OVERCONSTRAINED MECHANISMS BASED ON TRAPEZOHEDRA

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ABSTRACT: We start with a special polyhedron, the so-called ‘trapezohedron’ (see E. WEISSTEIN [11]). It is dual to an antiprism and can be generated as the intersection of two suitable congruent regular pyramids with regular *n*-gons as directing polygons ($n > 2$). Its faces are $2n$ congruent trapezoids (kites). We use this polyhedron to construct some generalisations of the Fulleroid-like mechanisms described by G. KIPER [2]-[3], K. WOHLHART [12]-[13] and the author [8]. These generalisations consist of $2n$ congruent special parallel four-bars in the faces of this trapezohedron which are interlinked by spherical 2R-joints (at some fixed angles). This mechanism consists of $8n$ rigid bodies interlinked via $8n$ 1R-joints and $4n$ 2R-joints. As the classical Grübler-Kutzbach-Chebyshev-formula gives a theoretical degree of freedom $F = - 6 - 8n$ at first sight this mechanism is supposed to be rigid. But owing to our special geometric dimensions physical models (for some values of $n > 2$) seem to admit at least a one-parametric self-motion. The particular case n=3 can also be viewed as some generalization of the Fulleroid-like-mechanism presented in [3]. The aim of this paper is to work out and elucidate the existence of highly symmetric one-parametric self-motions of the mechanisms for general values of n>2. All these different self–motions share two higher order singular positions.

Keywords: Kinematics, Robotics, Fulleroid-like-mechanisms, Overconstrained Mechanisms, Self-Motions.

1. INTRODUCTION

The aim of the paper is to present a new method of generating new and surprising examples of overconstrained polyhedral mechanisms. We start with a kinematic chain which partially allows reproducing the continuous scaling of a rectangle (factor depending of time *t*) from a center F^* (fixed in space). This chain later on is used for the construction of a wide (yet unknown) class of overconstrained polyhedral mechanisms. We will present some nice and surprising examples based on polyhedra known as *trapezohedra*. The presented mechanisms can be viewed as generalisations of the so-called Fulleroid-like mechanisms described by G. KIPER [3] and K. WOHLHART [11]. They belong to a class of interesting overconstrained ‘polyhedral’ mechanisms described by several authors (e.g. H. STACHEL [9], [10], K. WOHLHART [11], [12], G. KIPER [2], [3] and the author [4], [5], [7], [8]). Due to the generation of these mechanisms we work out a highly symmetric one-parametric self-motion of the mechanisms and display some of the states of their motions for an interesting example.

The paper is structured as follows: Section 2 is devoted to the study of some properties of a special planar parallel four-bar-mechanism which are used in chapter 3. Section 4 lists some geometric properties of trapezohedra. They are used for our construction of the corresponding overconstrained polyhedral mechanisms in Section 5 (see the example in Fig. 1). A one-parametric self-motion of these mechanisms is displayed. In Section 6 we describe properties of the relative motions of the rigid bodies. Section 7 lists some further self-motions of the mechanisms which are gained in a similar way. A conclusion will round off the paper.

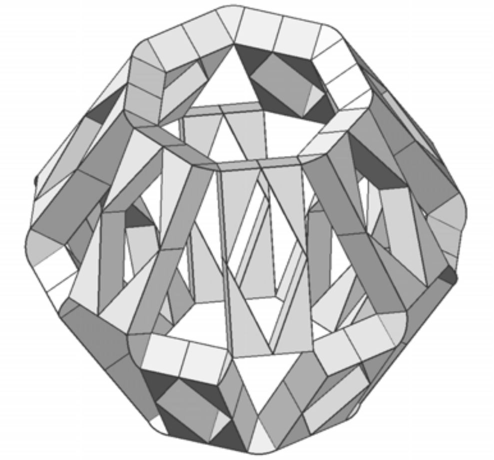

Figure 1: The mechanism based on a trapezohedron (n = 7).

