DEVELOPABLE $(1, n)$-BÉZIER SURFACES

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Abstract. *Rational $(1, n)$-Bézier surfaces are ruled surfaces: They are generated by a one parameter set of straight lines. Among the ruled surfaces the developable ones play a special role in technical use. In this paper we give a general characterisation of developable rational $(1, n)$-Bézier surfaces.*

Categories and Subject Descriptors: G.1.1 [**Numerical Analysis**]: Interpolational – spline and piecework polynomial interpolations; I.3.5 [**Computer Graphics**]: Computational Geometry and Object Modelling – curve, surface, solid, and object representations

General Terms: Algorithms, Design.

Additional Key Words and Phrases: Bézier surfaces, developable surfaces, spline surfaces.

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1. Introduction. Developable surfaces can be rolled out into a plane without stretching. For some technical applications objects made of paper, sheets of tin etc. consist of pieces of developable surfaces. Therfore technicians have always been interested in that type of surfaces.

The use of computers in technical construction led geometric investigations on so called *'free form surfaces'*. They allow a unique handling in Computer Aided Design. Especially for the class of *'Bézier surfaces'* algorithms have become international standard. G. AUMANN ([1], [2]) has studied certain types of developable Bézier patches.

In this paper we demonstrate for $n \in N$(*arbitrarily*), which of the rational $(1, n)$-Bézier surfaces are developable. This is a generalisation of G. AUMANN's papers in two directions: On the first hand metric restrictions are abandonned, on the second hand we study **rational** Bézier patches, which allow additional degrees of freedom at the phase of design. As an example the case $n = 3$ is regarded more closely, the case $n = 2$ is discussed in detail.

As known (see H. BRAUNER [5],p.293, and W. BLASCHKE/K. LEICHTWEISS [3],p.192) the devolopable surfaces (that are surfaces, which can be bended into a plane without stretching or pulling) are just those surfaces, which consist of parts of planes, cylinders, cones, and surfaces generated by the tangents of a (in general twisted) curve. This curve is called *'edge of regression'*. Every developable surface is generated of a one parameter set of straight lines (*'generators'*). Every one of these lines has the property, that there exists one and only one tangent plane for all of its points.
So we can say: A necessary condition of a Bézier surface for being developable is the existence of a one parameter set of straight lines on the surface. In this paper we investigate $(1, n)$-Bézier surfaces which surely fulfill this precondition[1].

2. Rational (m, n)-Bézier Surfaces. In the real 3-dimensional projective space $P_3(\Re)$ we describe points X by homogeneous coordinates $(x_0, x_1, x_2, x_3) \neq (0,0,0,0)$ and write them as vectors $\mathbf{x} = (x_0, x_1, x_2, x_3)^t$. Vectors of the same direction determine the same point in $P_3(\Re)$.
Let $\mathbf{p}^{ij} := (p_0^{ij}, p_1^{ij}, p_2^{ij}, p_3^{ij})$ with $i = 0, \ldots, m$ and $j = 0, \ldots, n$ be the control points of a so called *'control net of a rational (m, n)-Bézier surface Φ'*. Φ is described by the representation

$$\mathbf{x}(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{p}^{ij} B_i^m(u) B_j^n(v) \qquad (u, v) \in [0,1] \times [0,1]. \tag{1}$$

The polynomials

$$B_i^m(u) = \binom{m}{i} (1-u)^{m-i} u^i \tag{2}$$

are the *'Bernstein polynomials of degree m'*. Those of degree n are defined in the same way. The parameter curves $u = const.$ and $v = const.$ are rational Bézier curves of order n, m, respectively.

It is usual (see HOSCHEK J./LASSER D. [6]) to define an *'affine part'* of the projective space by picking out a *'plane at infinity ω'*. We choose $\omega \ \ldots \ x_0 = 0$; so we can introduce affine coordinates in the affine part $A_3(\Re)$. These coordinates are $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}$.

[1]The question for general (m, n)-Bézier surfaces $(m, n > 1)$ with a continuous set of straight lines seems to be open until now. Examples for such surfaces are certain surfaces of order 2. As is known, however, surfaces of order 2 in general are not developable.

Every control point $\mathbf{p}^{ij}$ not in ω determines an affine position vector $\mathbf{b}^{ij} = (x^{ij}, y^{ij}, z^{ij})$. If the coordinate p_0^{ij} is called β_{ij} (the *'weight of the control point* $\mathbf{p}^{ij}$*'*), we can write $\mathbf{p}^{ij} = \beta_{ij}(1, \mathbf{b}^{ij})^t \quad (\beta_{ij} \neq 0)$.

