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GEOMETRY AND ARCHITECTURALLY SHAKY PLATFORMS

S. MICK and O. RÖSCHEL
Institute of Geometry, Technical University Graz
Kopernikusgasse 24, A-8010 Graz, Austria
email: roeschel@geometrie.tu-graz.ac.at

Abstract. A Stewart-Gough platform is architecturally shaky iff it remains shaky while subject to arbitrary displacements of the planes ε and φ, wherein the anchor points of the platform are situated. We show that a smaller subset of displacements of ε and φ exists which characterises the architectural shakiness of the platform. This subset consists of those displacements which transform ε and φ into two fixed orthogonal planes.

1. Introduction

The paper deals with six legged Stewart-Gough platforms (see [Gough 56], [Stewart 65], [Karger-Husty 96], [Karger 97]). The geometry of these manipulators is uniquely determined by six pairs of points X_i and Y_i ($i = 1, \dots 6$), each set on a plane ε and φ, resp. (see figure 1). Every pair of displacements

$$(1) \qquad \alpha^*: \begin{array}{c} \varphi \to \varphi^* \\ Y_i \to Y_i^* \end{array} \quad \text{and} \quad \beta^*: \begin{array}{c} \varepsilon \to \varepsilon^* \\ X_i \to X_i^* \end{array}$$

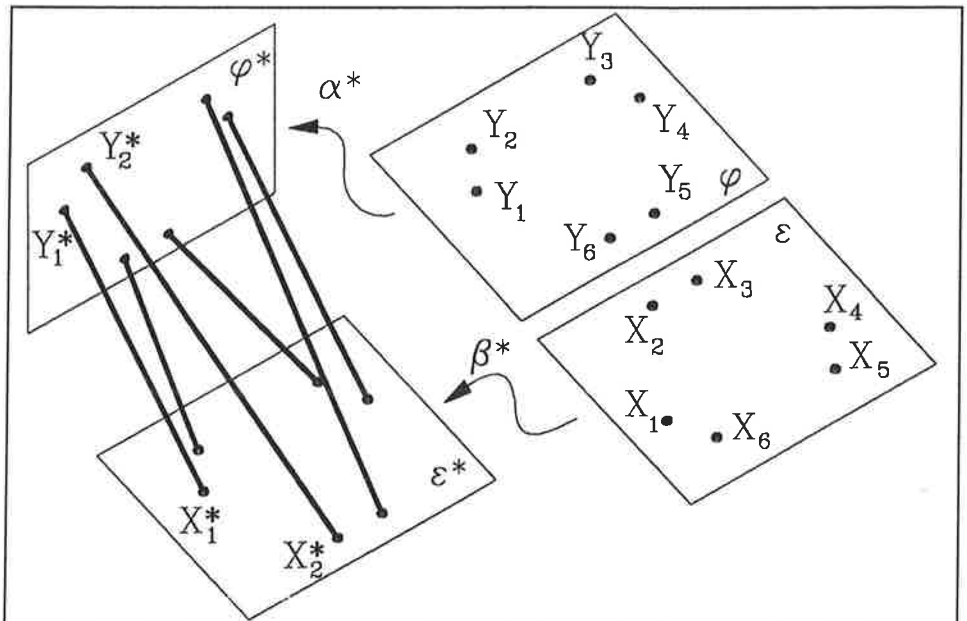

Figure 1: The geometry of a Stewart-Gough platform - pairs of anchor points $\{X_i, Y_i\}$

moves X_i and Y_i to new positions $X_i{}^*$ and $Y_i{}^*$, resp.[1] The six legs of the manipulator span the connections of the points $X_i{}^* = \beta^*(X_i)$ and $Y_i{}^* = \alpha^*(Y_i)$.

Definitions: The set of points $\{X_i, Y_i\}$ is called **shaky with respect to α^* and β^***, for short **α^*, β^*- shaky**, iff the lines $X_i{}^* Y_i{}^*$ belong to a linear line complex $L^*(\alpha^*,\beta^*)$ for a pair of displacements α^* and β^*. If the set of points $\{X_i, Y_i\}$ is α^*, β^*- shaky for every pair of displacements α^* and β^*, we call it **architecturally shaky**.