2. SOME PROPERTIES OF A SPECIAL PLANAR PARALLEL 4-BAR MOTION

We start our considerations with the study of the so-called planar CARDAN-motion (see O. BOTTEMA – B. ROTH [1]): Two distinct points A_1 and A_2 of a rigid body are compelled to stay on two orthogonal straight lines (see Fig. 2). We use a Cartesian frame $\{O x_1, y_1\}$ and the notations of Figure 2. If we put $d(A_1,A_2) = 2$ we can parametrize the resulting CARDAN-motion via

$$x' = x_1 \cos t + y_1 \sin t \qquad y' = 2\sin t - x_1 \sin t + y_1 \cos t \tag{1}$$

with $t \in [0, +2\pi)$.

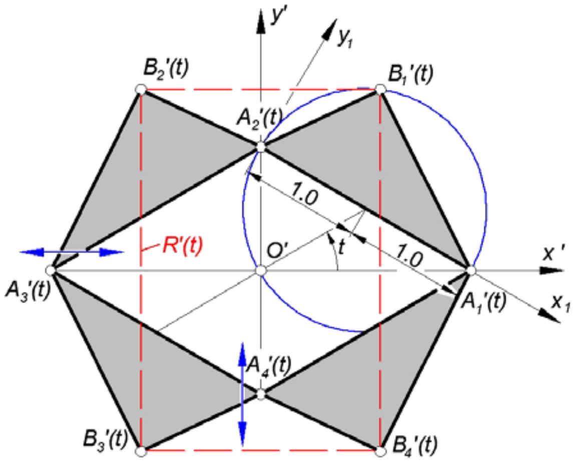

Figure 2: The basic CARDAN motions.

It is well-known that the orbit of any point B_1 on the circle with diameter A_1A_2 under this CARDAN motion is part of a straight line through *O'*. We can parametrize this circle by

$$x_1 = 1 + \cos u \qquad y_1 = \sin u \tag{2}$$

with $u \in [0, 2\pi)$. For the following considerations we fix the value u $(u \in (0,2\pi), u \neq \pi)$ and therefore the point B_1 on that circle. As we have $u \notin \{0, \pi\}$ this point together with A_1 and A_2 determines a rigid triangle with a right angle at B_1. According to (1) the straight line point path $B_1'(t)$ of the point B_1 has the parametrisation

$$x' = \cos t + \cos(u - t) \qquad y' = \sin t + \sin(u - t). \tag{3}$$

Now we reflect and copy this CARDAN – motion (and the poses of the chosen triangle $A_1A_2B_1$) in the lines $x' = 0$ and $y' = 0$, such that we gain a *special parallel four-bar mechanism* with equal side lengths (see Fig. 2). It consists of four rigid bodies materialized by the *right-angled triangles* $A_1A_2B_1$, $A_3A_2B_2$, $A_3A_4B_3$ and $A_1A_2B_1$. The triangles are linked via rotational joints. Its parallel self-motion can be performed moving the points A_1, A_3 and A_2, A_4 on the x'- and the y'-axis, resp.

A parametrization of this *parallel self-motion* $\eta_0(t)$ of the other parts can be achieved by change of some signs and by reflections of (1) in the lines $x' = 0$ and $y' = 0$.

The poses of the point B_1 are denoted by $B_1'(t)$. Together with its reflected counterparts $B_2'(t)B_3'(t)B_4'(t)$ they determine rectangles $R'(t)$ centered at *O'*. The side lengths of these rectangles are

$$l_1'(t,u) = 2[\cos t + \cos(u - t)] \quad \text{and} \qquad l_2'(t,u) = 2[\sin t + \sin(u - t)]. \tag{4}$$

As the point paths of the points B_i are straight lines through *O'*, the rectangles $R'(t)$ are similar to each other at any moment t. The ratio of its side lengths takes on the value

$$r(u):=l_1'(t,u)/l_2'(t,u)=\cot(u/2). \quad (5)$$

For any fixed value *u* it does not depend on the time *t*. The dilations with center *O'* of the rectangles from an initial position *R'(0)* to the position *R'(t)* have the factor

$$\lambda(t):=[\sin t+\sin(u-t)]/\sin u \quad (6)$$

with the constant $u\neq 0,\pi$. We have $\lambda(0)=1$ and $\lambda(\frac{\pi-u}{2})=\lambda(\frac{3\pi-u}{2})=0$.

