Frameworks generated by the Octahedral Group

Dedicated to Prof. Hans VOGLER at the occasion of his 70^{th} birthday

S. Mick, O. Röschel
Institute of Geometry
TU Graz
Kopernikusgasse 24
A-8010 Graz
Austria.
E-mail: mick@tugraz.at, roeschel@tugraz.at

Abstract The paper is devoted to the generation of frameworks of polyhedra as ornaments of the octahedral group O. An hierarchical block structure is used to implement the action of O in a CAD-package. The framework is generated by a starting (prismatic) rod as the motif. We give a series of examples and discuss the symmetry of the corresponding ornaments.

Keywords: Polyhedra, Octahedral Group, Frameworks, Geometric Modeling in CAD-packages
AMS-Class: 51N05,68U07, 97D30

1. Introduction. Any polyhedron of the 3-dimensional Euclidean space determines a group of symmetry G. Its elements are direct automorphic displacements of the polyhedron. We start with an object M, called *motif*. The orbit of M with respect to the group G is called *ornament* (G, M) *with motif* M.
There are many interesting publications with fascinating figures dealing with such ornaments (see [1]-[9]). We will present an approach to teach regular polyhedra and their ornaments even for undergraduates. It is an interesting topic to visualize the action of G with CAD-packages (see [10], [11]). We will work with an hierarchic block structure as implementation of the corresponding group of symmetry G. Additionally, the design of the motives trains geometric modeling and needs familiarity with spatial congruence transformations. All considerations can be performed directly in the 3-dimensional space. Former efforts of drawing of these ornaments are replaced by using a CAD-package.[1]
In this paper we restrict our examples to the group $G = O$: This group O is the *set of direct automorphic displacements of a regular octahedron* (or equivalently a cube).

[1]The figures of this paper are produced in the CAD-packages AutoCAD and MicroStation.

2. Frameworks as Ornaments. In the introduction we defined ornaments (G, M). We will present examples by using one prismatic rod as motif M. They will be defined by a regular polygon p as profile, which is extruded along its axis g (orthogonal to the plane of the profile).

We will present frameworks, where the axes and the edges of the rods form closed rings. They are gained, if g has at least two intersecting neighbors (positions under G). This will happen in the following two cases[2] (which can be mixed):

Type A: The axis g of the rod is orthogonal to at least one rotational axis of the group G.

Type B: The axis g of the rod meets at least two rotational axes of the group G.

3. The Octahedral Group. We will use some basic properties of the *octahedral group* O. We will give a list of the elements of O. [3] O contains the following 24 direct displacements (see figure 1):

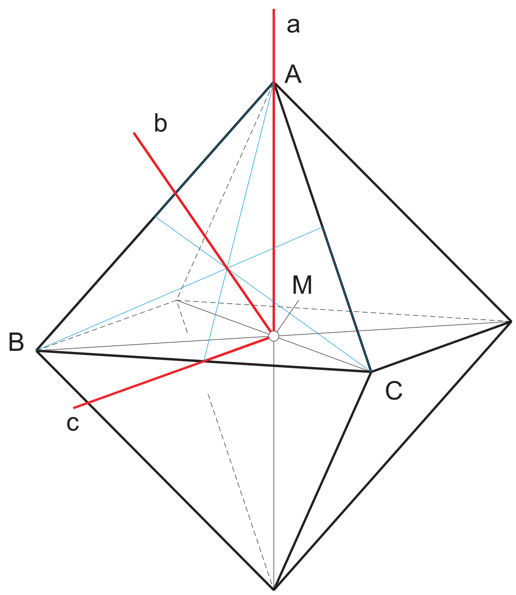

Figure 1: Some axes of rotation of the group O

- The identity id.
- Rotations about 4-fold axes a connecting opposite vertices of the octahedron.
- Rotations about 3-fold axes b connecting the centers of opposite faces.
- Rotations about 2-fold axes c connecting midpoints of opposite edges.

[2]There are more possibilities if G is containing reflections, too.

[3]For more details see the textbooks [1], [3], [5], [6], [7].