3. Developable $(1,n)$-Bézier Surfaces. In (1) we set $m = 1$. A $(1,n)$-Bézier surface Φ is a ruled surface: Its u-curves $v = const.$ are straight lines (generators). Φ is developable if and only if (see E. KRUPPA [7], p.64) along any one of these lines there exists one and only one tangent plane. Necessary and sufficient for that is, that on any generator there exist two points with the same tangent plane (E. KRUPPA [7]). Along the border curves $k_0 \ldots u = 0$ and $k_1 \ldots u = 1$ the tangent planes of the surface are determined by the vectors[2]

$$\mathbf{x}(0,v), \mathbf{x}_u(0,v), \mathbf{x}_v(0,v) \tag{3}$$

and by

$$\mathbf{x}(1,v), \mathbf{x}_u(1,v), \mathbf{x}_v(1,v), \tag{4}$$

respectively. We have

$$\mathbf{x}_u(0,v) = \sum_{j=0}^{n} B_j^n(v)(\mathbf{p}^{1j} - \mathbf{p}^{0j}) = \mathbf{x}_u(1,v) = \mathbf{x}(1,v) - \mathbf{x}(0,v).$$

This is a point on the line $v = const.$, which is independent of the parameter u. The coincidence of the tangent planes (3) und (4) therefore is characterized by condition

$$\det(\mathbf{x}(0,v), \mathbf{x}(1,v), \mathbf{x}_v(0,v), \mathbf{x}_v(1,v)) = 0. \tag{5}$$

The ruled surface Φ is developable iff (5) holds for all parameter values $v \in [0,1]$.

For the evaluation of condition (5) it is of advantage to represent the Bézier surface Φ with the help of *'shift operators'* (see HOSCHEK J./LASSER D. [6], p.134) E, F in the following way[3]

$$\mathbf{x}(u,v) = (1 - u + uE)(1 - v + vF)^n \mathbf{p}^{00}. \tag{6}$$

Then we have

$$\begin{aligned} \mathbf{x}_v(0,v) &= n(1 - v + vF)^{n-1}(F - 1)\mathbf{p}^{00} \\ \mathbf{x}_v(1,v) &= n(1 - v + vF)^{n-1}(F - 1)\mathbf{p}^{10}. \end{aligned}$$

We define:

$$\begin{aligned} \delta_{ijkl} &:= \det(F^i\mathbf{p}^{00}, F^j\mathbf{p}^{00}, EF^k\mathbf{p}^{00}, EF^l\mathbf{p}^{00}) \\ &= \det(F^i\mathbf{p}^{00}, F^j\mathbf{p}^{00}, F^k\mathbf{p}^{10}, F^l\mathbf{p}^{10}) = \det(\mathbf{p}^{0i}, \mathbf{p}^{0j}, \mathbf{p}^{1k}, \mathbf{p}^{1l}) \end{aligned} \tag{7}$$

[2]Subscripts in $\mathbf{x}_u, \mathbf{x}_v$ will denote partial derivatives.

[3]The operators E, F are defined by $E^iF^j\mathbf{p}^{00} := \mathbf{p}^{ij}$ for $i \in \{0,1\}, j \in \{0,\ldots,n\}$.

and after short calculation we get out of (5):

$$\sum_{i,j,k,l=0}^{n-1} \binom{n-1}{i}\binom{n-1}{j}\binom{n-1}{k}\binom{n-1}{l}(1-v)^{4n-4-i-j-k-l}v^{i+j+k+l}\delta_{i,j+1,k,l+1} = 0 \quad for \;\; all \;\; v \in [0,1]. \tag{8}$$

For $s = const.$ the polynomials $(1-v)^{4n-4-s}v^s$ represent a basis in the vector space of all polynomials of degree $4n-4$. Comparing the coefficients in (8) for $i+j+k+l = s$ we gain the elegant conditions:

$$\sum_{\substack{i,j,k,l=0 \\ i+j+k+l=s}}^{n-1} \binom{n-1}{i}\binom{n-1}{j}\binom{n-1}{k}\binom{n-1}{l}\delta_{i,j+1,k,l+1} = 0. \quad for \;\; all \;\; s \in \{0,\ldots,4n-4\}. \tag{9}$$

This yields:

Theorem 1: *Conditions (9) characterize the control nets (and weights) of developable $(1,n)$-Bézier surfaces.*

For example for $n = 3$ this characterisation (9) (we have to use $\delta_{jjkl} = \delta_{ijkk} = 0$) shows:

$$\begin{aligned}
s=0: &\quad \delta_{0101} = 0 \\
s=1: &\quad \delta_{0102} + \delta_{0201} = 0 \\
s=2: &\quad \delta_{0103} + \delta_{0301} + 3(\delta_{0112} + \delta_{1201}) + 4\delta_{0202} = 0 \\
s=3: &\quad \delta_{0203} + \delta_{0302} + \delta_{0113} + \delta_{1301} + 3(\delta_{0212} + \delta_{1202}) = 0 \\
s=4: &\quad \delta_{0123} + \delta_{2301} + \delta_{0303} + 9\delta_{1212} + 3(\delta_{1203} + \delta_{0312}) + 4(\delta_{0213} + \delta_{1302}) = 0 \\
s=5: &\quad \delta_{0223} + \delta_{2302} + \delta_{0313} + \delta_{1303} + 3(\delta_{1213} + \delta_{1312}) = 0 \\
s=6: &\quad \delta_{0323} + \delta_{2303} + 3(\delta_{1223} + \delta_{2312}) + 4\delta_{1313} = 0 \\
s=7: &\quad \delta_{1323} + \delta_{2313} = 0 \\
s=8: &\quad \delta_{2323} = 0.
\end{aligned} \tag{10}$$

If the control points (and weights) of a (rational) $(1,3)$-Bézier surface are given, we can check with the help of the conditions (10), whether the given net leads to a **developable** Bézier surface.