We can sum up:

Lemma 1: *The described parallel self-motion* $\eta_0(t)$ *of the special parallel four-bar mechanism moves the four special points* B_i *in a way that the rectangles R'(t) formed by the four points* $B_1'(t)B_2'(t)B_3'(t)$ *and* $B_4'(t)$ *are homothetic for any moment t.*

3. EXTENSION TO THE THREE - DIMENSIONAL SPACE

The idea of this Section is the following: We fix the value *u* as before. Then the rectangles *R'(t)* from Section 2 are homothetic (we will refer to it as a '*scaling*') for all *t* (center of the scaling is the point *O'*, scaling factor to *R'(0)* is $\lambda(t)$ given by (6)). Now we embed the rectangle *R'(0)* into a 3-dimensional '*world-space*' (*world coordinate frame {O*, x*, y*, z*}*). Then the scaling from above can be performed with respect to any center *F**, fixed in this space. This way the positions *R'(t)* of the scaled rectangles can be used to embed the positions of the parallel 4-bar at time *t* into the 3-dimensional space. Figure 3 displays this idea.

These considerations can also be performed analytically. The intial CARDAN-motion yields a one-parametric spatial motion $\xi_0(t)$. It can be parametrized by

$$\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} = (1-\lambda(t))\begin{pmatrix} f_1^* \\ f_2^* \\ f_3^* \end{pmatrix} + \begin{pmatrix} 0 \\ 2\sin t \\ 0 \end{pmatrix} + \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}. \quad (7)$$

There $\lambda(t)$ has to be taken from (6); the point *F** has the coordinates (f_1^*, f_2^*, f_3^*). The motions of the other parts of the extended 4-bar are gained by changing some signs in (7).

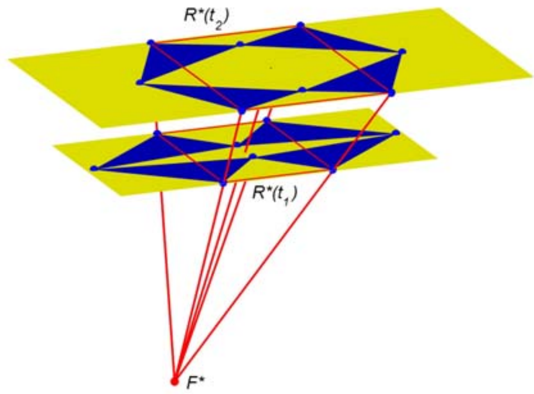

Figure 3: The embedded 4-bar-mechanisms.

Remarks:

1. The function $\lambda(t)$ is a linear combination of {1, *cos t*, *sin t*}. Thus, the one-parametric motion displayed in (7) is a DARBOUX-motion (see [1]). All general point-paths under this motion are ellipses. Of course, the pivot point B_1 moves on a straight line through *F**.
2. As stated before, the spatial motions of the other parts of the initial 4-bar are gained by interchanging some signs in (7). The induced spatial one-parametric motions of these rigid bodies are again DARBOUX-motions.
3. Extrusion of the triangles orthogonal to the plane of the 4-bar-mechanism yields prismatic parts. The former planar rotational joints become rotary (1R -) joints now (see Figure 4).

We can sum up:

Theorem 1: *The described one-parametric motion $\xi_0(t)$ consists of interlinked DARBOUX-motions. The point paths of the points B_1, B_2, B_3 and B_4 are straight lines through the given center F^*. At any moment these 4 points form a rectangle $R^*(t)$ which is a scaled copy of the initial one. The center of all these dilations is the point F^*.*

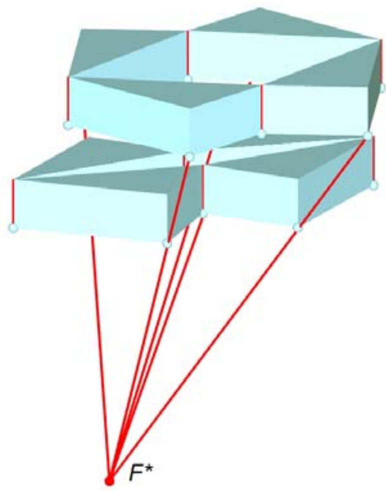

Figure 4: The extruded 4-bar-linkage in two of its spatial positions; the rotary-joints are denoted by small cylinders.