O is a normal subgroup of index 2 in the *full octahedral group* O_h. O_h is the union of O and the coset $O_\rho = O \circ (\rho)$, where (ρ) denotes a reflection in a plane of symmetry ρ of the octahedron.
An investigation of the structure of O leads to various implementations of the action of O in a CAD-package. We suggest to use the following hierarchical procedure, which implements O as a sequence of blocks (or models):

Motif = *Identity* - *Triple* - *Pair* - *Group* = **Ornament**

There the block *Triple* contains 3 rotated copies of the block *Identity* (axis of rotation is the 3-fold axis b of the face A, B, C of the octahedron).
The block *Pair* contains 2 rotated copies of the block *Triple* (axis of rotation is the 2 fold axis c - c contains the midpoint of the edge $[B, C]$.
The block *Group* contains 4 rotated copies of the block *Pair* (axis of rotation is the 4-fold axis a through the vertex A.
Our motif M is used as input into the block *Identity*. The resulting ornament is generated (as output) in the block *Group*. The following figure 2 displays the situation for an arrow as motif M.

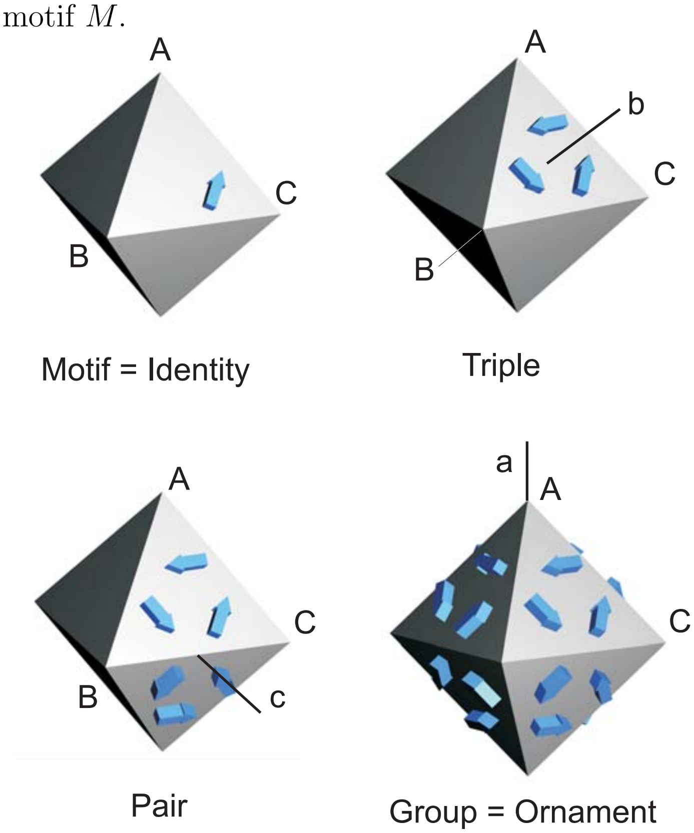

Figure 2: The blocks of our implementation of the ornaments (M, O)

These preparations allow to generate various ornaments by the choice of suitable motives. [4]

4. Hinged polygons as ornaments of case A. Firstly, we present two nice examples for case A. We will use a cylindric rod as the motif. According to section 2 its axis g is chosen orthogonal and skew to one of the 3- or 4-fold axis of the octahedron.

Example A1 (Hinged triangles): The axis g of the rod is orthogonal to the 3-fold axis b (see figure 3). The ornament is gained by rotating a rod along an edge of the octahedron with respect to the axis b (angle $2\pi/9$).