4. Developable $(1,2)$-Bézier surfaces. The case $n = 2$ should be considered in

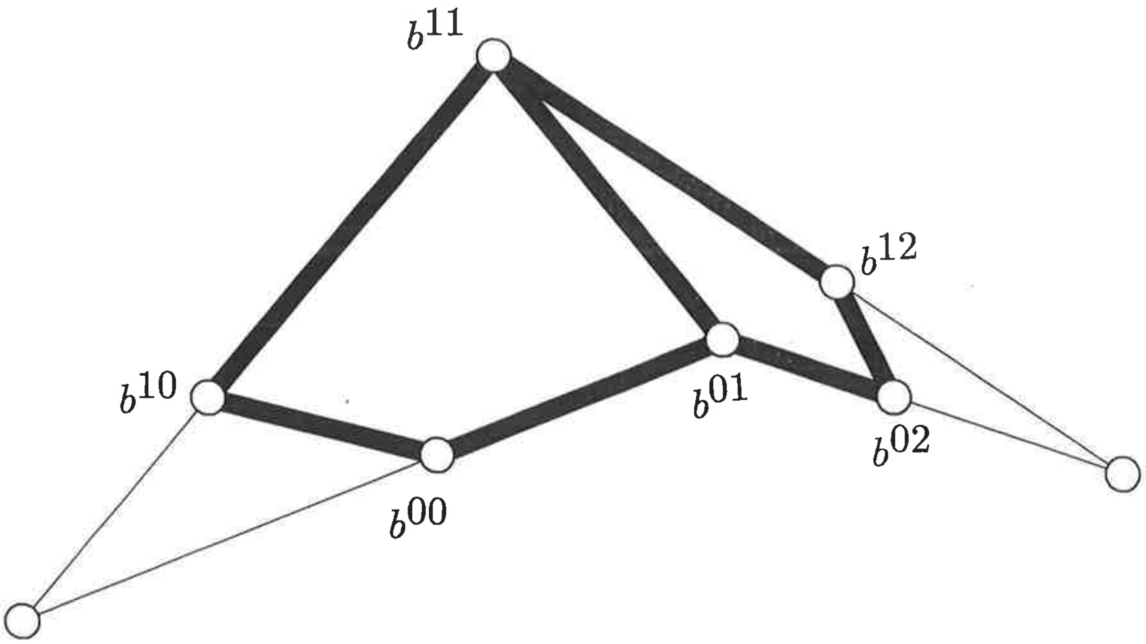

FIGURE 1: PLANARITY CONDITIONS FOR A DEVELOPABLE (1, 2)-BÉZIER SURFACE

detail. Out of (9) we gain

$$\begin{aligned} s=0: &\quad \delta_{0101}=0 \\ s=1: &\quad \delta_{0102}+\delta_{0201}=0 \\ s=2: &\quad \delta_{0202}+\delta_{0112}+\delta_{1201}=0 \\ s=3: &\quad \delta_{0212}+\delta_{1202}=0 \\ s=4: &\quad \delta_{1212}=0. \end{aligned} \tag{11}$$

As in the general case (9) the equations for $s=0$ and $s=4n-4$ (here $s=0$ and $s=4$) guarantee, that the control points $\mathbf{p}^{00}, \mathbf{p}^{01}, \mathbf{p}^{10}, \mathbf{p}^{11}$ and on the other hand $\mathbf{p}^{0n}, \mathbf{p}^{0(n-1)}, \mathbf{p}^{1n}, \mathbf{p}^{1(n-1)}$ are lying in one plane (see figure 1). These conditions (*'planarity conditions'*) have a very obvious geometric interpretation: They guarantee, that along every one of the border generators ($v=0$, $v=1$, resp.) the surface determines one and only one tangent plane.

We note:

Theorem 2: *A* $(1,2)$*-Bézier surface is developable iff the control net fulfills the conditions (11) .*

In the following we only regard the case $\beta_{ij} \neq 0$; we write $\mathbf{p}^{ij} = \beta_{ij} \begin{pmatrix} 1 \\ \mathbf{b}^{ij} \end{pmatrix}$ and so we have

$$\delta_{ijkl} = \beta_{0i}\beta_{0j}\beta_{1k}\beta_{1l} \ \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mathbf{b}^{0i} & \mathbf{b}^{0j} & \mathbf{b}^{1k} & \mathbf{b}^{1l} \end{pmatrix}.$$

For that we note

$$\delta_{ijkl} = \beta_{0i}\beta_{0j}\beta_{1k}\beta_{1l} \quad B_{ijkl}.$$

The real number B_{ijkl} is determined if the control points $\mathbf{b}^{0i}, \mathbf{b}^{0j}, \mathbf{b}^{1k}, \mathbf{b}^{1l}$ in E_3 are given.

In general it will not be possible to construct a developable rational Bézier surface, which is spanned between two given rational Bézier curves of order 2. If all the weights β_{ij} are free and the control points $\mathbf{b}^{00}, \mathbf{b}^{01}, \mathbf{b}^{10}, \mathbf{b}^{11}$ and $\mathbf{b}^{0n}, \mathbf{b}^{0(n-1)}, \mathbf{b}^{1n}, \mathbf{b}^{1(n-1)}$ respectively are in one plane (planarity conditions), in (10) there remain still three essential conditions for $s = 1, 2, 3$.