These 4 interlinked DARBOUX-motions determine 4 pivot points (stemming from the positions of the points B_i), which form a rectangle $R^*(t)$ at any moment. These rectangles are similar for all values of t.

This fact will be used to generate new closed *kinematic chains* based on some special polyhedra. In order to be able to apply the former considerations these polyhedra have to contain rectangles in their faces. The vertices of these rectangles should form *closed chains of rectangles*.

The first few examples for such a class are studied in this paper.

4. TRAPEZOHEDRA

The intersection of volume-models of a regular pyramid with a regular n-gon ($n > 2$) as directing polygon and a further suitable congruent one is the so-called '*trapezohedron*' (see E. WEISSTEIN [11]). It is dual to some antiprism. It has dihedral symmetry and consists of $2n$ congruent trapezoids (kites). It has $2n+2$ vertices and $4n$ edges. Figure 5 shows an example.

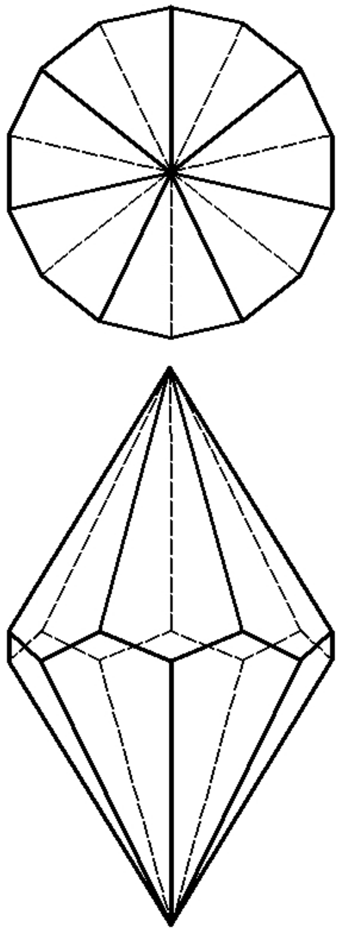

Figure 5: A trapezohedron in top and front view ($n = 7$).

Considering a face of the trapezohedron we can determine the midpoints of its edges. They determine a rectangle. As all $2n$ faces of the polyhedron are congruent, we get $2n$ congruent rectangles in the faces of this polyhedron, all linked to its neighbors via the midpoint of the corresponding edge (see Figure 6).

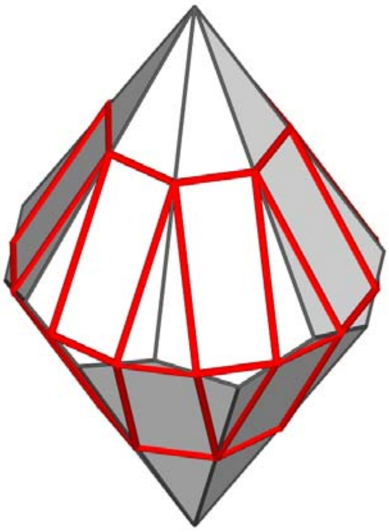

Figure 6: A trapezohedron and the $2n$ congruent rectangles in its faces (n = 7).

The shape of these rectangles is determined by the shape of the trapezohedron. The side lengths of the rectangles determine some ratio $r^* \in \Re - \{0\}$. This will be combined with considerations from Chapters 2 and 3 in the following Section.

5. PUTTING THINGS TOGETHER

We now are given some trapezohedron. Then $r^* \in \Re - \{0\}$ shall denote the ratio of the side lengths of the rectangles inscribed into its faces.