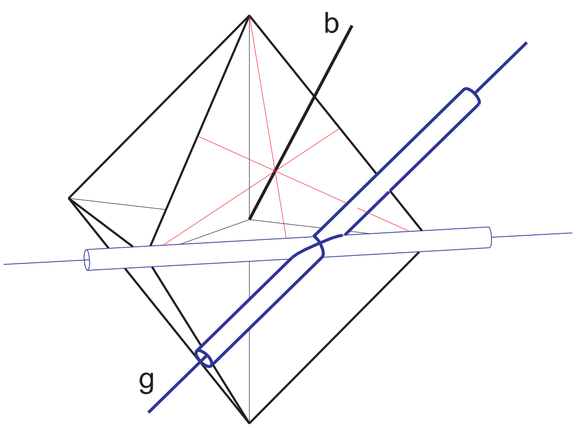

Figure 3: Motif for figures 4 and 5

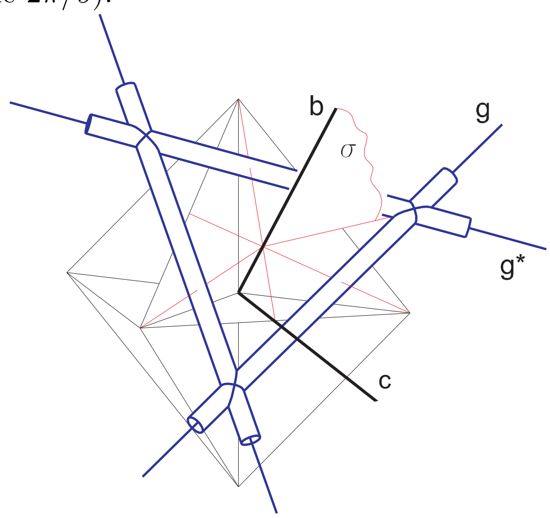

Figure 4: The block *Triple*

The block *Triple* contains 3 rotated copies of this motif, which form a triangle. The intersection of two neighbor rods (axis g and g^* respectively) splits into two ellipses. Therefore it is quite natural to use a miter cut in order to get fitting rods. This miter cut follows the plane of symmetry σ of g and g^* through the center M of the polyhedron. The following figure 5 shows the result with an inscribed sphere, which is used to hide some parts in the back and to highlight the structure of the framework. There are 8 triangles, each hinged with 3 neighbors. It needs some attempts to gain a solution without self-intersections. Figure 4 displays the necessary conditions: In order to get the ornament the triangle (included in the block *Triple*) is rotated about the 2-fold axis c. To guarantee that the original and the rotated triangle are hinged,

[4]The reader is invited to experiment with different motives. Even without deeper geometric considerations there can be gained fascinating ornaments.

the two-fold axis c has to intersect the initial triangle in an inner point. The radius of the rod has to be smaller than the distance of g and of c (see figure 4). Based on these ornaments there exist art objects (see H.S.M. COXETER [4]).

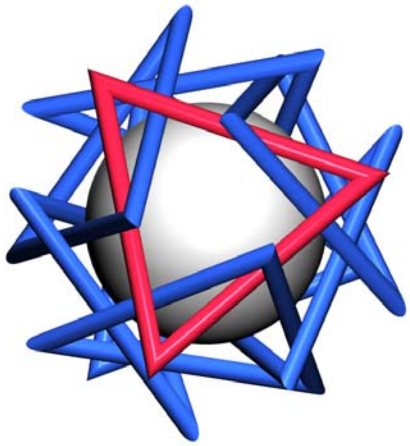

Figure 5: Hinged triangles

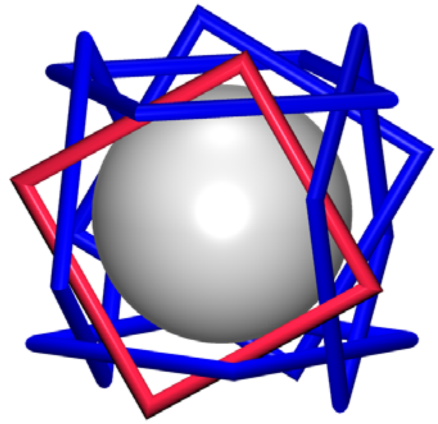

Figure 6: Hinged squares

Example A2 (Hinged Quadrangles): The axis g of the rod is orthogonal to the 4-fold axis a. For a suitable choice of the rod we gain the ornament of figure 6.

5. Frameworks as ornaments of case B. Secondly, we turn to case B and we will use prismatic rods. The axis g of the rod meets two (or more) axes of the octahedron. We generate a prismatic rod R by extrusion of a regular hexagon along the axis g. As in case A the rod R should fit to its neighbors. Additionally, we want to get intersecting edges for intersecting prismatic rods.