One possibility of a practical strategy to generate a developable $(1, 2)$-Bézier surface is the following:

1. Let the control points $\mathbf{b}^{ij} \in E_3$ be given the way that the planarity conditions hold. So we can be sure that in (11) the equations for $s = 0$ and $s = 4$ are fulfilled.

2. The remaining conditions for $s = 1, 2, 3$ are homogeneous quadratic equations in the weights $(\beta_{00}, \beta_{01}, \beta_{02})$ and $(\beta_{10}, \beta_{11}, \beta_{12})$, respectively. With any triple of solution every proportional triple also is a solution. Thus we choose $\beta_{00}{:}\,\beta_{02}$ arbitrarily and get out of (11) for $s = 1, 2, 3$:

$$\begin{aligned} \beta_{00}\beta_{10}\left[\beta_{01}\beta_{12}B_{0102} + \beta_{02}\beta_{11}B_{0201}\right] &= 0 \\ \beta_{00}\beta_{02}\beta_{10}\beta_{12}B_{0202} + \beta_{00}\beta_{01}\beta_{11}\beta_{12}B_{0112} + \beta_{01}\beta_{02}\beta_{10}\beta_{11}B_{1201} &= 0 \\ \beta_{02}\beta_{12}\left[\beta_{00}\beta_{11}B_{0212} + \beta_{01}\beta_{10}B_{1202}\right] &= 0. \end{aligned} \tag{12}$$

   Elimination of β_{01} yields two equations for $\beta_{10}{:}\,\beta_{11}{:}\,\beta_{12}$.

In the general case we can gain β_{01} as well as $\beta_{10}{:}\,\beta_{11}{:}\,\beta_{12}$ out of it. The results are

$$\beta_{01}^2 = \beta_{00}\beta_{02}\frac{B_{0201}B_{0202}B_{0212}}{B_{0201}B_{0112}B_{1202} + B_{1201}B_{0212}B_{0102}} \tag{13}$$

and

$$\beta_{10}{:}\,\beta_{11}{:}\,\beta_{12} = \beta_{00}B_{0212}B_{0102} : -\beta_{01}B_{0102}B_{1202} : \beta_{02}B_{0201}B_{1202}. \tag{14}$$

Real solutions only exist iff $\beta_{00}\beta_{02}\frac{B_{0201}B_{0202}B_{0212}}{B_{0201}B_{0112}B_{1202}+B_{1201}B_{0212}B_{0102}} \geq 0$ holds.

Figures 2 and 3 show some examples of rational $(1, 2)$-Bézier surfaces; we have put $\beta_{00} = \beta_{02} = 1$.

For $\beta_{00}, \beta_{02} > 0$ we suggest to choose the root $\beta_{01} \geq 0$ in condition (13). We get a rational Bézier patch Φ_1, which naturally is part of an algebraic surface Φ. If we take - for comparison - the second root $\beta_{01} < 0$ in (14) only β_{11} changes its sign.