According to Equation (5) the value $r^* \neq 0$ can be achieved by two values of the angle u:

$$u = 2 \operatorname{arc\,cot} r^* \quad (u \in (0, 2\pi), u \neq \pi). \qquad (6)$$

We recommend using the solution $u^* = 2 \operatorname{arc\,cot} r^*$ for $u^* \in (0, \pi)$. For this solution we construct the offset triangles (point $B_1, \ldots$) of a corresponding parallel 4-bar-mechanism. The self-motion of this 4-bar-mechanism will move the pivot points B_i in such a way that the corresponding rectangles will be homothetic to those in the faces of the trapezohedron.

Now we use the center of the trapezohedron as the center of scaling (the point F^* from Chapter 3 is the polyhedron's center) and put the 4-bar with its triangular offsets into the corresponding positions. Figure 7 displays the situation. There the input data from Figures 3 and 4 (for the correct value u^*) are presented for two values of t, again.

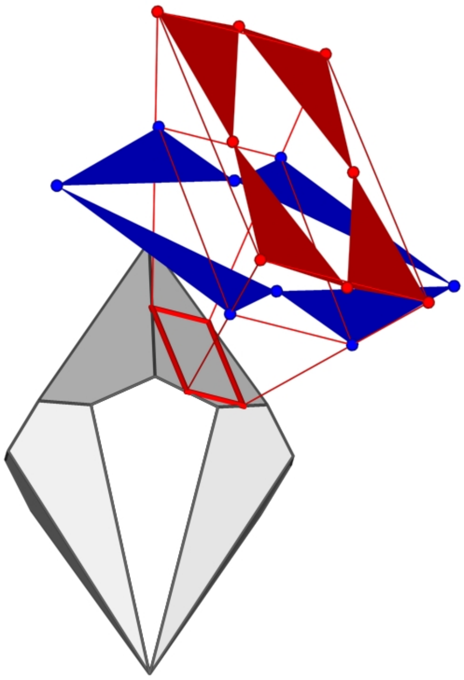

Figure 7: The trapezohedron and two poses of the corresponding special parallel 4-bar-mechanism (n = 7).

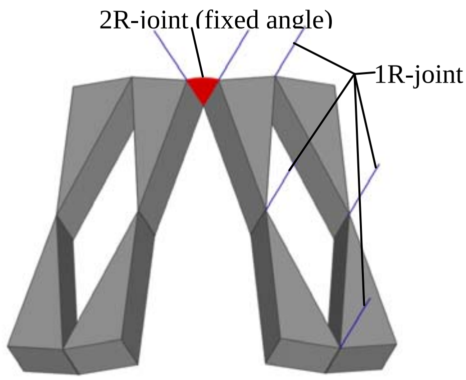

Figure 8: A spherical 2R-joint and some of the rotary joints.

As mentioned before, we will extrude the triangular bodies to get rigid parts of some mechanism. The planes of neighboring rectangles keep their angle even for varying parameter *t* (dilation does not change the angle). Therefore, we can interlink two neighbor 4-bars by spherical 2R-joints at the pivot points. These spherical 2R-joints have two rotary axes which intersect at constant angle. Figure 8 shows the different joints.

Figure 9 displays the corresponding closed chain (for *n* =7) at some time *t*.

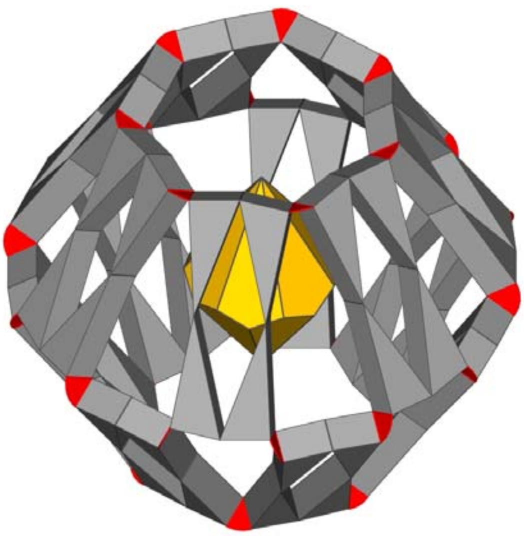

Figure 9: The trapezohedron and the corresponding closed mechanism (*n* = 7).