We consider an $n-$fold axis z intersecting g at a point Z (see figure 7). g^* is the axis of the neighbor rod R^* (rotation ρ of g about the axis z through the angle $2\pi/n$). This rotation can be generated as composition of two reflections $(\sigma) \circ (\varepsilon)$: The first plane is $\varepsilon := [g, z]$, the second plane σ is the plane of symmetry of g and g^* through z. We get intersecting edges on R and R^* if the rotation ρ and the reflection in the plane σ yield the same rod R^*. This fact is guaranteed if the rod R is plane symmetric with respect to the plane ε (σ is used as the plane of a miter cut). As the $n-$fold axis z contains the center M of the polyhedron we have $\varepsilon = [g, M]$. Hence ε is independent from the axis z. The rod R has a second neighbor R^{**} (rotated copy of R, rotation about z through the angle $-2\pi/n$). As before, we put a second miter cut through R. Figure 8 shows the result. In general the two miter cuts are different - in the case of a 2-fold axis ($n = 2$) the two cuts are coincident.

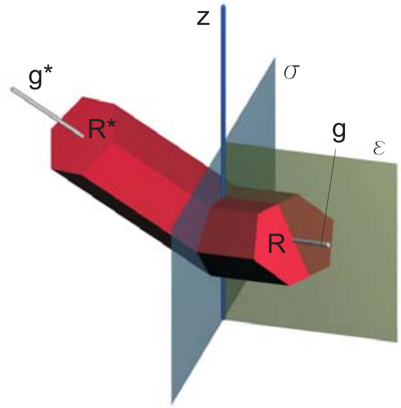

Figure 7: Fitting edges

g meets a second axis $\bar{z}$ of the octahedron. Therefore there exist two further miter cuts (fitting to the new neighbors). $\bar{z}$ is a line in ε, too. The symmetry of the rod R with respect to ε guarantees that the edges of R and its new neighbors are intersecting in points of the corresponding miter cut. To get ornaments with closed loops we chose as motif a rod with axis $g = [1, 2]$ where 1 and 2 are points of intersection from g and two axes of the octahedron. The planes of the miter cuts are planes through 1 and 2, respectively (see figure 9).

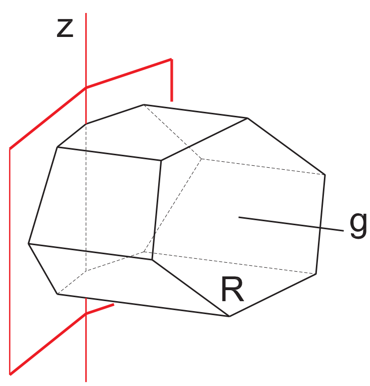

Figure 8: The rod with 2 miter cuts

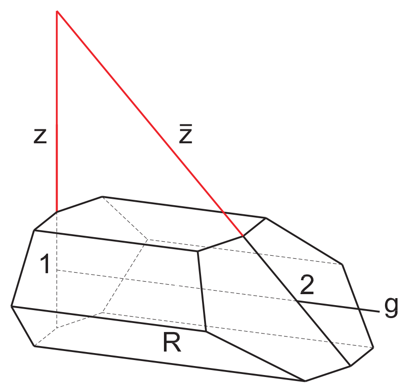

Figure 9: The rod with 4 miter cuts

Remarks: a) As a consequence of these considerations we will use **prismatic rods** R (axis g) **symmetric with respect to the plane** $\varepsilon := [g, M]$ henceforth.
b) If this reflection of the rod R is an automorphic reflection of the octahedron, we gain ornaments from the full octahedral group O_h. [5]

[5]Our next examples show, that there are examples for frameworks as ornaments in the group O which do not belong to the full group O_h even if our motif has a reflection symmetry.

In order to get a structured presentation for examples in case B we will give a complete list of different types of these planes of symmetry ε with respect to the octahedron.[6] We study the bundle of its axes. All plane sections of this bundle are projectively equivalent. We will use the section with a plane π orthogonal to the 2-fold axis c_1 (see figure 10). The points A_1, A_2 and A_3 denote the intersections of the 4-fold axes, B_1, B_2, B_3, B_4 and $C_1, C_2, C_3, C_4, C_5, C_6$ those with 3-fold and 2-fold axes, respectively. The points A_3, B_3, B_4 and C_6 are points at infinity.