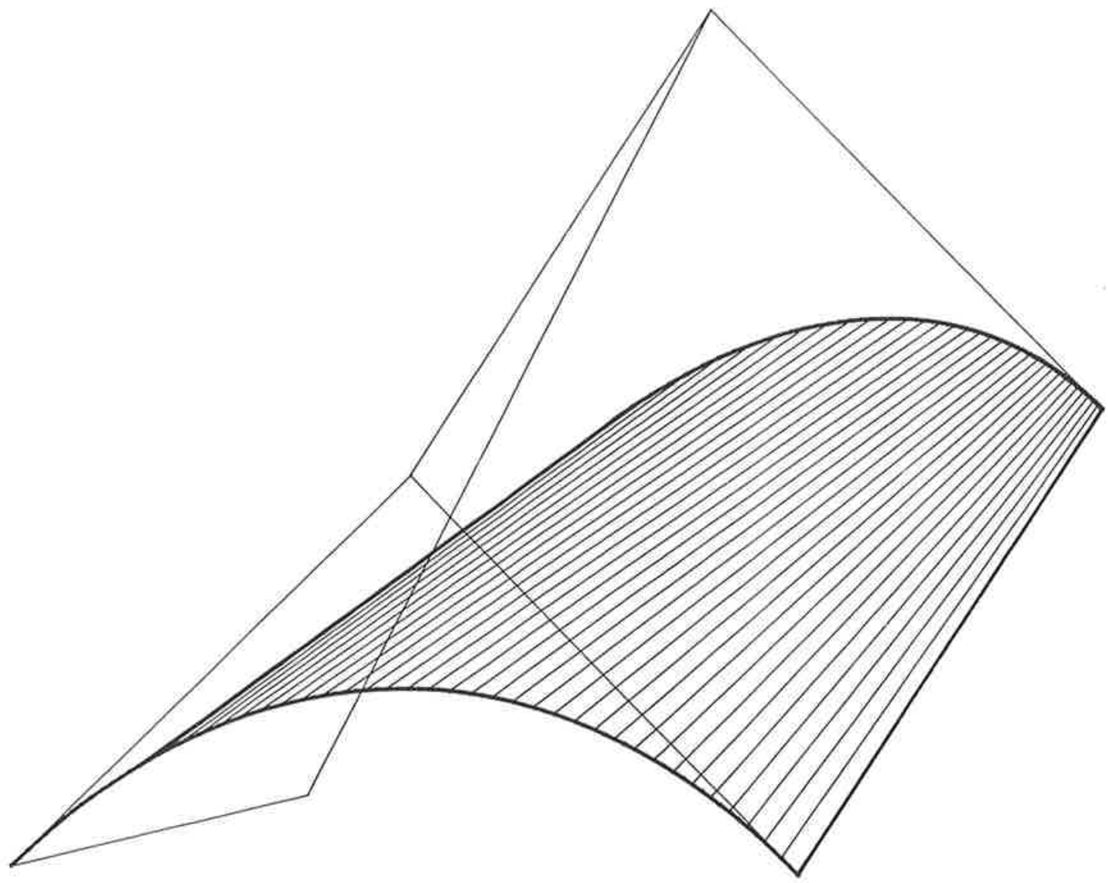

FIGURE 2: AN EXAMPLE OF A DEVELOPABLE RATIONAL (1, 2)-BÉZIER SURFACE

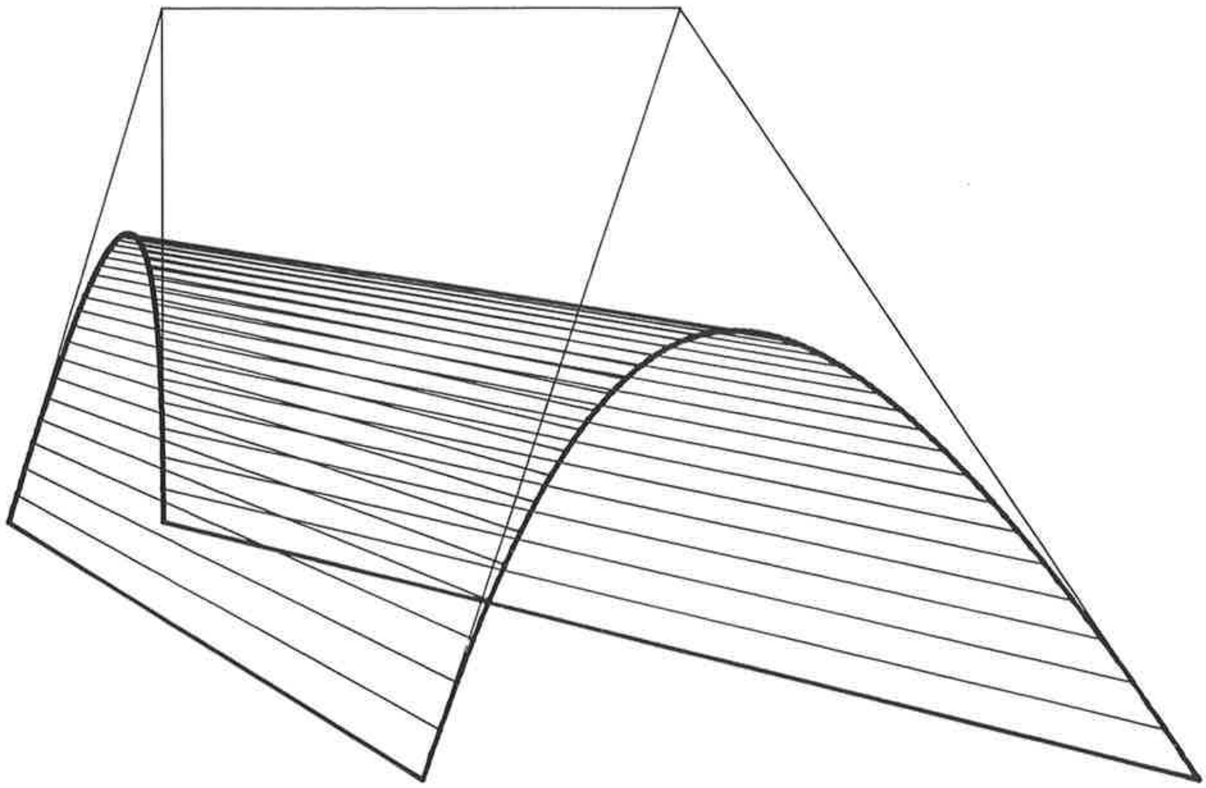

FIGURE 3: AN EXAMPLE OF A DEVELOPABLE RATIONAL $(1,2)$-BÉZIER SURFACE

The corresponding patch Φ_2 then is part of the same algebraic surface Φ. Figure 4a,b illustrates that.

On every nonconical developable surface the generators are tangent to the edge of regression (see 1.). Patches containing parts of this curve in general have no technical application. The algorithm described above also works in this case. Figure 5 shows such a Bézier patch.

5. An example of a developable $(1,3)$-Bézier surface. Equation (10) characterizes the control points of developable $(1,3)$-Bézier patches. We now treat with an example in order to show the use of these conditions.