This mechanism consists of 8*n* rigid bodies which are linked via 8*n* rotary (1R-) joints and by 4*n* spherical 2R-joints.

The theoretical degree of freedom *F* of such a linkage is determined via the classical GRÜBLER-KUTZBACH-CHEBYSHEV-formula. It counts the number of theoretical restrictions of the mechanism with respect to the number of rigid bodies without allowing for geometric and topologic specialties etc.

For these special mechanisms it conveys the value

$$\begin{aligned} F &= 6\times(2n-1) - 6\times 12n + 8n + 2\times 4n \\ &= -8n - 6. \end{aligned} \quad (6)$$

Remarks:

1. Any mechanism of this type admits at least the one-parametric self-motion generated by the basic special parallel 4-bar motion. As *F*<0 for all *n*>2 we have generated new examples of overconstrained mechanisms.

2. These examples can be viewed as some generalisations to the so-called FULLEROID-like-mechanisms (see G. KIPER [3], K. WOHLHART [11] and the author [8]). As these mechanisms are based on trapezohedra we will refer to them as ‘*Trapezo – Fulleroid - mechanisms*’ here.

3. The one-parametric self-motion is a finite motion, but restricted by some physical limitations of the rigid bodies.

4. The Trapezo – Fulleroid - mechanism displayed in Figure 9 belongs to *n* = 7. Therefore its theoretical degree of freedom is *F* = - 62. But it admits at least the presented one-parametric self-motion.

5. It is far from being obvious that Trapezo – Fulleroid - mechanisms do not admit further self-motions. The paper [8] of the author can be interpreted as a case study to a very special case with a cube as basic trapezohedron (*n* = 3). For this very special example there even exist four different one-parametric self-motions of the mechanisms (see [8]). In Chapter 7 we will describe further one-parametric self-motions of the Trapezo - Fulleroid - mechanisms. But it seems to be far beyond this paper to determine all possible self-motions of general Trapezo – Fulleriod – mechanisms.

We can sum up:

Theorem 2: *The kinematic chains generated on the grounds of Chapters 2, 3, 4 and 5 yield overconstrained mechanisms. They are called Trapezo – Fulleroid - mechanisms and consist of 8n rigid bodies linked via 8n 1R-joints and 4n spherical 2R-joints. Though the classical theoretical degree of freedom of these mechanisms takes on the amazing value F = -8n-6, the Trapezo – Fulleroid - mechanism admits at least a one-parametric self-motion for all n>2.*

