpha_1)$ along the conics $k_2(\alpha_2)$.

5. Surfaces $F_4(d_1,d_2)$ of type 7. These surfaces have the normal form (1.14) and their surface of principle axes is the plane ω $(x_o = 0)$ counting twice.
The intersections of these surfaces with the planes

$$x_1 = (-B\pm\sqrt{B^2-b_{22}})x_o \text{ and} \tag{5.1}$$
$$x_2 = (-A\pm\sqrt{A^2-b_{11}})x_o \tag{5.2}$$

decompose in nothing but straight lines. Without any loss of generality we may choose them symmetrically to the point (1:0:0:0) (A = B = 0). Then we get out of (1.14) the following equation of our surface

$$(x_1{}^2+b_{22}x_o{}^2)(x_2{}^2+b_{11}x_o{}^2) = x_o{}^2[(b_{11}b_{22}-b_{oo})x_o{}^2- \\ -2b_{o1}x_ox_1-2b_{o2}x_ox_2-2b_{12}x_1x_2-2b_{o3}x_ox_3]. \tag{5.3}$$

The projective collineation

$$(x_o:x_1:x_2:x_3) \longrightarrow (x_o:x_1:x_2:(b_{11}b_{22}-b_{oo})x_o-2b_{o1}x_1- \\ -2b_{o2}x_2-2b_{o3}x_3) \tag{5.4}$$

transforms (5.3) into the normal form

$$x_o^2(\alpha x_1x_2+x_ox_3)=(x_1^2+Ex_o^2)(x_2^2+Fx_o^2) \qquad (5.5)$$

with $\alpha = -2b_{12}$, $E = b_{11}$ and $F = b_{22}$. Discussing the position and the reality cf the lines (5.1) we get the following types of surfaces :

$$\begin{aligned}
&(a1) \quad x_o^2(\alpha x_1x_2+x_ox_3) = (x_1^2-x_o^2)(x_2^2-x_o^2)\\
&(a2) \quad x_o^2(\alpha x_1x_2+x_ox_3) = (x_1^2-x_o^2)(x_2^2+x_o^2)\\
&(a3) \quad x_o^2(\alpha x_1x_2+x_ox_3) = (x_1^2+x_o^2)(x_2^2+x_o^2)\\
&(a4) \quad x_o^2(\alpha x_1x_2+x_ox_3) = (x_1^2+x_o^2)x_2^2\\
&(a5) \quad x_o^2(\alpha x_1x_2+x_ox_3) = (x_1^2-x_o^2)x_2^2\\
&(a6) \quad x_o^2(\alpha x_1x_2+x_ox_3) = x_1^2x_2^2.
\end{aligned} \qquad (5.6)$$

These are 12 types; 6 for $\alpha \neq 0$ and 6 for $\alpha = 0$. All together we can formulate

THEOREM 5 : The surfaces $F_4(d_1,d_2)$ of type 7 consist of twelfe projective different classes of surfaces. Their normal forms are given by formula (5.6).

6. Generation of the surfaces $F_4(d_1,d_2)$ of type 7. We change the oneparameter motion (3.5) by using the parabola

$$y'=0, \ z' = \beta x'^2 + \gamma \qquad (\beta,\gamma \in \mathbb{R}, \beta \neq 0) \qquad (6.1)$$

instead the hyperbola (3.4) as path of the point (0,0,1). (3.5) gets the form

$$\varphi'(u) = \begin{bmatrix} a_{11}(u) & 0 & 0 \\ 0 & 1 & 0 \\ a_{31}(u) & \alpha u & \beta u^2+\gamma \end{bmatrix} \varphi + \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} . \qquad (6.2)$$

The point paths of the parabola

$$x = 0,\ z = \delta y^2 + \varepsilon \qquad (\delta,\varepsilon \in \mathbb{R},\ \delta \neq 0) \qquad (6.3)$$

lie on an algebraic surface with the equation

$$z' - \alpha x'y' = (\beta x'^2+\gamma)(\delta y'^2+\varepsilon). \qquad (6.4)$$

Comparing (5.6) and (6.4) we see that all these surfaces can be generated in this way (by a suitable choice of the parameters $\alpha,\beta,\gamma,\delta,\varepsilon$). Thus we have

THEOREM 6 : All surfaces $F_4(d_1,d_2)$ of type 7 can be generated by pseudoisotropic oneparameter motions. These motions move all points of a conic $k_1(\alpha_1)$ on the conics $k_2(\alpha_2)$.

