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# CHARACTERISATION OF ARCHITECTURALLY SHAKY PLATFORMS

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**Abstract.** We study architecturally shaky Stewart-Gough platforms and characterize the geometry of its anchor points. The main tool is the connection of geometric objects with mappings which preserve the structure of the problem. Here, the geometric way is the use of linear manifolds of correlations and quadratic transformations. By these methods we show that the anchor points have to be conjugate points with respect to 3-dimensional linear manifolds of correlations. This result is used to give all possible configurations of anchor points of architecturally shaky Stewart-Gough platforms.

## 0. Introduction

The paper is sequel to paper [Mick -Röschel 98]. Therefore references to it are prefixed "I". It deals with six legged Stewart-Gough platforms given by six pairs of anchor points $X_i$ and $Y_i$ $(i = 1, ...,6)$, each set on a plane $\varepsilon$ and $\varphi$, resp. (see figure I.1). In theorem I.4.1 we gave a characterization of architecturally shaky platforms of this type. Now we discuss these results in a geometric context: Linear manifolds of correlations are used to give a geometric characterization of the six pairs of anchor points $X_i$ and $Y_i$ $(i = 1, ...,6)$. It is shown the theory of $n$-fold conjugate pairs of points illuminates Karger's characterisation of architecturally shaky Stewart-Gough platforms [Karger 97].

The paper is organized as follows: In the first chapter well known geometric results on linear manifolds of correlations between planes are used. The next one gives an example. In chapter 3 it is shown that, barring a few exceptions, the set of points $\{X_i, Y_i\}$ is architecturally shaky, iff it consists of **four-fold conjugate points** with respect to a certain set of correlations.

## 1. Linear manifolds of correlations

The following is a brief summary of well known algebraic facts concerning linear manifolds of correlations. For further background one may consult [Berzolari 32]. Thus, we work in the projective space and assume the field algebraically closed. As in I.2B an arbitrary (3,3)-matrix $A := (a_{kl})$ describes a correlation

(1) $$\kappa : X \in \varepsilon \rightarrow \kappa(X) \subset \varphi$$

between the points $X\ (x_0 : x_1 : x_2)$ of the plane $\varepsilon$ and the lines of the plane $\varphi$. The coordinates of the points $Y\ (y_0 : y_1 : y_2)$ of the line $\kappa(X)$ satisfy the linear equation

(2) $$0 = (y_0, y_1, y_2)\, A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = (y_0, y_1, y_2) \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

We study an $r$-dimensional linear manifold

$$(3) \qquad A = \sum_{j=1}^{r} \lambda_j C_j \quad (r \geq 3) \quad (\lambda_j \in \Re)$$

of matrices spanned by linearly independent matrices $C_j$ ($j = 1,...,r$). This is an analytic representation of an ($r$-1)-parametric linear manifold of correlations $K_{r-1} := [\kappa_1, \ldots, \kappa_r]$, which is spanned by the (linearly independent) correlations $\kappa_j$ defined by the matrices $C_j$. We determine pairs of points $X$ and $Y$, which are conjugate points with respect to all correlations $\kappa$ of the linear manifold $K_{r-1} := [\kappa_1, \ldots, \kappa_r]$. This is equivalent to the fact, that $X$ and $Y$ are conjugate points with respect to the correlations $\kappa_j$ ($j = 1, ..., r$). An arbitrary point $X$ of $\varepsilon$ is mapped into $r$ lines $\kappa_j(X)$. In general they determine an $r$-sided polygon. If the lines $\kappa_j(X)$ for $j = 1, ..., r$ pass through one point $Y$ in $\varphi$, $X$ and $Y$ are called ***r*-fold conjugate points.**

1) First we investigate two-fold conjugate points. Each pair $\kappa_k, \kappa_l$ $(k \neq l)$ of different correlations defines a mapping

$$(4) \qquad Q_{k,l} : X \in \varepsilon \ \rightarrow \ Q_{k,l}(X) := \kappa_k(X) \cap \kappa_l(X) \in \varphi .$$

In general it is a **quadratic mapping** of the points of $\varepsilon$ to those of the plane $\varphi$ with three **singular points** in $\varepsilon$. In special cases the quadratic mapping can degenerate into a collineation, which may be a singular one. For all correlations $\kappa \in [\kappa_k, \kappa_l]$, the line $\kappa(X)$ contains the image point $Q_{k,l}(X)$. Therefore two independent correlations from $[\kappa_k, \kappa_l]$ produce the same quadratic transformation.