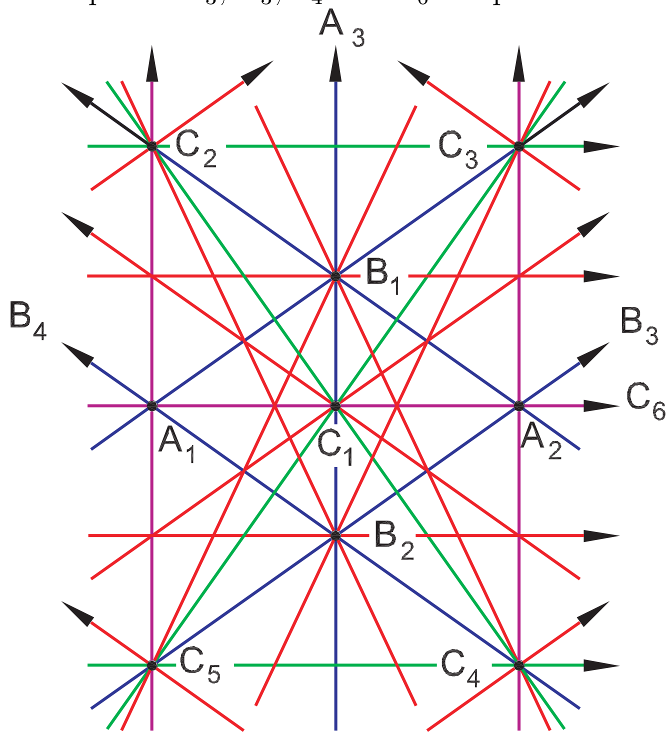

Figure 10: The configuration of the intersections of the axes from O in π

The different types of planes of symmetry ε have lines of intersection in π, which are displayed with different colors in figure 10:

B1. ε contains exactly two axes of the octahedron (red): There ε is spanned exactly by a 3-fold axis (e.g. b_2) and a 2-fold axis (e.g. c_2) which do not belong to one triangular face of the octahedron. ε is no plane of symmetry for the octahedron!

B2. ε contains exactly three axes of the octahedron (green): There ε contains three 2-fold axes (e.g. c_1, c_2, c_4). ε is no plane of symmetry for the octahedron!

B3 ε contains exactly four axes of the octahedron: Here we have two distinct possi-

[6]Note that ε has to contain at least two axes of the octahedron.

bilities:

B3a. ε contains two 4-fold and two 2-fold axes of the octahedron (blue, e.g. a_1, c_1, a_2, c_6). ε is a plane of symmetry containing 4 edges of the octahedron.

B3b. ε contains one 4-fold, two 3-fold and one 2-fold axes of the octahedron (magenta, e.g. a_3, b_1, c_1, b_2). ε is a plane of symmetry of the octahedron.

In the cases B3a and B3b the plane ε is a plane of symmetry of the octahedron. Hence, in these cases our procedure yields ornaments from the full octahedral group O_h.
Now we present some *examples*:
Case B1: g intersects a 3-fold axis b and a 2-fold axis c of the octahedron, which do not belong to one face of the octahedron (see figure 11)[7]. The two axes b and c are orthogonal lines. Therefore it is natural to take our line g on one side 12 of a quadrangle (green) with vertices on b and c, respectively. The corresponding ornament is shown in figure 12. It consists of 12 "half - quadrangles", which form closed loops of 6 rods. In order to get its structure they are displayed in different colors.