The considerations in [Roeschel-Mick 98] show that for all Stewart-Gough platforms there exist special displacements α^*, β^*, such that the configuration becomes shaky. Then the platform is in a singular position. At this position the stiffness matrix of the platform becomes singular. If we start with architecturally shaky points, then for all displacements α^*, β^* the configuration is shaky and all positions are singular. Such platforms represent the worst possible case: For all positions the stiffness matrix (the inverse Jacobian matrix) remains singular. If we fix the leg lengths in this case the manipulator admits self-motions at every position and is uncontrollable.

A. Karger [Karger 97] characterised architecturally shaky Stewart-Gough platforms. But thousands of terms occured in the expansion of a key determinant. As Karger's result is geometric, a more geometric path to it may be deemed interesting. For this, some classical, well known properties of linear line complexes and linear manifolds of correlations between two planes are useful. Such methods of geometric deduction seem applicable to many robotic problems (see [Dandurand 84], [Husty-Zsombor-Murray 94], [Merlet 89], [Mohammed - Duffy 85] and [Zsombor-Murray et al. 95]).

The paper is organized as follows: In chapter 2 we consider linear line complexes, null correlations and correlations between the two planes of the platform and outline the projective invariance of shakiness. In chapter 3 and 4 we develope a sufficient condition (theorem 4.2) of architecturally shaky Stewart-Gough platforms. In [Röschel-Mick 98] geometric results on linear manifolds of correlations between planes are used. It is shown that, barring a few exceptions, the set of points $\{X_i, Y_i\}$ is architecturally shaky, iff it consists of **fourfold conjugate points** with respect to a certain set of correlations.[2]

2A. Linear line complexes and null-correlations

In the Euclidean 3-space we use cartesian coordinates (x^*,y^*,z^*) and corresponding homogeneous cartesian coordinates $(x_0{}^*: x_1{}^*: x_2{}^*: x_3{}^*) = (1:x^*:y^*:z^*)$ to describe points X^*. For a line g^* containing two different points X^* $(x_0{}^*: x_1{}^*: x_2{}^*: x_3{}^*)$ and Y^* $(y_0{}^*: y_1{}^*: y_2{}^*: y_3{}^*)$ we define

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[1] It would be sufficient to move one of planes, but our procedure is a more symmetric one. The relative motion of one plane with respect to the other can be computed by combining displacement α^* with β^{*-1}.

[2] A correlation maps points into lines. Our set of correlations is spanned by four linearly independent correlations. If all four image lines of a point X contain a point Y, X and Y are called fourfold conjugate with respect to this set of correlations.

(2) $$p_{ij} := \det\begin{pmatrix} x_i{}^* & x_j{}^* \\ y_i{}^* & y_j{}^* \end{pmatrix} \quad (i, j = 0, ..., 3).$$

Six of these terms determine g^* uniquely. They are the well known **homogeneous Plücker line-coordinates**[3] of g^*, denoted by

(3) $$p_1 : p_2 : p_3 : p_4 : p_5 : p_6 := p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12} \neq 0{:}0{:}0{:}0{:}0{:}0\,.$$

They satisfy the *Plücker-condition*

(4) $$p_1\, p_4 + p_2\, p_5 + p_3\, p_6 = 0\,.$$

Special 3-dimensional line manifolds are the **linear line complexes** L. They contain all lines with Plücker-coordinates $(p_1{:}...{:}p_6)$ which satisfy a linear equation

(5) $$a_4\, p_1 + a_5\, p_2 + a_6\, p_3 + a_1\, p_4 + a_2\, p_5 + a_3\, p_6 = 0\,.$$

$(a_1{:}...{:}a_6) \neq (0{:}\ ...{:}\ 0)$ are the **homogeneous coordinates of the linear line complex** L.

Remarks: 1) According to (5) five linearly independent lines determine a linear line complex and its coordinates $(a_1{:}...{:}a_6)$.

2) Pitch $\sigma = 0$ (rotations) characterises **singular linear line complexes** wherein all lines intersect the axis of rotation. In this case the coordinates of L satisfy the Plücker-conditions (4).