2) Next we consider the linear manifold of correlations spanned by three linearly independent correlations $\kappa_k, \kappa_l, \kappa_m$: Points $X \in \varepsilon$ with images $\kappa_k(X), \kappa_l(X), \kappa_m(X)$ passing through a point $Y$ are situated on a **plane cubic curve** $c_{k,l,m} \subset \varepsilon$, which can degenerate. It contains the singular points of all quadratic transformations which are generated by two independent correlations from $K_2 := [\kappa_k, \kappa_l, \kappa_m]$. For example, the singular points of $Q_{k,l}$ and $Q_{k,m}$ are on $c_{k,l,m}$. These points $X \in c_{k,l,m}$ and their images $Y = Q_{k,l}(X) = Q_{k,m}(X)$ are **triple conjugate points.** The image points $Y$ of $X \in c_{k,l,m}$ in general are situated on a plane cubic curve in $\varphi$.

Consider a special degenerate case of interest: A correlation $\kappa$ with matrix $C$ of rank 1 maps all points of $\varepsilon$ into one line of $\varphi$. For a given line $l$ in $\varphi$ we get a two-dimensional linear manifold $K_2 := [\kappa_k, \kappa_l, \kappa_m]$ of correlations. It is spanned by three linearly independent correlations $\kappa_k, \kappa_l, \kappa_m$ which map $\varepsilon$ to $l$. Then each point on $\varepsilon$ is conjugate to each point on $l$ with respect to $K_2$.

3) Finally, we consider the linear manifold of correlations spanned by four linearly independent correlations $\kappa_k, \kappa_l, \kappa_m, \kappa_n$: We look for **four-fold conjugate points** with respect to $\kappa_k, \kappa_l, \kappa_m, \kappa_n$. Each four-fold conjugate pair of points is triple conjugate.

According 2), the points we seek belong to every cubic curve for all $K_2 \subset K_3 := [\kappa_k, \kappa_l, \kappa_m, \kappa_n]$. We consider a special pair of cubics, for instance $c_{k,l,m}$ and $c_{k,l,n}$. These cubics have 9 points in common: Three of them are the singular points of the quadratic transformation $Q_{k,l}$ belonging to $[\kappa_k, \kappa_l]$ (4). For the remaining 6 points $X_i$ ($i$ = 1, ..., 6) we have: $\kappa_k(X_i), \kappa_l(X_i), \kappa_m(X_i), \kappa_n(X_i)$ are concurrent lines. The common point is denoted by $Y_i$. As mentioned before, $Y_i$ is the image of $X_i$ under the quadratic transformation $Q_{k,l}$. Thus, in general there exist exactly **six pairs of four-fold conjugate points** $\{X_i, Y_i\}$ ($i$ = 1, ..., 6). The other cubics $c_{k,m,n}$ and $c_{l,m,n}$ pass through $X_1, ..., X_6$, too. But they do not contain the singular points of the quadratic transformation $Q_{k,l}$.

We choose three arbitrary linearly independent correlations from $K_3$ and determine the corresponding cubic of triple conjugate points: Then in the general case all these cubics are on the six points $X_i$ ($i$ = 1, ..., 6).

**Remark:** These properties are **projectively invariant**. If we apply (independent) regular plane collineations to $\varepsilon$ and $\varphi$, correlations are mapped to correlations of image data. Conjugate points with respect to given correlations are mapped into conjugate points with respect to the images of these correlations.

**These results are well known in plane kinematics in another context** (see [Bottema-Roth 79, pp. 219]): We study four or more different positions of a moving plane $\varepsilon$ in a fixed plane $\varphi$. Each point $X \in \varepsilon$ has positions $X^0, ..., X^r$ ($r \geq 3$) in $\varphi$. The perpendicular bisector of $X^0$ and $X^j$ is a line in $\varphi$ denoted by $\kappa_j(X)$ ($j$ = 1,...,$r$). It is well known in kinematics, that in general this map $\kappa_j$ from the points of $\varepsilon$ to these perpendicular bisectors is a correlation. All circles containing the positions $X^0, X^1$ have their centers on $\kappa_1(X)$. Thus, the **quadratic mapping** (4) has its three-position kinematic counterpart in the fact, that the map of the points of the moving plane to the centers of the circumcircle of the 3 positions $X^0, X^1, X^2$ in general is a quadratic one. The respective poles of these three positions are its singular points. The **plane cubic curves** of triple conjugate points with respect to 3-dimensional manifolds of correlations from 2) have a four-position kinematic

