THE TEN^TH INTERNATIONAL CONFERENCE ON GEOMETRY AND GRAPHICS

UKRAINE, KYIV, 2002, JULY 28 – AUGUST 2

Published in: Proc. of the 10th International Conference on Geometry and Graphics (ISGG) (V. Mykhailenko et al. edts.), Vol. 1, 89 – 93 (2002).

INDUCED VELOCITY VECTOR FIELDS

Sybille MICK and Otto ROESCHEL
University of Technology
Graz, AUSTRIA

PROCEEDINGS

VOLUME 1

SPONSORED BY:

ISGG
International Society for Geometry and Graphics

SINCE 1967
JSGS
Japan Society for Graphic Science

Published in: Proc. of the 10th International Conference on Geometry and Graphics (ISGG) (V. Mykhailenko et al. edts.), Vol. 1, 89 – 93 (2002).

INDUCED VELOCITY VECTOR FIELDS

Sybille MICK and Otto ROESCHEL
University of Technology
Graz, AUSTRIA

ABSTRACT

We start with a given velocity-distribution of a one-parametric Euclidean motion and restrict ourselves to the points of a surface. According to the concepts of differential geometry these velocity vectors can be split into a normal and a tangent component with respect to this surface. The tangent components build up a velocity vector field on the surface. If the surface is a sphere, there are appropriate results by O. Roeschel (1995). In this paper velocity-distributions of projective motions and some different types of quadrics are being studied. The result for the sphere leads to a characterisation of infinitesimal Euclidean motions. Additionally, the analogous considerations for cones and cylinders of revolution are presented.

1. INTRODUCTION

In 1995 O. Roeschel restricted the velocity-distribution of a one-parametric Euclidean motion to the points of a sphere κ: Following the concepts of differential geometry these velocity vectors can be split into a normal and a tangent component with respect to the sphere. The tangent components themselves build up a velocity vector field tangent to the sphere, which can be interpreted as a velocity vector field induced by an infinitesimal Non-Euclidean motion with κ as absolute quadric. These considerations can be generalized in different ways: We can firstly replace the starting velocity-distribution of a Euclidean motion by the velocity points of a projective motion. The induced velocities on the sphere κ then, in general, will build up a tangent point distribution of κ, which does not belong to an infinitesimal projective transformation keeping the sphere fixed.

Secondly, the sphere κ can be replaced by another type of quadric. Then, in general, an analogous procedure will result in a velocity vector field, which does not fit an infinitesimal Non-Euclidean motion. We offer some examples of this nonlinear behavior in the cases of cylinders and cones of revolution.

2. BASIC NOTATIONS

In the threedimensional Euclidean space $E_3 \subset P_3$ we use homogeneous Cartesian coordinates $(x_0 : x_1 : x_2 : x_3)^t$ to describe points. All vectors proportional to the vector $\mathbf{X} = (x_0, x_1, x_2, x_3)^t$ belong to the same point. Sometimes it is convenient to put $\mathbf{X} = (x_0, \mathbf{x})^t$. The points in the plane at infinity are given by $(0 : x_1 : x_2 : x_3)^t$. The unit sphere κ with center $\mathbf{M} = (1, 0, 0, 0)^t$ has the equation

$$x_1^2 + x_2^2 + x_3^2 = x_0^2. \tag{1}$$

A one-parametric projective motion of the projective space P_3 is given by

$$\mathbf{Y} = A_4(t)\mathbf{X} = \begin{pmatrix} a_0(t) & \hat{\mathbf{a}}^t(t) \\ \mathbf{a}(t) & A_3(t) \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix}, \tag{2}$$

where $A_4(t)$ denotes a 4×4- matrix of the class C^1.

At any moment this one-parametric projective motion determines a mapping which assigns the tangent point $\dot{\mathbf{Y}}$ to the position $\mathbf{Y}$. It is called *infinitesimal transformation* of the motion (see J. Toelke 1975) and is given by

$$\dot{\mathbf{Y}} = B_4\mathbf{Y} = \begin{pmatrix} b_0 & \hat{\mathbf{b}}^t \\ \mathbf{b} & B_3 \end{pmatrix} \begin{pmatrix} y_0 \\ \mathbf{y} \end{pmatrix}. \tag{3}$$

The corresponding one-parametric subgroups are gained by integration of (3) using constant matrices B_4

$$A_4(t) = exp(tB_4). \tag{4}$$

Those keeping κ invariant are characterized by the following: For all points $\mathbf{Y} \in \kappa$ the tangent point $\dot{\mathbf{Y}}$ lies in the plane tangent to κ at $\mathbf{Y}$. Computation shows that the corresponding constant matrices can be displayed as

$$C_4 = \begin{pmatrix} \gamma & \mathbf{c}^t \\ \mathbf{c} & \gamma I_3 + S_3 \end{pmatrix}, \tag{5}$$

where I_3 and S_3 denote the 3×3 unit matrix and a 3×3 skew-symmetric matrix, respectively.