3) If the coordinates $(a_1{:}...{:}a_6)$ of a given linear line complex L do not satisfy the Plücker-condition (4), L is a **regular linear line complex.** L is built up by the normals of the screw with axis a and pitch σ (see [Hunt 90, pp. 314-319]). The line a and the screw pitch are given by

(6) $$a\ \ (a_1 : a_2 : a_3 : a_4 - \sigma a_1 : a_5 - \sigma a_2 : a_6 - \sigma a_3),\ \ \sigma := \frac{(a_1\, a_4 + a_2\, a_5 + a_3\, a_6)}{a_1{}^2 + a_2{}^2 + a_3{}^2}.$$

All lines of the complex L (5) on a given point X^* $(x_0{}^* : x_1{}^* : x_2{}^* : x_3{}^*)$ are in a pencil situated in a plane $\pi(X^*)$. Its vertex is X^*. The map of the points X^* to these planes $\pi(X^*)$ is a **null-correlation** π in the 3-space. A null correlation is a linear mapping and has a representation

(7) $$(y_0{}^*, y_1{}^*, y_2{}^*, y_3{}^*)\begin{pmatrix} 0 & -a_4 & -a_5 & -a_6 \\ a_4 & 0 & -a_3 & a_2 \\ a_5 & a_3 & 0 & -a_1 \\ a_6 & -a_2 & a_1 & 0 \end{pmatrix}\begin{pmatrix} x_0{}^* \\ x_1{}^* \\ x_2{}^* \\ x_3{}^* \end{pmatrix} = 0\,.$$

(7) gives the equation of $\pi(X^*)$ for unknowns $(y_0{}^*, y_1{}^*, y_2{}^*, y_3{}^*)$. If the coordinates of two points X^* and Y^* satisfy equation (7) they are **conjugate points with respect to** π **and the line** $X^*\,Y^*$ **belongs to the linear line complex** L.

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[3] A good survey of line geometry is given in [Hoschek 71]. The following is a short outline of the methods we use for our considerations.

2B. Correlations between orthogonal planes connected with linear line complexes

We restrict this null-correlation π to the points X^* of the plane $\varepsilon^* \ldots\ x_3{}^* = 0$ and intersect the image planes $\pi(X^*)$ with the plane $\varphi^* \ldots\ x_1{}^* = 0$, perpendicular to ε^*. The linear mapping

(8) $\quad \gamma: X^* \in \varepsilon^* \to \gamma(X^*) := \pi(X^*) \cap \varphi^* \subset \varphi^*$

is a **correlation from the points of** ε^* **to the lines of** φ^*. The points X^* $(x_0{}^*: x_1{}^*: x_2{}^*: 0)$ are mapped to lines (see figure 2). According to (7) with $x_3{}^* = y_1{}^* = 0$ the coordinates of its points $Y^*\,(y_0{}^*:0:y_2{}^*:y_3{}^*)$ satisfy the linear equations

(9) $$0 = (y_0{}^*, y_2{}^*, y_3{}^*)\begin{pmatrix} 0 & -a_4 & -a_5 \\ a_5 & a_3 & 0 \\ a_6 & -a_2 & a_1 \end{pmatrix}\begin{pmatrix} x_0{}^* \\ x_1{}^* \\ x_2{}^* \end{pmatrix} := (y_0{}^*, y_2{}^*, y_3{}^*)\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_0{}^* \\ x_1{}^* \\ x_2{}^* \end{pmatrix}, \quad y_1{}^* = 0 .$$

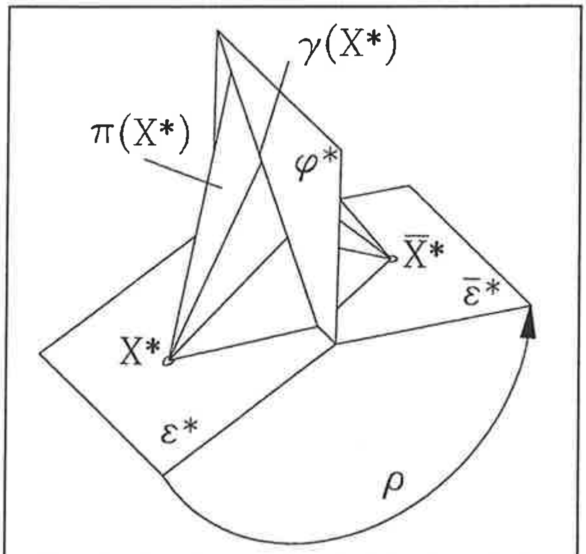

Figure 2: The null correlation, the induced correlation between planes and projective invariance