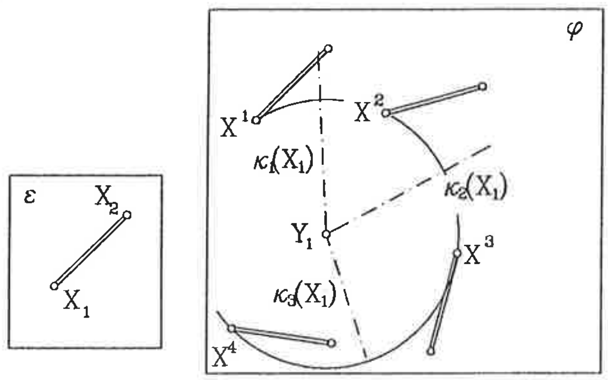


Figure 1: Four positions of a moving plane

counterpart. Those points $X$ of $\varepsilon$ with four positions on a circle in general are situated on plane cubic curve (triple conjugate points with respect to $\kappa_1,\kappa_2,\kappa_3$). In Figure 1 there is shown one triple conjugate pair of points $\{X_1, Y_1\}$ and a point $X_2$ which does not have this property . The **six pairs of four-fold conjugate points** from 3) belong to five-position theory: We look for points $X$ in $\varepsilon$ with five positions on a circle. They correspond to the four-fold conjugate points with respect to $\kappa_1,\kappa_2,\kappa_3,\kappa_4$. They are the intersections of two cubics. In plane kinematics only 4 such points are real, the Burmester-points. The remaining two are always complex circular points.

**2. An example.** To clarify the results of chapter 1 we give an example. Let 5 pairs of points of $\varepsilon$ and $\varphi$ be given by coordinates

$$\text{(5)}\qquad \begin{array}{lllll} X_1\,(1,0,0) & X_2\,(0,1,0) & X_3\,(0,0,1) & X_4\,(1,1,1) & X_5\,(x_{0,5},x_{1,5},x_{2,5}) \\ Y_1\,(1,0,0) & Y_2\,(0,1,0) & Y_3\,(0,0,1) & Y_4\,(1,1,1) & Y_5\,(y_{0,5},y_{1,5},y_{2,5}) \end{array}$$

We compute representations of all correlations (2), which have $\{X_i, Y_i\}$ $(i = 1, ..., 5)$ as conjugate points. With the abbreviations

$$\text{(6)}\qquad \mu_{k,l} := y_{k,5}\,x_{l,5} \quad \text{for } k,\ l = 0,1,2.$$

the 3-dimensional linear manifold $K_3$ of correlations spanned by $[\kappa_1, \kappa_2, \kappa_3, \kappa_4]$ with matrix representation (3) and matrices

$$\text{(7)}\qquad C_1 = \begin{pmatrix} 0 & \mu_{1,0}-\mu_{0,2} & \mu_{0,1}-\mu_{1,0} \\ \mu_{0,2}-\mu_{0,1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad C_3 = \begin{pmatrix} 0 & \mu_{1,2}-\mu_{0,2} & \mu_{0,1}-\mu_{1,2} \\ 0 & 0 & \mu_{0,2}-\mu_{0,1} \\ 0 & 0 & 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 0 & \mu_{2,1}-\mu_{0,2} & \mu_{0,1}-\mu_{2,1} \\ 0 & 0 & 0 \\ 0 & \mu_{0,2}-\mu_{0,1} & 0 \end{pmatrix} \quad C_4 = \begin{pmatrix} 0 & \mu_{2,0}-\mu_{0,2} & \mu_{0,1}-\mu_{2,0} \\ 0 & 0 & 0 \\ \mu_{0,2}-\mu_{0,1} & 0 & 0 \end{pmatrix}.$$

Thus, our given set of points $\{X_i, Y_i\}$ $(i = 1, ..., 5)$ consists of **four-fold conjugate points with respect to** $K_3 := [\kappa_1, \kappa_2, \kappa_3, \kappa_4]$. But there is at least a sixth pair $\{X_6, Y_6\}$ with this property:

For the following we assume $\mu_{0,1} \neq \mu_{0,2}$.[1] Then $C_1, C_2$ define the **quadratic transformation** $Q_{1,2}$, which maps the point $X(x_0,x_1,x_2) \in \varepsilon$ into $Y := Q_{1,2}(X)(y_0,y_1,y_2) \in \varphi$ with

$$\text{(8)}\qquad \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_0\,x_1\,(\mu_{0,2}-\mu_{0,1}) \\ x_1\,[x_1\,(\mu_{0,2}-\mu_{1,0}) + x_2\,(\mu_{1,0}-\mu_{0,1})] \\ x_0\,[x_1\,(\mu_{0,2}-\mu_{2,1}) + x_2\,(\mu_{2,1}-\mu_{0,1})] \end{pmatrix}$$

The **singular points of this transformation** are

[1] This condition is valid if neither $X_5$ lies on $X_1, X_4$ nor $Y_5$ lies on $Y_2, Y_3$.

(9) $\quad I := X_1\ (1,0,0),\ \ II := X_3\ (0,0,1)$ and $III(0,\mu_{1,0}-\mu_{0,1},\mu_{1,0}-\mu_{0,2})$.

Now we follow chapter 1, 3) and determine the **four-fold conjugate points** with respect to $K_3$. The equations of the curves $c_{k,l,m}$ are

$$
(10)\qquad
\begin{aligned}
c_{1,2,3}\ \ldots\ldots\ 0 &= x_1\,[x_1x_2(\mu_{1,0}-\mu_{0,2})+x_2^2(\mu_{0,1}-\mu_{1,0})+\\
&\quad + x_0x_1(\mu_{0,2}-\mu_{1,2})+x_0x_2(\mu_{1,2}-\mu_{0,1})]\\
c_{1,2,4}\ \ldots\ldots\ 0 &= x_0\,[x_1x_2(\mu_{0,1}-\mu_{2,0})+x_1^2(\mu_{2,0}-\mu_{0,2})+\\
&\quad + x_0x_1(\mu_{0,2}-\mu_{2,1})+x_0x_2(\mu_{2,1}-\mu_{0,1})]\\
c_{1,3,4}\ \ldots\ldots\ 0 &= x_0\,[x_1x_2(\mu_{1,0}-\mu_{0,2})+x_2^2(\mu_{0,1}-\mu_{1,0})+\\
&\quad + x_0x_1(\mu_{0,2}-\mu_{1,2})+x_0x_2(\mu_{1,2}-\mu_{0,1})]\\
c_{2,3,4}\ \ldots\ldots\ 0 &= x_2\,[x_1x_2(\mu_{0,1}-\mu_{2,0})+x_1^2(\mu_{2,0}-\mu_{0,2})+\\
&\quad + x_0x_1(\mu_{0,2}-\mu_{2,1})+x_0x_2(\mu_{2,1}-\mu_{0,1})]
\end{aligned}
$$

Thus, the cubics of the general case split into straight lines and (two distinct) conic sections $k_1, k_2$.

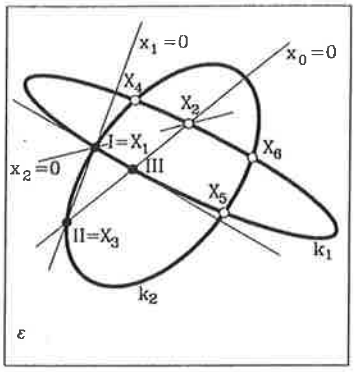

Figure 2: The general case in $\varepsilon$

There exists a unique collineation $\omega$ from $\varepsilon$ to $\varphi$ which transforms $X_i$ into $Y_i$ for $i$ = 1,2,3,4. We discuss two different cases:

**A)** $(x_{0,5}: x_{1,5}: x_{2,5}) \neq (y_{0,5}: y_{1,5}: y_{2,5})$ - i.e. $\omega$ does not transform $X_5$ into $Y_5$ : Then the two conic sections are different. There are exactly six points on all $c_{k,l,m}$. Thus, there are exactly 6 four-fold pairs of conjugate points with respect to the linear manifold $K_3$: In $\varepsilon$ they are the points $X_1,\ldots,X_5$ from (5) and a sixth point

$$
(11)\qquad X_6 = \begin{pmatrix} (y_{2,5}-y_{1,5})(\mu_{1,0}-\mu_{0,1})(\mu_{2,0}-\mu_{0,2}) \\ (y_{2,5}-y_{0,5})(\mu_{1,0}-\mu_{0,1})(\mu_{2,1}-\mu_{1,2}) \\ (y_{1,5}-y_{0,5})(\mu_{2,0}-\mu_{0,2})(\mu_{2,1}-\mu_{1,2}) \end{pmatrix},
$$

which lies on all 4 curves (10), too. Via the quadratic transformation (8) it is mapped to the point

$$
(12)\qquad Y_6 = \begin{pmatrix} (x_{2,5}-x_{1,5})(\mu_{1,0}-\mu_{0,1})(\mu_{2,0}-\mu_{0,2}) \\ (x_{2,5}-x_{0,5})(\mu_{1,0}-\mu_{0,1})(\mu_{2,1}-\mu_{1,2}) \\ (x_{1,5}-x_{0,5})(\mu_{2,0}-\mu_{0,2})(\mu_{2,1}-\mu_{1,2}) \end{pmatrix},
$$

which together with $X_6$ forms the sixth pair of four-fold conjugate points with respect to $K_3$. It is easy to see that the collineation $\omega$ maps the connecting line $X_5 X_6$ into

$Y_5\ Y_6$. Figure 2 shows this constellation for the plane $\varepsilon$ - in $\varphi$ we get an equivalent situation.

**B)** $(x_{0,5}: x_{1,5}: x_{2,5}) = (y_{0,5}: y_{1,5}: y_{2,5})$ : This condition is equivalent to $\mu_{0,1} - \mu_{1,0} = \mu_{0,2} - \mu_{21,0} = 0$ - i.e. $\omega$ transforms $X_5$ into $Y_5$ : Then $k_1 = k_2$ contains the given points $X_1,\ldots,X_5$. Thus **all points on this conic section** $k_1 = k_2$ with equation

(13) $$x_1 x_2 (\mu_{1,0} - \mu_{0,2}) + x_0 x_1 (\mu_{0,2} - \mu_{1,2}) + x_0 x_2 (\mu_{1,2} - \mu_{0,1}) = 0$$

and their images under $Q_{1,2}$ (4) are **four-fold conjugate points with respect to** $K_3$. As $k_1$ contains two singular points of $Q_{1,2}$ the images are on a conic section again. Moreover, the restriction of $Q_{1,2}$ to $k_1$ determines a **projectivity from** $k_1$ to $Q_{1,2}(k_1)$.

### 3. Architecturally shaky platforms

We consider a given platform with anchor points

(14) $\{X_i,\ Y_i\} \in \varepsilon \times \varphi$ with coordinates $(x_{0,i}, x_{1,i}, x_{2,i})$ $(y_{0,i}, y_{1,i}, y_{2,i})$

($i = 1,\ldots,6$) as in (I.16) (see figure I.1). As stated in I, chapter 4, architecturally shaky platforms have anchor points with coordinates satisfying (I.19). This is a system of 6 homogeneous linear equations for the 9 elements $a_{jk}$ of the matrix $(a_{jk})$. We have

(15) $$T\left(a_{00}, a_{01}, a_{02}, a_{10}, a_{11}, a_{12}, a_{20}, a_{21}, a_{22}\right)^t = (0,\ldots,0)^t$$

with coefficient matrix $T$ given by

(16) $$T := \begin{pmatrix} y_{01}x_{01} & y_{01}x_{11} & y_{01}x_{21} & y_{11}x_{01} & y_{11}x_{11} & y_{11}x_{21} & y_{21}x_{01} & y_{21}x_{11} & y_{21}x_{21} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{06}x_{06} & y_{06}x_{16} & y_{06}x_{26} & y_{16}x_{06} & y_{16}x_{16} & y_{16}x_{26} & y_{26}x_{06} & y_{26}x_{16} & y_{26}x_{26} \end{pmatrix}.$$

The elements of $T$ depend of the coordinates of the six anchor points $\{X_i,\ Y_i\}$. The system (15) can be written as

(17) $$\left(y_{0,i}, y_{1,i}, y_{2,i}\right) \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \left(x_{0,i}, x_{1,i}, x_{2,i}\right) = 0 \quad \text{for } i = 1, \ldots, 6.$$

Matrix $(a_{jk})$ determines a correlation of the points of $\varepsilon$ to the lines of $\varphi$. (17) characterises **the anchor points** $\{X_i,\ Y_i\}$ as **conjugate points with respect to this correlation.**

For the following considerations **we exclude the most singular cases**: If one (or both) series of anchor points $\{X_i\}$ or $\{Y_i\}$ ($i$=1,...,6) is on a line (in $\varepsilon$ or $\varphi$) we get an architecturally shaky case: All connecting lines belong to a singular linear line complex. We will not consider this trivial case.