The corresponding subgroups can be interpreted as one-parametric subgroups of the 3-dimensional hyperbolic CAYLEY-KLEIN space of index 1 with κ as absolute quadric (see O. Giering, 1982).

We also need the infinitesimal transformations of a Euclidean motion. Their matrices are

$$E_4 = \begin{pmatrix} 0 & \mathbf{o}^t \\ \mathbf{e} & E_3 \end{pmatrix} \tag{6}$$

with an arbitrary skew-symmetric 3×3 matrix E_3.

3. THE INDUCED TANGENT FIELD

We watch the sphere κ (1) and an arbitrary projective infinitesimal transformation Φ (3). Φ can be restricted to the points P of κ. The path tangent of P is orthogonally projected into a tangent line of κ at P. This 'tangential component' generates a tangent field Φ^* on κ. In general, Φ^* is not linear, i.e. Φ^* will not belong to a one-parametric hyperbolic subgroup with respect to κ. The aim of the following considerations is to characterize those projective infinitesimal transformations Φ with Φ^* match a one-parametric hyperbolic subgroup with respect to κ.

Figure 1 shows some points of the sphere κ, their path tangents assigned by Φ (belonging to a screw with the indicated axis) and the orthogonally projected tangents.

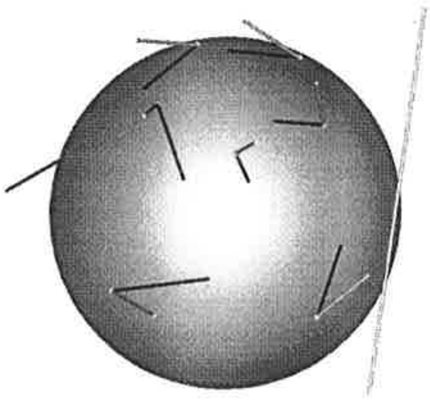

Figure 1: Points of the sphere κ, their path tangents assigned by Φ and the induced tangents.

Examples emerge from the results given by O. Roeschel (1995): Any Euclidean infinitesimal transformation Φ with a matrix E_4 (6) induces a hyperbolic infinitesimal transformation of κ. Figure 2 gives an example: The starting Euclidean infinitesimal transformation is the one of figure 1. Some integral curves of the induced hyperbolic infinitesimal transformation of κ are sketched.

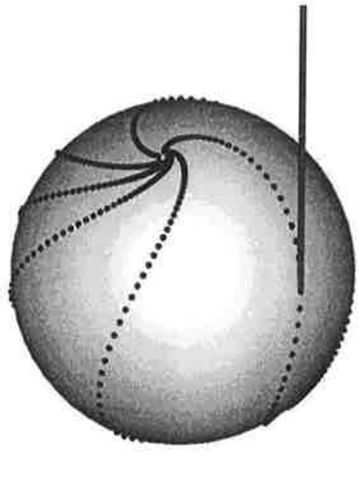

Figure 2: Some integral curves of the hyperbolic transformation induced by a Euclidean screw motion

Infinitesimal perspective collineations with centre M are given by transformation matrices

$$(7) \qquad Z_4 = \begin{pmatrix} \alpha & \beta \quad \gamma & \delta \\ \mathbf{o} & O_3 & \end{pmatrix} + \varepsilon\, I_4$$

with arbitrary real constants $\alpha, \cdots \varepsilon$, the 3×3 zero-matrix O_3 and the 4×4 identity matrix I_4. The plane of the fixed points of this transformation has the equation

$$(8) \qquad \alpha y_0 + \beta y_1 + \gamma y_2 + \delta y_3 \;=\; 0.$$

In a trivial way the infinitesimal transformations Z_4 induce the identity on κ. Therefore, all arbitrary one-dimensional projective subgoups $B_4 \in [E_4, Z_4]$ (different from Z_4) induce the same hyperbolic infinitesimal transformation on κ as E_4. This manifold of matrices are discribed by

$$(9)\ [E_4, Z_4] = \{\rho \begin{pmatrix} 0 & \mathbf{o}^t \\ \mathbf{e} & E_3 \end{pmatrix} \sigma \begin{pmatrix} \alpha & \beta \quad \gamma & \delta \\ \mathbf{o} & O_3 & \end{pmatrix} + \\ + \tau\, I_4, \rho, \sigma, \tau \in \Re\}$$

and is being built up by the infinitesimal transformations of perspective collineations with centre M and of Euclidean motions.