Remarks: 1) Every bilinearform with arbitrary (3,3)-matrix (a_{ij}) represents a correlation from ε^* to φ^*. As proportional matrices define the same correlation, they form an 8-parametric linear manifold K. The elements of the (3,3)-matrix are homogeneous coordinates of these correlations.

2) (9) is a special bilinearform. It represents the correlation γ (8). Only the correlations with (3,3)-matrices (9, left part) pertain to linear line complexes. As the shape of the matrix (9, left part) shows, we get a 5-parametric linear submanifold K_L of all correlations K. The three conditions are

(10) $\quad a_{00} = 0, \qquad a_{12} = 0, \qquad a_{02} + a_{10} = 0.$

3) All lines of complex L are connecting lines of conjugate points with respect to γ and intersecting lines of $\varepsilon^* \cap \varphi^*$.

2C. Maps in the manifold *L* of linear line complexes induced by rotations

Given are two orthogonal planes ε^* and φ^*. First, we rotate the plane ε^* with respect to the axis $r^* := \varepsilon^* \cap \varphi^* \ldots x_1^* = x_3^* = 0$ through angle $\delta \in [0, 2\pi)$ and get

(11) $\rho\colon \varepsilon^* \to \bar{\varepsilon}^*$ with $X^*(x_0^*, x_1^*, x_2^*, 0) \to \bar{X}^*(x_0^*, x_1^* \cos\delta, x_2^*, -x_1^* \sin\delta)$.

Second, we define an affinity of the space given by $\kappa\colon E_3 \to E_3$ with

$$(12)\qquad X^*\ldots\begin{pmatrix} x_0^* \\ x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} \to \bar{X}^*\ldots\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\delta & 0 & 1 \end{pmatrix}\begin{pmatrix} x_0^* \\ x_1^* \\ x_2^* \\ x_3^* \end{pmatrix}.$$

The points of ε^* are mapped by affinity (12) in the same way as they are mapped by rotation ρ (11). The elements of φ^* are fixed (see figure 2). The connecting lines of points $\bar{X}^*\ldots(x_0^*, x_1^* \cos\delta, x_2^*, -x_1^* \sin\delta) \in \bar{\varepsilon}^*$ to points $Y^*\ldots(y_0^* : 0 : y_2^* : y_3^*) \in \gamma(X^*) \subset \varphi^*$, conjugate to X^* with respect to correlation γ (8) have Plücker-coordinates

$$(13)\qquad \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} -x_1^* y_0^* \cos\delta \\ x_0^* y_2^* - x_2^* y_0^* \\ x_0^* y_3^* + x_1^* y_0^* \sin\delta \end{pmatrix}, \begin{pmatrix} p_4 \\ p_5 \\ p_6 \end{pmatrix} = \begin{pmatrix} x_2^* y_3^* + x_1^* y_2^* \sin\delta \\ -x_1^* y_3^* \cos\delta \\ x_1^* y_2^* \cos\delta \end{pmatrix}.$$

It is easy to verify, that these lines belong to a new linear line complex $\bar{L}$ with coordinates

$$(14)\qquad (\bar{a}_1 : \bar{a}_2 : \bar{a}_3 : \bar{a}_4 : \bar{a}_5 : \bar{a}_6) = (a_1 \cos\delta : a_2 : a_3 - a_1 \sin\delta : a_4 + a_6 \sin\delta : a_5 \cos\delta : a_6 \cos\delta).$$

We denote the set of linear complexes by ***L*** and state our result.