In our example we restrict the weights $\beta_{ij} = 1$ $(i = 0,1,\ j = 0,\ldots,3)$. The edge generators $e_0 \ldots \mathbf{x}(u,0), e_1 \ldots \mathbf{x}(u,1)$, resp., shall not be situated in one plane. The tangent planes τ_0, τ_1 along e_0, e_1 shall not be parallel; the line of intersection is called s. Excluding some special cases we place affine coordinates $\{O,x,y,z\}$ the way that $\tau_0\ \ldots\ y=0,\ \tau_1\ \ldots\ x=0,\ e_0\ \ldots\ y=z=0$ and $e_1\ \ldots\ x=y-z=0$ hold. As in

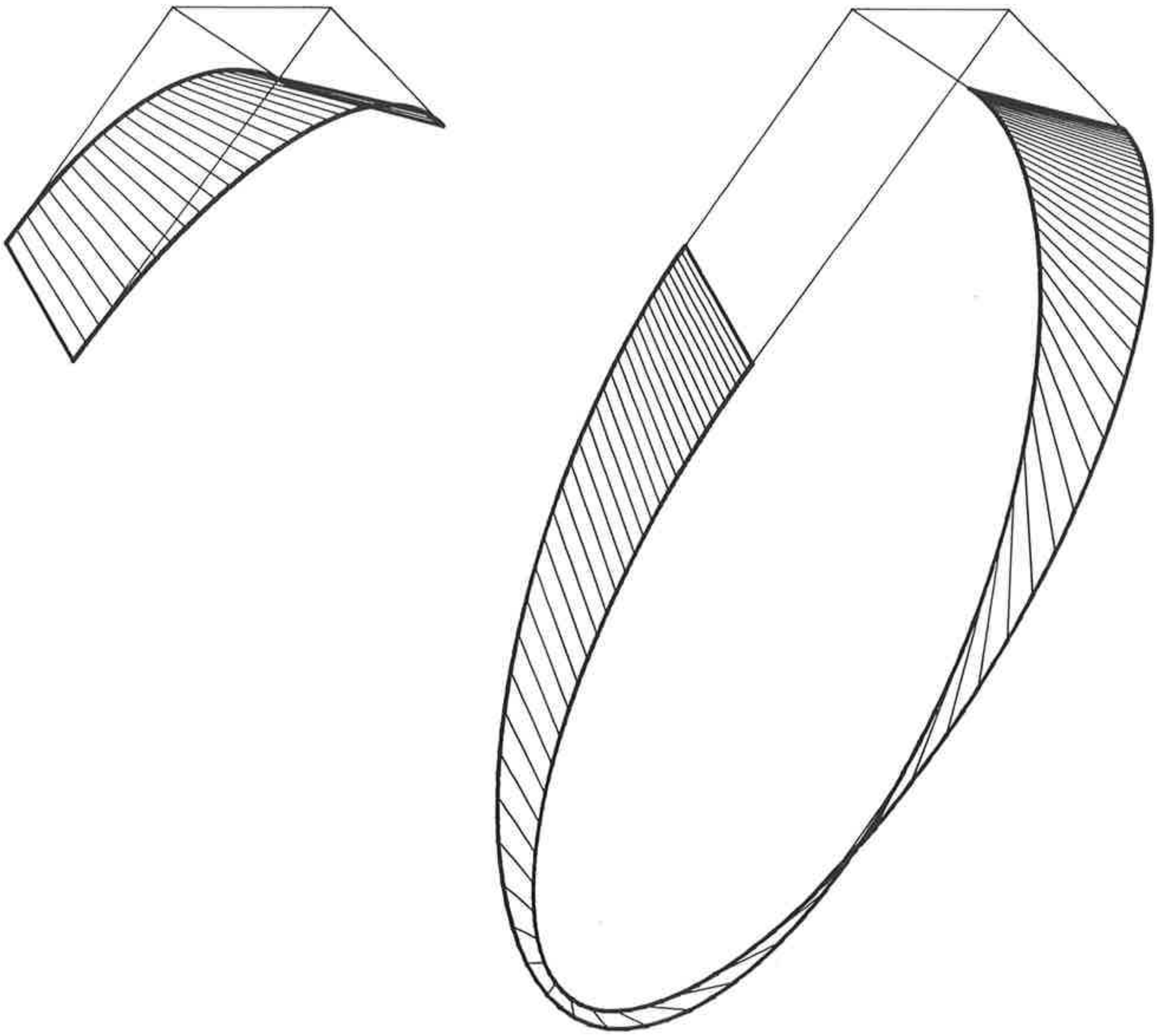

FIGURE 4A,B: CHANGING THE SIGN OF β_{01}

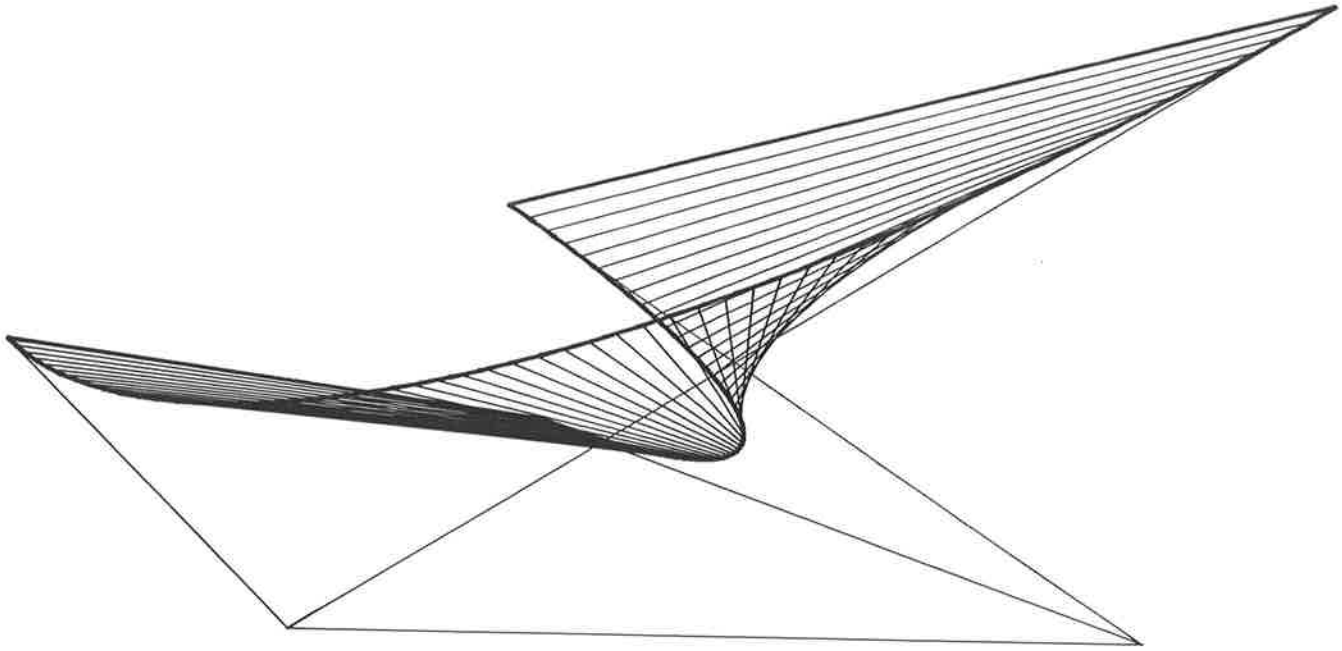

FIGURE 5: A PATCH CONTAINING A PART OF THE EDGE OF REGRESSION

section 4 we put $\mathbf{p}^{ij} = \begin{pmatrix} 1 \\ \mathbf{b}^{ij} \end{pmatrix}$ with $\mathbf{b}^{ij} := (x^{ij}, y^{ij}, z^{ij})^t$. Then we have $x^{02} = x^{12} = 0$, $y^{01} = y^{11} = 0$, $z^{10} = 0$, $z^{13} = 1 - y^{13}$ and we may put $\mathbf{b}^{00} = (1,0,0)^t$, $\mathbf{b}^{03} = (0,1,0)^t$ (see fig. 6).