Figure 2

Figure two shows the surface (5.6)(a1) with $\alpha = 0.25$ in an axonometric mapping $((x,y,z)\in[-2,2]^3)$. We remark the great similarity of this surface to the surface (1.19a) out of [10](figure 2); the essential difference are the four knots existing on that surface. Here we have no singularities than the double lines d_1 and d_2 resp. their intersection D.

Figure 3

Figure three shows the surface (5.6)(a6) with $\alpha = 0$ in an axonometric mapping $((x,y,z) \in [-2,2]^3)$. The surface touches the plane $z = 0$ along the x - and the y - axis.

Remark : The surfaces $F_4(d_1,d_2)$ of type 8 are cones; all

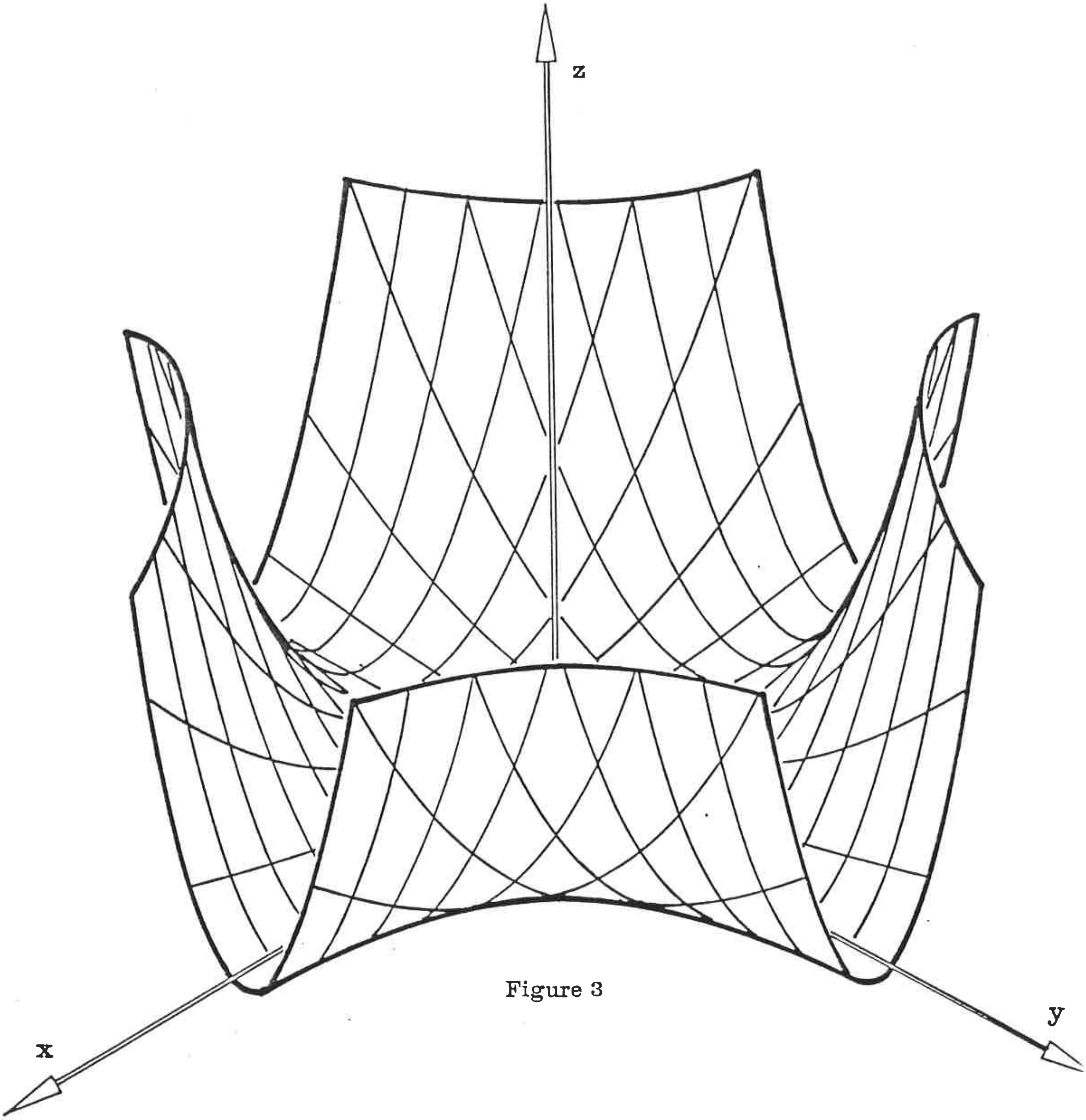

Figure 3

conics $k_1(\alpha_1)$ resp. $k_2(\alpha_2)$ decompose into straight lines containing the vertex D. So there exists no generation in the way we have shown for the types 5 and 7.

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