Lemma 2.1: We consider the lines of each linear line complex L as connections of the piercing points X^* and Y^* in the two intersecting planes ε^* and φ^*. If we rotate X^* according to (11) and connect $\bar{X}^*$ with the fixed points Y^* we get a new line complex $\bar{L}$. **Thus, a rotation** (11) **with axis** $r^* := \varepsilon^* \cap \varphi^*$ **gives a map in the set *L*** described by (14).[4]

Remarks: 1) This is a special case of the well known fact that nonsingular projective collineations transform one linear line complex into another one. Nonlinear line complexes are transformed into nonlinear ones.

2) The lines of L intersecting $\varepsilon^* \cap \varphi^*$ are mapped into lines in $\bar{L}$, which intersect $\varepsilon^* \cap \varphi^*$.

3) Only in the cases $\delta = \pi/2, 3\pi/2$ is the affinity (12) a singular one. In these cases our connecting lines are mapped into all lines on the plane $\varphi^* = \bar{\varepsilon}^*$. The corresponding linear line complex becomes singular.

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[4] Including singular linear line complex with axis $\varepsilon^* \cap \varphi^*$.

2D. Correlations between parallel planes connected with linear line complexes

The considerations of 2B can be similarly carried out for parallel planes. We restrict the null-correlation π (7) to the points X^* of the plane ε^* ... $x_3{}^* = 0$ and intersect the image planes $\pi(X^*)$ with the plane φ^* ... $x_3{}^* = x_0{}^*$ parallel to ε^*. The mapping $\gamma: X^* \in \varepsilon^* \to \gamma(X^*) := \pi(X^*) \cap \varphi^* \subset \varphi^*$ is a **correlation** from the points of ε^* to the lines of φ^*. The points X^* $(x_0{}^*: x_1{}^*: x_2{}^*: 0) \in \varepsilon^*$ and Y^* $(y_0{}^*: y_1{}^*: y_2{}^*: y_0{}^*) \in \varphi^*$ are conjugate with respect to γ, iff their coordinates satisfy condition

$$(15) \qquad 0 = (y_0{}^*, y_1{}^*, y_2{}^*) \begin{pmatrix} a_6 & -a_2 - a_4 & a_1 - a_5 \\ a_4 & 0 & -a_3 \\ a_5 & a_3 & 0 \end{pmatrix} \begin{pmatrix} x_0{}^* \\ x_1{}^* \\ x_2{}^* \end{pmatrix}.$$

The remarks of 2B hold analogously. Now the procedure of 2C for rotations can be done for parallel translations of ε^*. We sum up.

Lemma 2.2: We consider the lines of each linear line complex L as connections of points X^* and Y^* of the two different parallel planes ε^* and φ^*. An arbitrary translation of ε^* moves its points X^* to $\overline{X}^* \in \overline{\varepsilon}^*$. If we connect $\overline{X}^*$ with the fixed points Y^* we get the lines of a new linear line complex. **Thus, these translations induce a map in the set of linear line complexes L.**

3. Conditions for architectural shakiness

Now we transfer the results on linear line complexes from chapter 2 into properties of shakiness. We use two homogenous cartesian coordinate systems[5] to describe our anchor points $\{X_i, Y_i\}$ in the planes ε and φ of the platform.

$$(16) \qquad X_i \ldots (x_{0,i} : x_{1,i} : x_{2,i}) \text{ and } Y_i \ldots (y_{0,i} : y_{1,i} : y_{2,i}) \qquad (i = 1,\ldots, 6).$$

Given are a fixed set of architecturally shaky points $\{X_i, Y_i\}$ $(i = 1,\ldots,6)$, and a fixed pair of orthogonal planes $\varepsilon_1{}^*$, $\varphi_1{}^*$, and a fixed pair of parallel planes $\varepsilon_2{}^* \neq \varphi_2{}^*$ in 3-space. According to definition of architecturally shaky points the set $\{X_i, Y_i\}$ is shaky with respect to every pair of displacements α^*, β^*. In particlar $\{X_i, Y_i\}$ $(i = 1,\ldots,6)$ is $\alpha_j{}^*$, $\beta_j{}^*$ - shaky with respect to those pairs of displacements

$$(17) \qquad \alpha_j{}^*: \begin{matrix} \varphi \to \varphi_j{}^* \\ Y_i \to Y_i{}^* \end{matrix} \text{ and } \beta_j{}^*: \begin{matrix} \varepsilon \to \varepsilon_j{}^* \\ X_i \to X_i{}^* \end{matrix}, \text{ with } j = 1,2$$

which move ε and φ to the special fixed planes $\varepsilon_1{}^*$, $\varphi_1{}^*$ and $\varepsilon_2{}^*$, $\varphi_2{}^*$ resp. This property of architecturally shaky points is sufficient, too. We prove

[5] Asterisks (*) on the coordinates of points of 3-space differentiate these from coordinates on our planes.