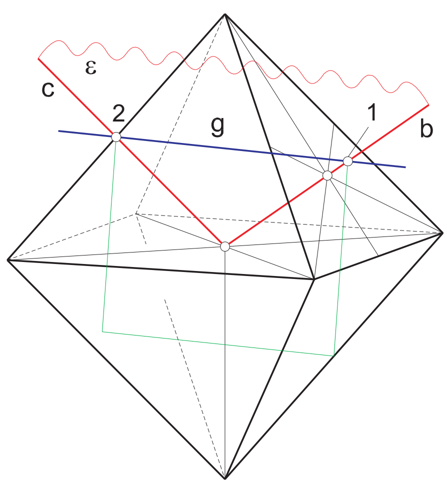

Figure 11: g meeting a 2- and a 3-fold axis

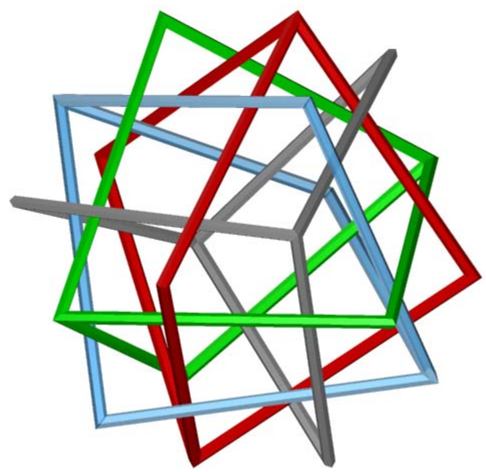

Figure 12: Ornament

Case B2: g intersects two 2-fold axes c_1 and c_2 of one face of the octahedron (figure 13). The plane $\varepsilon = [g, M] = [c_1, c_2]$ is orthogonal to a 3-fold axis b. As a first choice we take the axis g_1 like in figure 13. The ornament is displayed in figure 14. It consists of four triangular stars". A very special version is displayed in figure 15:

[7]If there is only one axis of the same type we omit the indices of the axis.

Here the axis g_2 of the rod is orthogonal to the 2-fold axis c_2. Then the two rods meeting at c_2 have the same axis - hence the *star* degenerates into a triangle. The ornament consists of 4 hinged triangles (each is built up from 6 copies of the motif). Again we gain an ornament known from art (see H.S.M. COXETER [4]).

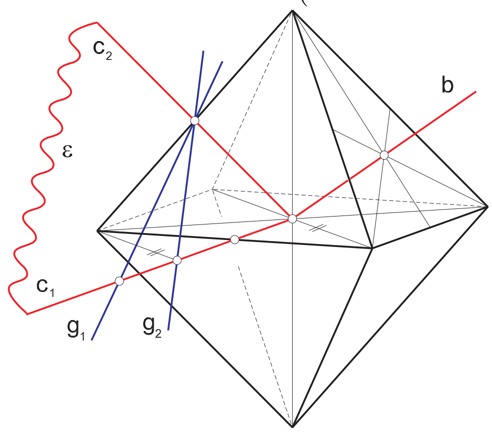

Figure 13: g_1 and g_2 intersecting two two-fold axes

In this case we have a third 2-fold axis c_3 in ε. The point of intersection of g and c_3 can be an outer or an inner point of 12. The latter case gives ornaments with self-intersections (no example displayed).

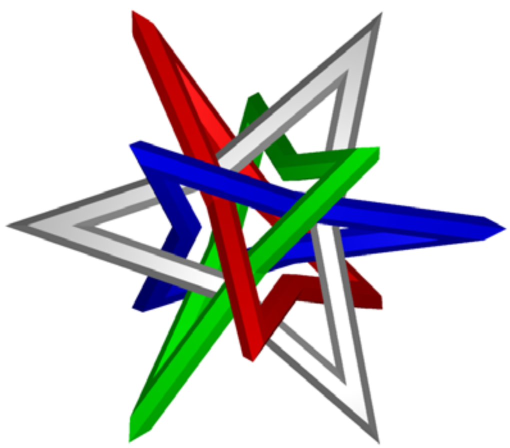

Figure 14: Triangular stars

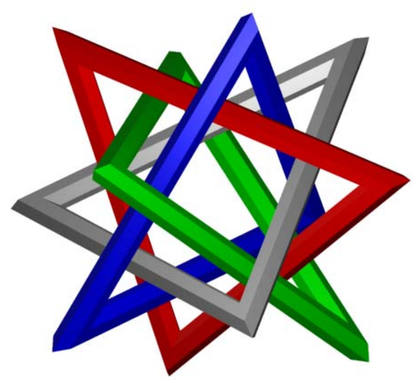

Figure 15: Hinged triangles

<u>Case B3a</u>: g is a line in the plane $\varepsilon = [a_1, c_1, a_2, c_6]$ (see figure 16). Our rod is restricted to two of these axes (points $1, 2$). If the initial rod follows an edge of the octahedron from the vertex to its midpoint we gain a framework ornament of the octahedron. If one of the other axes intersects $[1, 2]$ in an inner point we get self-intersecting ornaments. The following example (figure 17) is gained by using $1, 2$ as end-points on a 2-fold and a 4-fold axis. The second 2-fold and the second 4-fold axis

intersect $[1, 2]$ in inner points. Therefore we get 2-fold and 4-fold self-intersections of the ornament. Figure 18 displays another example of this case with 2-fold self-intersections: The end-points 1 and 2 are chosen on the 4-fold axes a_1 and a_2 (see figure 16).