According to remark 2 in chapter I.4 matrices $(a_{jk})$ determine corresponding linear line complexes, iff the additional 3 linear and homogenous equations (I.22) (resp. (I.23) with matrix $U$) hold. The elements of $U$ depend on the variable displacement parameters $\alpha, \beta, a, b, A, B$. Thus, corresponding linear line complexes exist only when

$$(18) \quad rank \begin{pmatrix} T \\ U \end{pmatrix} \leq 8 .$$

Without any conditions for the design of a Stewart-Gough platform we can choose certain values of $\alpha, \beta, a, b, A, B$ such that (18) holds. Thus, each Stewart-Gough platform has singular positions. Architecturally shaky platforms are characterised by condition (18) independent of $\alpha, \beta, a, b, A, B$. We give a geometric interpretation: The second part $U\left(a_{00}, a_{01}, a_{02}, a_{10}, a_{11}, a_{12}, a_{20}, a_{21}, a_{22}\right)^t = (0,0,0)^t$ of (18) (resp. (I.23)) are three homogenous linear equations of the form (17) for $i = 7,8,9$, if we put

$$(19) \quad \begin{array}{ll} X_7\left(0, \sin\beta, \cos\beta\right), & Y_7\left(0, \cos\alpha, -\sin\alpha\right) \\ X_8\left(1, -a\cos\beta + (A-b)\sin\beta,\ a\sin\beta + (A-b)\cos\beta\right), & Y_8\left(-1, B\sin\alpha, B\cos\alpha\right) . \\ X_9\left(1, -a\cos\beta,\ a\sin\beta\right), & Y_9\left(-1, B\sin\alpha + (A-b)\cos\alpha, B\cos\alpha - (A-b)\sin\alpha\right) \end{array}$$

Therefore shakiness of the system of points is equivalent to the existence of a correlation $\kappa$ (depending on $A, B, a, b, \alpha, \beta$), which has $\{X_i,\ Y_i\}$ for $i = 1, \dots, 9$ as conjugate points. Furthermore, points $X_7, X_8, X_9$ and $Y_7, Y_8, Y_9$ are on lines $g_X$ and $g_Y$ in $\varepsilon$ and $\varphi$ with parametrisation

$$(20) \quad \begin{array}{ll} X(t) \quad \begin{pmatrix} 1 \\ -a\cos\beta \\ a\sin\beta \end{pmatrix} + t\,(A-b)\begin{pmatrix} 0 \\ \sin\beta \\ \cos\beta \end{pmatrix} & \text{on } g_X \text{ and} \\ Y(t) \quad \begin{pmatrix} -1 \\ B\sin\alpha \\ B\cos\alpha \end{pmatrix} + (1-t)\,(A-b)\begin{pmatrix} 0 \\ \cos\alpha \\ -\sin\alpha \end{pmatrix} & \text{on } g_Y \text{ with } t \in \Re . \end{array}$$

(20) determines a projectivity between $g_X$ and $g_Y$. Correlations, which have $\{X_i,\ Y_i\}$ ($i$= 7,8,9) as conjugate points, have all pairs $\{X(t),\ Y(t)\}$ as conjugate points too. The points $X_7, X_8, X_9$ and $Y_7, Y_8, Y_9$ and the lines $g_X$ and $g_Y$ depend on the displacement data $a, (A-b), \beta, B, \alpha$. Thus we have

**Theorem 3.1:** A Stewart-Gough platform with six given pairs of anchor points $\{X_i,\ Y_i\}$ $(i = 1, \dots, 6)$ is architecturally shaky, iff for all possible displacement parameters $a, (A-b), \beta, B, \alpha$ there exist correlations $\kappa(a, (A-b), \beta, B, \alpha)$ with $\{X_i,\ Y_i\}$ $(i = 1, \dots, 6)$ and for all $t \in \Re$ $\{X(t),\ Y(t)\}$ (3.7) as conjugate pairs of points.