4. THE CHARCTERISATION OF INFINITESIMAL PROJECTIVE TRANSFORMATIONS WHICH FIT HYPERBOLIC INFINITESIMAL TRANSFORMATIONS OF κ

Now we start with an arbitrary projective infinitesimal transformation Φ with given data B_4 and look for an infinitesimal transformation keeping κ invariant. We want to determine data C_4 (5) such that this transformation for all points of κ fits the induced tangent field Φ^*. This yields the condition

$$(10) \qquad det\ (B_4\mathbf{Y},\ C_4\mathbf{Y},\ \mathbf{Y},\ \mathbf{M}) = 0$$

for all points $\mathbf{Y} \in \kappa$. Applying (3) and (5) we modify (10) to

$$(11) \qquad \begin{aligned} & det\ (\mathbf{b}y_0 + B_3\mathbf{y},\ \mathbf{c}y_0 + S_3\mathbf{y},\ \mathbf{y}) = \\ & = y_0^2\ det\ (\mathbf{b},\ \mathbf{c},\ \mathbf{y}) + \\ & + y_0\ [det\ (\mathbf{b},\ S_3\mathbf{y},\ \mathbf{y}) + det\ (B_3\mathbf{y},\ \mathbf{c},\ \mathbf{y})] + \\ & + det\ (B_3\mathbf{y},\ S_3\mathbf{y},\ \mathbf{y}) = 0 \end{aligned}$$

for all points $\mathbf{Y} \in \kappa$. This equation is a homogenous polynomial of degree 3 in the variables y_0, y_1, y_2, y_3. Now we have to consider 2 cases:

Case A: (11) holds identically for all points $\mathbf{Y} \in P_3$.

Case B: (11) holds for all points $\mathbf{Y} \in \kappa$.

Case A: Equation (11), viewed as a polynomial condition for all $\mathbf{y}$, holds identically. Therfore we have

$$(12) \qquad \begin{aligned} & det\ (\mathbf{b},\ \mathbf{c},\ \mathbf{y}) = 0 \\ & det\ (\mathbf{b},\ S_3\mathbf{y},\ \mathbf{y}) + det\ (B_3\mathbf{y},\ \mathbf{c},\ \mathbf{y}) = 0 \\ & det\ (B_3\mathbf{y}\ S_3\mathbf{y},\ \mathbf{y}) = 0 \end{aligned}$$

for all $\mathbf{y}$. A careful discussion of all different subcases gives the result:

All matrices B_4 satisfying the condition (12) belong to the manifold (9).

Case B: If the equation (11) holds for all $\mathbf{y} \in \kappa$, but not identically, it is bound to be the product of the equation of the sphere (1) with some linear polynomial:

$$(13) \qquad \begin{aligned} & y_0^2\ det\ (\mathbf{b},\ \mathbf{c},\ \mathbf{y}) + \\ & + y_0\ [det\ (\mathbf{b},\ S_3\mathbf{y},\ \mathbf{y}) + det\ (B_3\mathbf{y},\ \mathbf{c},\ \mathbf{y})] + \\ & + det\ (B_3\mathbf{y},\ S_3\mathbf{y},\ \mathbf{y}) = \\ & = (y_0^2 - y_1^2 - y_2^2 - y_3^2)(l_0y_0 + l_1y_1 + l_2y_2 + l_3y_3) \end{aligned}$$

with constants l_0, l_1, l_2, l_3 for all points $\mathbf{Y} \in P_3$. Comparison of the coefficients yields

$$(14) \qquad \begin{aligned} & l_0 = 0 \\ & det\ (\mathbf{b},\ \mathbf{c},\ \mathbf{y}) = l_1y_1 + l_2y_2 + l_3y_3 \\ & det\ (\mathbf{b},\ S_3\mathbf{y},\ \mathbf{y}) + det\ (B_3\mathbf{y},\ \mathbf{c},\ \mathbf{y}) = 0 \\ & det\ (B_3\mathbf{y},\ S_3\mathbf{y},\ \mathbf{y}) = -(y_1^2 + y_2^2 + y_3^2) det\ (\mathbf{b},\ \mathbf{c},\ \mathbf{y}) \end{aligned}$$

for all $\mathbf{y}$. Here there are many subcases which have to be considered. Again the discussion shows that all matrices B_4 satisfying the condition (14) belong to the manifold (9). Thus we have gained the somewhat puzzling result