So equations $s = 0$ and $s = 8$ in (10) hold. The remaining equations $s = 1, \dots, 7$ represent a linear system for the unknowns z^{ij} $(i = 0,1,\ j = 1,2)$. Solving the equations $s = 1,2,6,7$ we get solutions z^{ij} $(i = 0,1,\ j = 1,2)$, depending on the values of $x^{01}, x^{11}, x^{10}, y^{02}, y^{12}, y^{13}$. Then the other equations $s = 3,4,5$ represent a single condition[4] on $x^{01}, x^{11}, x^{10}, y^{02}, y^{12}, y^{13}$.

As an example we choose $x^{01} = \frac{3}{4}, x^{10} = \frac{1}{2}$, $y^{02} = \frac{3}{4}$, $y^{12} = \frac{1}{4}$, $y^{13} = \frac{1}{2}$. One solution of the mentioned condition is given by $x^{11} = \frac{25}{36}$. The corresponding values for the unknown z^{ij} are: $z^{01} = -\frac{6}{13}$, $z^{11} = -\frac{3}{13}$, $z^{02} = -\frac{59}{52}$, $z^{12} = \frac{83}{156}$, $z^{13} = \frac{1}{2}$. Fig. 7 shows the resulting developable $(1,3)$-Bézier surface.

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[4]This equation is excessively long and therefore should only be computed for given numerical input.