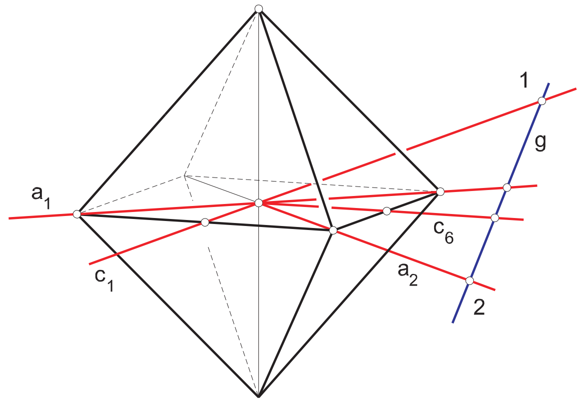

Figure 16: g for case B3a

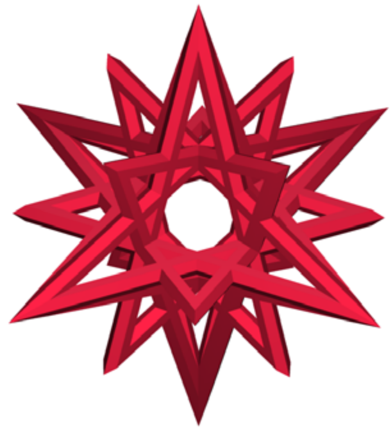

Figure 17: Example with 2-fold and 4-fold self-intersections

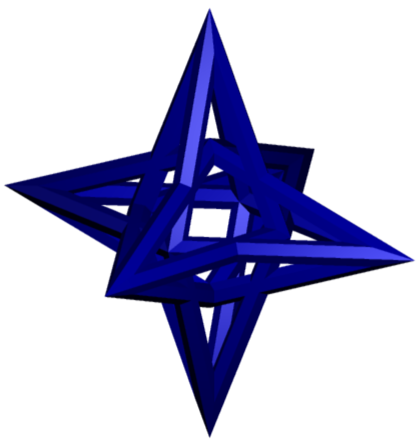

Figure 18: Example with 2-fold self-intersections

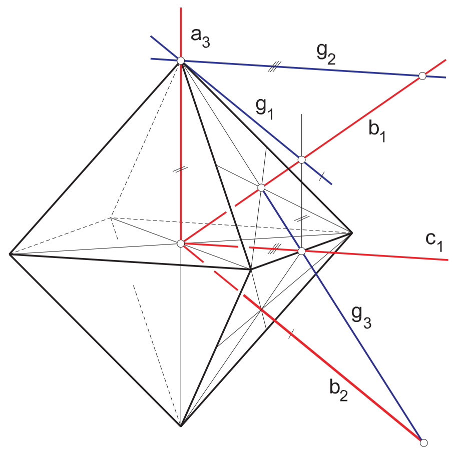

Figure 19: g_1, g_2 and g_3 for case B3b

Case B3b: g is a line in the plane $\varepsilon = [a_3, b_1, c_1, b_2]$ (see figure 19). We choose the points $1, 2$ on the 4-fold axis a_3 and the 3-fold axis b_1. The rod g_1 is parallel to the second 3-fold axis b_2 in ε, g_2 is parallel to the 2-fold axis c_1 (see figure 19). The corresponding ornaments are the Rhombic Dodecahedron (figure 20) and the famous Stella Octangula of J. KEPLER (figure 21). The latter is a compound of two congruent regular tetrahedra, which are displayed in different colors.

Figure 20: Rhombic Dodecahedron

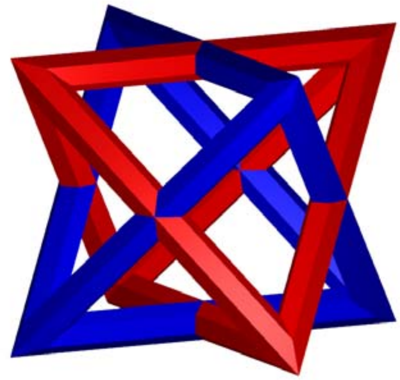

Figure 21: Stella Octangula

There are various cases with further intersections of the rods. Figure 22 displays an example with 2-fold self-intersections. The initial rod has the axis g_3 (see figure 19). It is orthogonal to the 3-fold axis b_1.