Theorem 3.1: Given are two orthogonal planes $\varepsilon_1 *$, $\varphi_1 *$ and two parallel planes $\varepsilon_2 * \neq \varphi_2 *$, fixed in the space. If $\{X_i, Y_i\}$ $i = 1,...,6$ is a set of points, $\alpha_j *, \beta_j *$ - shaky ($j = 1,2$) with respect to all pairs of displacements (17), then it is already architecturally shaky.

Remark: This lemma makes clear, that we just have to test shakiness with respect to displacements for two fixed orthogonal planes and for two parallel planes. Later we will see, that the test for the parallel planes $\varepsilon_2 * \neq \varphi_2 *$ can be omitted.

To prove theorem 3.1 we use theorems 3.2 and 3.3. Lemma 2.1 and definition of shakiness result in:

Theorem 3.2: Given are two intersecting planes $\varepsilon_1 *$ and $\varphi_1 *$ and displacements $\alpha_1 *, \beta_1 *$ from the set (17), then $\alpha_1 *, \beta_1 *$ - shakiness of the system $\{X_i, Y_i\}$ is a property independent of the angle of intersection of $\varepsilon_1 *$ and $\varphi_1 *$.

From lemma 2.2 and the definition of shakiness follows:

Theorem 3.3: Given are two parallel planes $\varepsilon_2 *$ and $\varphi_2 *$ and displacements $\alpha_2 *, \beta_2 *$ from the set (17), then $\alpha_2 *, \beta_2 *$ - shakiness of the system $\{X_i, Y_i\}$ is a property independent of parallel translations of $\varepsilon_2 *$ with respect to $\varphi_2 *$.

This finishes the proof of theorem 3.1. Thus, these conditions are sufficient too. □

Remark: As we know (see [Liebmann 20], [Wegner 84] and [Wunderlich 83]) shakiness is projectivly invariant. Theorems 3.2 and 3.3 are conclusions of this fact for the situation of Stewart-Gough platfroms.

4. Consequences of theorem 3.1

Following theorem 3.1 we discuss two different cases:

Case a: Given are two orthogonal planes $\varepsilon_1 *$, $\varphi_1 *$, fixed in space. The displacements $\alpha_1 *$ and $\beta_1 *$ have representationes

$$\alpha_1 *: \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \to \begin{pmatrix} y_0 * \\ 0 \\ y_2 * \\ y_3 * \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ A & \cos\alpha & -\sin\alpha \\ B & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

(18) and

$$\beta_1 *: \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_0 * \\ x_1 * \\ x_2 * \\ 0 \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 \\ a & \cos\beta & -\sin\beta \\ b & \sin\beta & \cos\beta \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

The points $X_i * := \beta_1 * (X_i)$ and $Y_i * := \alpha_1 * (Y_i)$ have connecting lines. Because of (18) and (9) the connecting lines $X_i * Y_i *$ belong to a linear line complex, iff

$$(y_0, y_1, y_2)\begin{pmatrix} 1 & 0 & 0 \\ A & \cos\alpha & -\sin\alpha \\ B & \sin\alpha & \cos\alpha \end{pmatrix}^t \begin{pmatrix} 0 & -a_4 & -a_5 \\ a_5 & a_3 & 0 \\ a_6 & -a_2 & a_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & \cos\beta & -\sin\beta \\ b & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} :=$$