The equations (15) give a linear $r$-parametric solution for the unknown matrices $A = (a_{ij})$: The general solution can be written as

$$A = \sum_{j=1}^{r} \lambda_j \, C_j \quad (r = 9 - rank\ T) \quad (\lambda_j \in \Re) \tag{21}$$

with linearly independent matrices $C_j$ with constant elements. As $T$ and $U$ depend on different entities we discuss two different cases: A) $rank\ T \le 5$ (then condition (18) holds - thus, we can expect solutions) and B) $rank\ T = 6$.

**Case A**: $rank\ T \le 5$ - i.e. $r \ge 4$ in formula (21).

This is a condition for the set of anchor points $\{X_i, Y_i\}$ ($i$ = 1,...,6). We give a geometric characterisation. As $rank\ T \le 5$, at least one pair of points (say $\{X_6, Y_6\}$) is not necessary to establish the set (21). It can be computed from the information in $\{X_i, Y_i\}$ for $i$ = 1,...,5. The geometric discussion of (21) leads to the theory of correlations summarised in chapter 1.

We start our considerations with $rank\ T = 5$ (i.e. $r = 4$). Then the matrices $A$ determine a 3-parametric linear manifold of correlations $K_3$. Our six pairs of anchor points $\{X_i, Y_i\}$ ($i$=1,...,6) are conjugate points with respect to $K_3$, i.e. they are **four-fold conjugate points** (see figure 2). As these properties are projectivly invariant this example covers the general case for $rank\ T = 5$. This is the geometric background of Karger's construction [Karger 97]. In the special case mentioned there it is possible to use all points of the special conic section (13) and their images (on a conic section) as anchor points of the platform without losing architectural shakiness. The points of these conic sections are linked in a projectivity. Their connections establish platforms which admit selfmotions. This fact was already been known to R. Bricard [Bricard 06]. Of course, the conic sections can split into lines.

If $rank\ T < 5$ (i.e. $r > 4$) there exists at least a 4-parametric linear manifold of correlations $K_4$. Our six pairs of anchor points $\{X_i, Y_i\}$ are at least five-fold conjugate points with respect to $K_4$. According to our example this is impossible if there exist 4 pairs of anchor points which are not collinear.