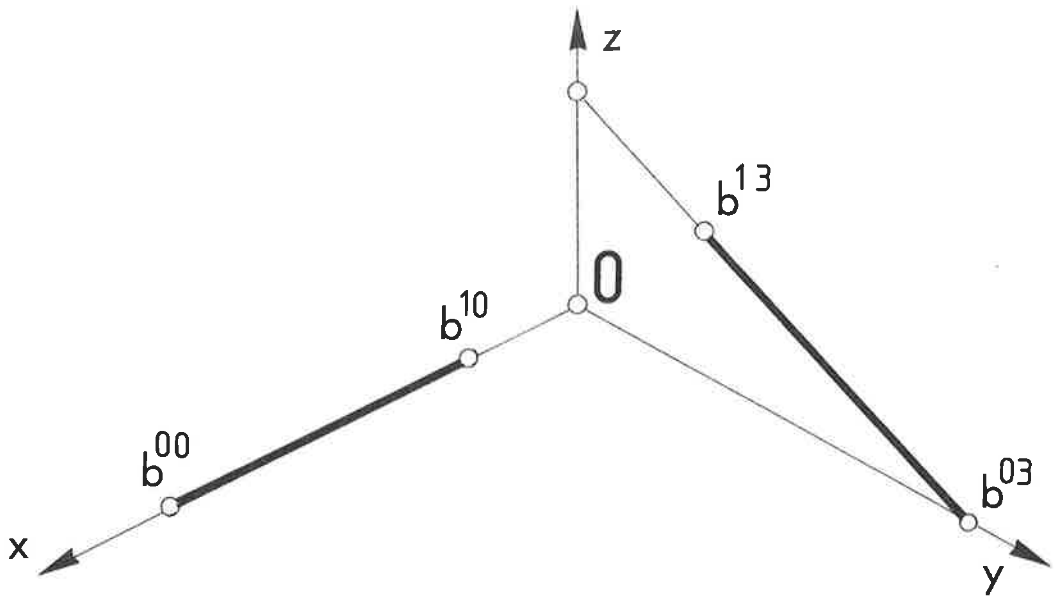

FIGURE 6: CONTROL POINTS OF THE EXAMPLE OF SECTION 5

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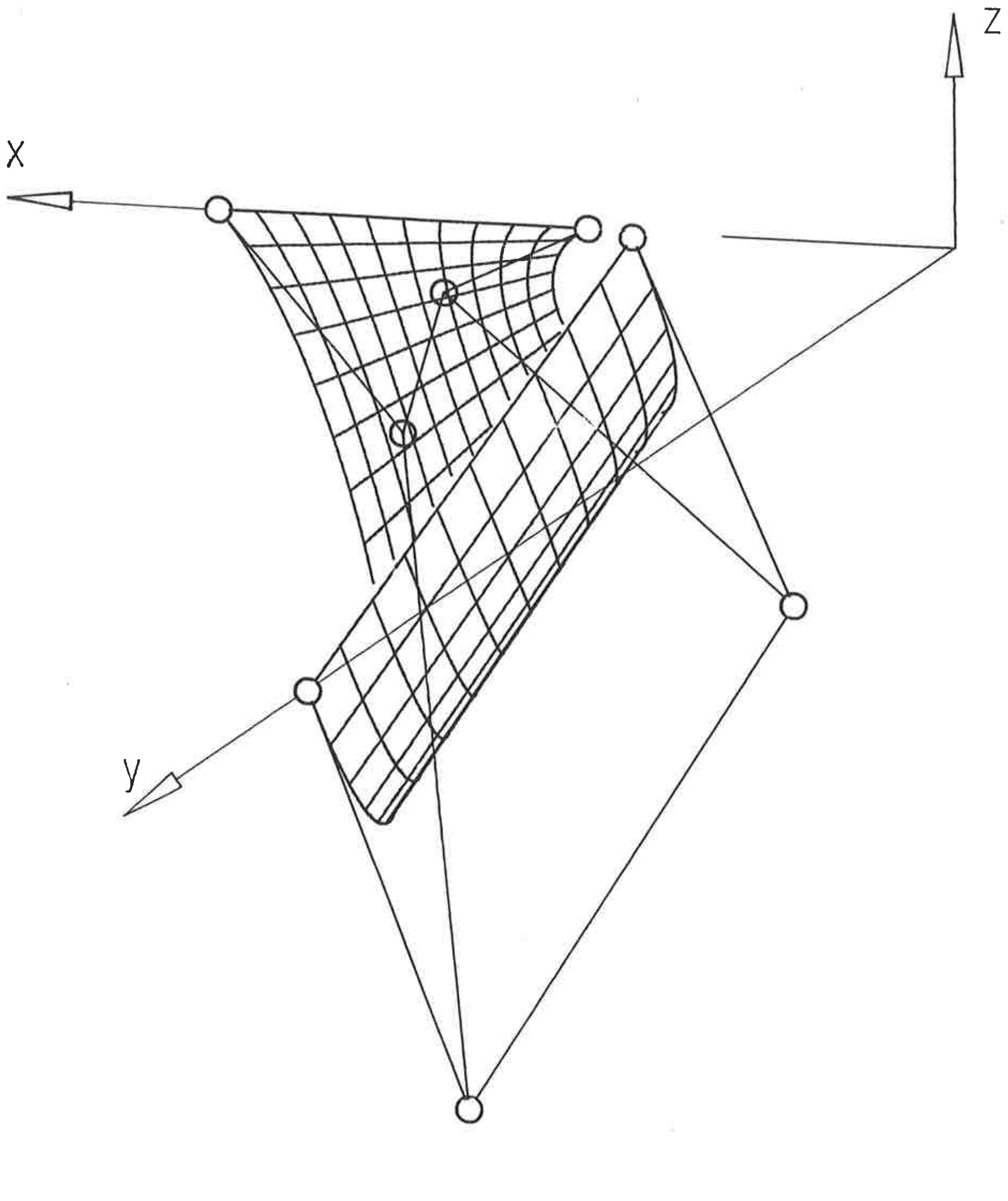

FIGURE 7: A DEVELOPABLE (1, 3)-BÉZIER PATCH