(19)

$$=(y_0, y_1, y_2)\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0$$

holds for the coordinates of $\{ X_i, Y_i \}$ $(i = 1, ...,6)$. There we have

(20)

$$\begin{aligned}
a_{00} &= a_1 b B - a_2 a B + a_3 a A - a_4 a + a_5 (A-b) + a_6 B \\
a_{10} &= a_1 b \sin\alpha - a_2 a \sin\alpha + a_3 a \cos\alpha + a_5 \cos\alpha + a_6 \sin\alpha \\
a_{20} &= a_1 b \cos\alpha - a_2 a \cos\alpha - a_3 a \sin\alpha - a_5 \sin\alpha + a_6 \cos\alpha \\
a_{01} &= a_1 B \sin\beta - a_2 B \cos\beta + a_3 A \cos\beta - a_4 \cos\beta - a_5 \sin\beta \\
a_{11} &= a_1 \sin\alpha \sin\beta - a_2 \sin\alpha \cos\beta + a_3 \cos\alpha \cos\beta \\
a_{21} &= a_1 \cos\alpha \sin\beta - a_2 \cos\alpha \cos\beta - a_3 \sin\alpha \cos\beta \\
a_{02} &= a_1 B \cos\beta + a_2 B \sin\beta - a_3 A \sin\beta + a_4 \sin\beta - a_5 \cos\beta \\
a_{12} &= a_1 \sin\alpha \cos\beta + a_2 \sin\alpha \sin\beta - a_3 \cos\alpha \sin\beta \\
a_{22} &= a_1 \cos\alpha \cos\beta + a_2 \cos\alpha \sin\beta + a_3 \sin\alpha \sin\beta
\end{aligned}$$

Remarks: 1) Translations of $\varepsilon_1{}^*$ with respect to $\varphi_1{}^*$ in the direction of $\varepsilon_1{}^* \cap \varphi_1{}^*$ can be fixed either by the translational component A of $\alpha_1{}^*$ or b of $\beta_1{}^*$. Thus, by (18) a five-parametric set of displacements is given.

2) The a_{ij} are homogenous linear functions of the coordinates a_i of the linear line complex. Thus, the set of linear line complexes determines a five-parametric manifold of correlations (see 2B). In the 8-parametric linear manifold of correlations described by arbitrary (3,3)-matrices a_{ij} formula (20) represents a parametrisation of those, which belong to linear line complexes. This linear manifold is described by three equations, which are equivalent to the conditions (10) in 2B. With the abbreviations

(21) $\quad f := a \cos\beta - (A-b) \sin\beta, \qquad g := a \sin\beta + (A-b) \cos\beta$

they become

(22)

$$\begin{aligned}
&a_{11} \cos\alpha \sin\beta + a_{12} \cos\alpha \cos\beta - a_{21} \sin\alpha \sin\beta - a_{22} \sin\alpha \cos\beta = 0 \\
&a_{01} \sin\beta + a_{02} \cos\beta + a_{10} \cos\alpha - a_{20} \sin\alpha = \\
&= a_{11} (B \sin\alpha \sin\beta + a \cos\alpha \cos\beta) + a_{12} (B \sin\alpha \cos\beta - a \cos\alpha \sin\beta)) + \\
&\quad + a_{21} (B \cos\alpha \sin\beta - a \sin\alpha \cos\beta) + a_{22} (B \cos\alpha \cos\beta + a \sin\alpha \sin\beta) \\
&f(a_{01} - B a_{11} \sin\alpha - B a_{21} \cos\alpha) = \\
&= g(a_{02} - B a_{12} \sin\alpha - B a_{22} \cos\alpha) + a_{00} - B a_{10} \sin\alpha - B a_{20} \cos\alpha
\end{aligned}$$

The three conditions (22) can be written in the alternate form

(23) $U\left(a_{00},a_{01},a_{02},a_{10},a_{11},a_{12},a_{20},a_{21},a_{22}\right)^t=(0,0,0)^t$ with the (3,9) - matrix

$$U=\begin{pmatrix} 0 & 0 & 0 & 0 & \cos\alpha\,\sin\beta & \cos\alpha\cos\beta & 0 & -\sin\alpha\sin\beta & -\sin\alpha\cos\beta \\ 0 & -\sin\beta & -\cos\beta & -\cos\alpha & B\sin\alpha\,\sin\beta + a\,\cos\alpha\,\cos\beta & B\sin\alpha\cos\beta - a\cos\alpha\sin\beta & \sin\alpha & B\cos\alpha\sin\beta - a\sin\alpha\cos\beta & B\cos\alpha\cos\beta + a\sin\alpha\sin\beta \\ -1 & f & -g & B\sin\alpha & -Bf\,\sin\alpha & Bg\sin\alpha & B\cos\alpha & -Bf\cos\alpha & Bg\cos\alpha \end{pmatrix}$$

All elements of this (3,9) - matrix depend on the motion parameters. As we look for a solution for arbitrary α,β,a,b,A,B, we have rank $U=3$.