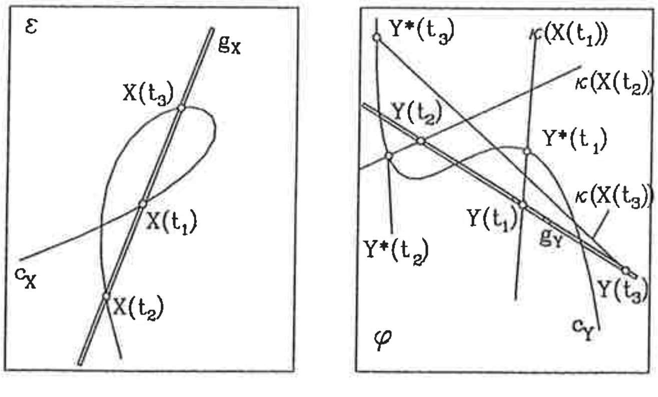


Figure 3: Geometric considerations in case B

**Case B:** *rank* $T = 6$. Linear homogenous system (15) then has a linear 3-parametric solution (21) with $r = 3$. They determine a 2-dimensional linear manifold of correlations $K_2$. According to chapter 1 there exists a plane cubic[2] $c_X := c_{1,2,3}$ in $\varepsilon$, such that all its points $X \in c_X$ and the images $Q_{1,2}(X) \in c_Y$ in $\varphi$ are triple conjugate points with respect to all correlations $\kappa \in K_2$. Now we consider an arbitrary position of the line $g_X$ (see figure 3). The points $\{X(t_1), X(t_2), X(t_3)\}$ (20) are the intersections of $g_X$ and $c_X$. We map the triple of points $\{X(t_1), X(t_2), X(t_3)\}$ in two ways: By $Q_{1,2}$ they have images $\{Y^*(t_1), Y^*(t_2), Y^*(t_3)\}$ on $c_Y$, by projectivity (20) $\{Y(t_1), Y(t_2), Y(t_3)\}$ on the line $g_Y$. Following theorem 3.1 we seek for correlations $\kappa \in K_2$ which have $\{X(t_i), Y^*(t_i)\}$ and $\{X(t_i), Y(t_i)\}$ as pairs of conjugate points for $i = 1,2,3$. Thus, correlation $\kappa$ maps $X(t_i)$ on the line $\kappa(X(t_i)) = Y^*(t_i), Y(t_i)$ $(i = 1,2,3)$. If $Y^*(t_i)$ and $Y(t_i)$ coincide, we can choose another position of the line $g_Y$ by variing $\alpha$ and $B$. As we can choose any position of $g_Y$, these three lines always must be concurrent, because they are the images of three collinear points. For a nondegenerate cubic $c_Y$ this is impossible. Geometric considerations show that $c_Y$ has to split into 2 different lines, one counting twice.[3] As mentioned in chapter 1 the same has to hold for the original cubic $c_X$. As $c_X$ contains the singular points of $Q_{1,2}$, one of these lines connects two of the singular points. All its points are mapped into the same image point by $Q_{1,2}$. The other line is mapped projectively into the image line. A short computation shows, that these conditions fix at least a 3-dimensional linear manifold of correlations. Thus, this doesnot belong to case B. Therefore, case B produces no additional solutions.

We sum up.

**Theorem 3.2:** The given set of points $\{X_i, Y_i\} \in \varepsilon \times \varphi$ $(i = 1, ..., 6)$ is architecturally shaky, iff one of the following statements holds:

a) $\{X_i, Y_i\}$ $(i = 1, ..., 6)$ are four-fold conjugate pairs of points with respect to a 3-dimensional linear manifold of correlations,

b) One or both sets of points $\{X_i\} \in \varepsilon$ and $\{Y_i\} \in \varphi$ $(i = 1, ..., 6)$ is situated on a line in $\varepsilon$ or $\varphi$, respectively. If this condition holds for exactly one set of points, the other one can be choosen arbitrarily in the plane of the platform.

**Remarks**: 1) For a given platform not of type b), the following test of architectural shakiness is valid: According to the proof of theorem 3.2 compute the rank of the matrix $T$ (16) - input data are the coordinates of the anchor points. *rank* $T \leq 5$ characterises architecturally shaky platforms.

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[2] Note, that we have excluded some singular cases!

[3] If $c_Y$ splits into one triple line we gain the excluded trivial case.

2) Theorem 3.2 gives the possibility to compute the complete set of architecturally shaky Stewart-Gough platforms. There is a great variety of socalled "degenerated" cases, if some anchor points in $\varepsilon$ or $\varphi$ coincide. A complete list[4] consists of many types depending on these coincidencies. It will not be given here.

3) If five arbitrary pairs of anchor points in the two planes $\varepsilon$ and $\varphi$, respectively, are given our construction allows to determine a sixth pair such that the platform becomes architecturally shaky.

4) Our characterisation is invariant with respect to (even different) nonsingular collineations in the planes $\varepsilon$ and $\varphi$, respectively.

5) It can happen that additional legs donot disturb architectural shakiness, for instance the conic section of the example of chapter 2.

6) If we fix the leg lengths of an architecturally shaky Stewart-Gough platform at any position we get: It is shaky from definition and does not loose this property in the next position. Thus, the manipulator admits self-motions.

7) According to chapter 2 we can compute numerical examples. One is used to build a model which is presented in the lecture.

**Conclusion.** We studied linear manifolds of correlations to characterise architecturally shaky Stewart-Gough platforms. The final result (theorem 3.2) shows that, barring one exception (theorem 3.2b), the anchor points $\{X_i, Y_i\}$ $(i = 1, ..., 6)$ of architecturally shaky Stewart-Gough platfroms consist of four-fold conjugate pairs of points with respect to a linear manifold of correlations. This property is invariant with respect to nonsingular collineations applied to the sets of points in $\varepsilon$ and $\varphi$, respectively. This result offers a simple possibility to test whether a platform is architecturally shaky or not. Theorem 3.2 and the remarks illuminate the geometric meaning of Karger's listing [Karger 97].

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For further literature see the references in [Mick-Röschel 98].


[4] See the list in [Karger 97].