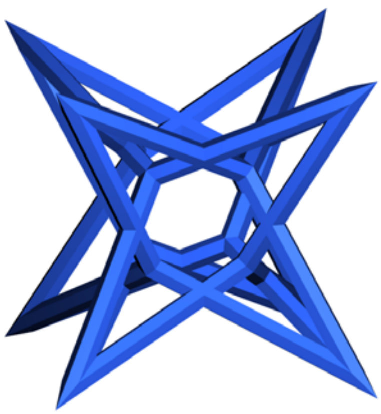

Figure 22: Example for case B3b

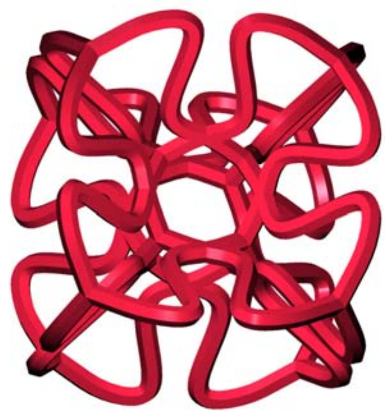

Figure 23: Curved rods

Of course, the prismatic rod can be replaced by any other motif. If we extrude the hexagon along a curved path we gain further ornaments. Figure 23 displays one example. In order to get fitting edges again we have to guarantee the existence of the corresponding and fitting miter cuts. This implies that the motif has to be symmetric with respect to the corresponding plane of symmetry ε (at least in the neighborhood of the end-points of the path). If the path is in the plane ε this condition holds automatically.

6. Conclusion. We presented a large scope of examples all generated as ornaments of the octahedral group. We started with prismatic rods as motives and used miter cuts in order to get fitting edges. In the last figure 23 we replaced the prisms by other geometric objects. Further generalizations of this step (with a skew path) will produce objects between geometry and art. Additionally, the presented geometric considerations and methods can be used for symmetric groups of other polyhedra, too.

The fascinating ornaments of the paper shall motivate the use of professional CAD-packages with geometric knowledge. This topic offers a good possibility to train spatial transformations and constructions. Without any structured use of blocks it is almost impossible to construct the ornaments.

We hope, the paper will stimulate the reader to its own experiments in this fascinating field.

References

[1] J. Böhm - E. Quaisser: Schönheit und Harmonie geometrischer Formen. Akademie Verlag, Berlin 1991.

[2] S. Bilinski: Über die Rhombenikosaeder. Glasnik Math. **15**, 251 - 262 (1960).

[3] H. S. M. Coxeter: Regular Polytopes. Dover, New York 1973.

[4] H. S. M. Coxeter: Symmetrical Combinations of Three or Four Holow Triangles. Math. Intelligencer **16/3**, 25 - 30 (1994).

[5] P. R. Cromwell: Polyhedra. Cambridge Univ. Press, Cambridge 1997.

[6] L. Fejes - Toth: Regular Figures. Int. Series of Monogr. in Pure and Appl. Math. 48, Pergamon, Oxford 1964.

[7] B. Grünbaum: Convex Polytopes. Pure and Applied Mathematics vol. XVI, Wiley, London - New York - Sidney, 1967.

[8] F. Hohenberg: Das abgestumpfte Dodekaeder des Archimedes und seine projektiven Eigenschaften. Sber. Österr. Akad. Wiss II, Bd. 192, 143 - 159 (1983).

[9] W. Jamnitzer: Perspectiva Corporum Regularium. Nberg 1568.

[10] S. Mick - O. Röschel: Ausgewählte Beispiele für Ornamente und Stabmodelle der Ikosaedergruppe. Informationsblätter Geometrie (IBDG), (in print).

[11] A. Wiltsche: Die Kunst ein 3D-CAD-Programm zu erlernen (part 1 and 2). Informationsblätter Geometrie (IBDG), **22/2**, 12 - 21 (2003) and **23/1**, 21 - 33 (2004).

Further references are listed in the textbooks [1], [3], [5], [6] and [7].