Case b: Given are two parallel planes $\varepsilon_2{}^*$ ... $x_3{}^*=0$ and $\varphi_2{}^*$ ... $y_3{}^*=y_0{}^*$, fixed in space. From theorem 3.3, without loss of generality, displacements $\alpha_2{}^*:\varepsilon\to\varepsilon_2{}^*$ and $\beta_2{}^*:\varphi\to\varphi_2{}^*$ can be taken without any translational component for our investigations. Furthermore, the relative motion of φ with respect to ε has only one essential rotational component. Thus we put

(24) $$\alpha_2{}^*:\begin{pmatrix}y_0\\y_1\\y_2\end{pmatrix}\to\begin{pmatrix}y_0{}^*\\0\\y_2{}^*\\y_3{}^*\end{pmatrix}:=\begin{pmatrix}1&0&0\\0&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}y_0\\y_1\\y_2\end{pmatrix},\quad \beta_2{}^*:\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}\to\begin{pmatrix}x_0{}^*\\x_1{}^*\\x_2{}^*\\0\end{pmatrix}:=\begin{pmatrix}1&0&0\\0&\sin\beta&\cos\beta\\0&-\cos\beta&\sin\beta\\0&0&0\end{pmatrix}\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}$$

The points $X_i{}^*=\beta_2{}^*(X_i)$ and $Y_i{}^*=Y_i$ have connecting lines $X_i{}^*,Y_i{}^*$. Because of (24) and (15) they belong to a linear line complex, iff

(25) $$(y_0,y_1,y_2)\begin{pmatrix}a_6 & -a_2-a_4 & a_1-a_5\\ a_4 & 0 & -a_3\\ a_5 & a_3 & 0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\sin\beta&\cos\beta\\0&-\cos\beta&\sin\beta\end{pmatrix}\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}:=$$
$$=(y_0,y_1,y_2)\begin{pmatrix}a_{00}&a_{01}&a_{02}\\a_{10}&a_{11}&a_{12}\\a_{20}&a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}=0$$

holds for the coordinates of $\{X_i,\ Y_i\}$ $(i=1,\ldots,6)$. As before, we get a 5-parametric manifold of correlations, which belong to linear line complexes. This manifold is described by three equations. They are

(26) $$\begin{aligned} &a_{11}\sin\beta+a_{12}\cos\beta=0\\ &a_{21}\sin\beta+a_{22}\cos\beta-a_{11}\cos\beta+a_{12}\sin\beta=0\\ &a_{21}\cos\beta-a_{22}\sin\beta=0 \end{aligned}$$

Additionally, conditions (26) are exactly the limits of the conditions (22), if we put $\alpha := 0$, $B := t$, $a := -t$ for lim $t \to \infty$. Thus, conditions (26) are algebraically included in those of (22). Theorem 3.1 can be replaced by

Theorem 4.1: Given are two fixed orthogonal planes $\varepsilon_1{}^*$, $\varphi_1{}^*$ in the space and displacements $\alpha_1{}^*: \varphi \to \varphi_1{}^*$, $\beta_1{}^*: \varepsilon \to \varepsilon_1{}^*$. If $\{X_i, Y_i\}$ $i = 1,\ldots,6$ is a set of points, $\alpha_1{}^*, \beta_1{}^*$- shaky with respect to all pairs $\alpha_1{}^*, \beta_1{}^*$, then the set is architecturally shaky.

Conclusion. The decision whether a Stewart-Gough platfrom is architecturally shaky or not, is substantially simplified by theorem 4.1. The key point of our proof is the projective invariance of architectural shakiness. This result is used in [Röschel-Mick 98] to give a geometric characterisation of architecturally shaky Stewart-Gough platforms.

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