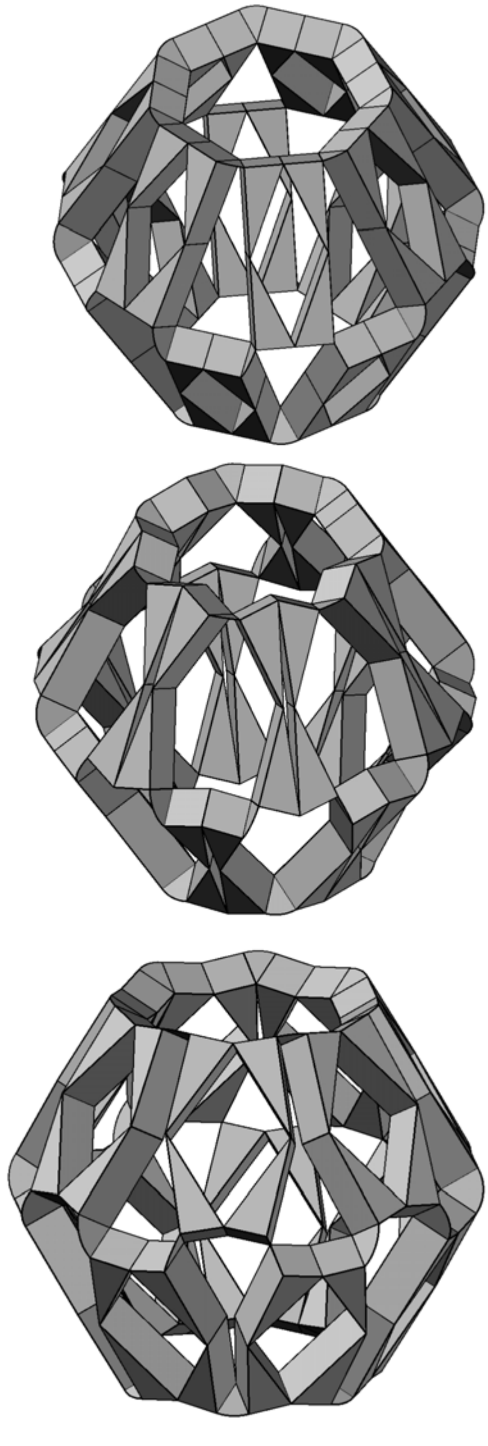

Figure 10: One of the overconstrained Trapezo – Fulleroid - mechanisms for *n* = 7 and 3 of its poses.

Figure 10 illustrates such a mechanism for *n* = 7 in three of its positions.

6. RELATIVE MOTIONS

The self-motion of our description can be viewed as a one-parametric motion of the Trapezo - Fulleroid - mechanism with respect to some 'world - coordinate frame' related to the given trapezohedron. With respect to this frame any single rigid body of the mechanism performs a one-parametric DARBOUX - motion (see (7)) parametrized by the same parameter *t*. The relative motion of one rigid body of the chain with respect to some other rigid body of the mechanism is therefore a one-parametric motion of a type studied in another paper by the author (see [6]). In all cases we get rational one-parametric motions with rational point-paths of degree ≤ 4. As for further properties of these motions we refer to [6]. In some cases they degenerate into pure spherical motions or pure translations (which are in all cases rational of degree ≤ 4).

We can summarize:

Theorem 3: *The overconstrained Trapezo – Fulleroid - mechanisms generated on base of Chapters 2, 3, 4 and 5 admit at least the one-parametric self-motions from its generation. The relative motion of any rigid body with respect to another one is a rational motion with rational point-paths of degree* ≤ 4.

7. FURTHER SELF-MOTIONS

In this chapter we demonstrate that the overconstrained Trapezo - Fulleroid - mechanisms admit even more than the single self-motion presented above. According to (4) we have

$$l_i'(\frac{u}{2}+\tau)=l_i'(\frac{u}{2}-\tau) \quad (7)$$
$$for\ i=1,2\ and\ \forall\tau\in[0,2\pi).$$

This yields $R'(\frac{u}{2}+\tau)=R'(\frac{u}{2}-\tau)$ for all values of τ. In analogy we have

$R'(\pi+\frac{u}{2}+\tau)=R'(\pi+\frac{u}{2}-\tau)$. At each of the the two instants $t=u/2$ and $t=\pi+u/2$ the mechanism is in a *high order singularity*: If the self-motion is stopped at these moments we can change the self-motion from chapter 5: For the parts of the mechanism corresponding to any face of the trapezohedron we either can proceed like for the self-motion from above or we can change the direction of the motion. This is possible without damaging our mechanism. As this can be done independently for all faces of the polyhedron, we have got 2^{2n-1} different one-parametric self-motions of the Trapezo – Fulleroid - mechanisms. They all share the two singular positions $t=u/2$ and $t=\pi+u/2$. Viewed from the world coordinate frame many of them are congruent, but viewed from the single rigid bodies they are different.

We can summarize:

Theorem 4: *The overconstrained Trapezo – Fulleroid – mechanisms based on the Trapezohedron with 2n faces in general admit at least 2^{2n-1} different one-parametric self-motions. All these self-motions share two high order singular positions.*

8. CONCLUSION

The paper has been dedicated to the generation of a new class of overconstrained mechanisms, the so-called Trapezo - Fulleroid - mechanisms. We presented a method constructing such mechanisms starting from polyhedra with specific closed chains of rectangles inscribed into their faces. These rectangles were used to extend some corresponding special parallel 4-bar-mechanisms into space. This new method highlights a new approach for implementing further highly overconstrained mechanisms based on polyhedra. The paper focused on examples generated with the so-called trapezohedra as basic polyhedra.

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