Theorem 1: *Be given the unit sphere κ with center M in the Euclidean 3-space embedded into the projective space. Then in terms of intrinsic geometry the infinitesimal transformations Φ of projective motions induce tangent fields Φ^* on κ. Each of these tangent fields Φ^* belongs to an infinitesimal hyperbolic transformation with the absolute κ iff Φ can be represented as a linear combination of an infinitesimal transformation of a perspective collineation with center M and of an Euclidean motion.*

Remark: The set of infinitesimal transformations mentioned in Theorem 1 includes the infinitesimal hyperbolic transformations with the absolute κ.

5. FURTHER EXAMPLES: CONES OF REVOLUTION

Surprisingly, the general result of theorem 1 does not hold, if we consider a cone of revolution γ instead of a sphere, even if we start with an infinitesimal transformation of a Euclidean motion.

We start with the cone of revolution γ given by

$$(15) \qquad x_1^2 + x_2^2 = R^2x_3^2.$$

Its axis coincides with the x_3-axis of the coordinate system. For any point $P \in \gamma$ given by $\mathbf{X} = (x_0, x_1, x_2, x_3)^t$ the normal to ζ contains the point $\mathbf{N} = (0, x_1, x_2, -R^2x_3)^t$.

All projective collineations keeping γ invariant are given by the infinitesimal transformations

$$\mathbf{Y} \in \zeta \longrightarrow \dot{\mathbf{Y}} = Q_4\mathbf{Y}$$

with

$$(16) \qquad Q_4 = \begin{pmatrix} a+\lambda & b & c & d \\ 0 & \lambda & e & fR^2 \\ 0 & e & \lambda & gR^2 \\ 0 & f & g & \lambda \end{pmatrix},$$

where $a, \cdots g, R, \lambda$ are arbitrary real numbers. Additionally we start with an arbitrary infinitesimal transformation of a Euclidean motion given by (6). Condition (10) of chapter 4 now delivers

$$(17) \qquad det\ (E_4\mathbf{Y},\ Q_4\mathbf{Y},\ \mathbf{Y},\ \mathbf{N}) = 0$$

for all points $\mathbf{Y} \in \gamma$. A computation similar to that for the sphere shows that exactely the Euclidean infinitesimal transformations (6) belonging to pure translations in the direction of the axis of the cone or pure rotations around this axis induce automorphic infinitesimal projective transformations of the cone γ.

Remarks: 1) Any other Euclidean infinitesimal transformation induces nonlinear infinitesimal transformations of γ.

2) Stunningly, even the infinitesimal screw round the axis of the cone γ does not induce a Non-Euclidean autocollineation of γ. Figure 3 shows dots on the integral curves of such a screw induced on γ.

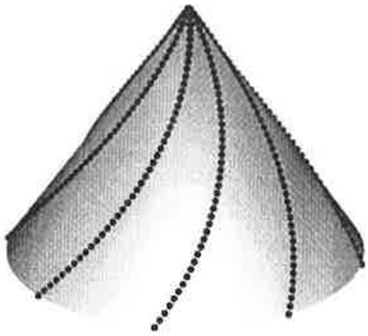

Figure 3: Some integral curves of the non-projective transformation induced by a Euclidean screw motion round the axis of the cone of revolution

We have got

Theorem 2: *Let be given the cone of revolution γ in the Euclidean 3-space embedded into the projective space. Then in terms of intrinsic geometry the infinitesimal transformations Φ of Euclidean motions induce tangent fields Φ^* on γ. These tangent fields Φ^* belong to infinitesimal Non-Euclidean transformations with the absolute γ iff Φ either belongs to a pure translation in the direction of the axis of γ or to a pure rotation round the axis of γ.*

6. FURTHER EXAMPLES: CYLINDERS OF REVOLUTION

As a last example we consider a cylinder of revolution ζ and an infinitesimal transformation of a Euclidean motion. ζ be determined by

$$x_1^2 + x_2^2 = R^2 x_0^2. \tag{18}$$

Its axis coincides with the x_3-axis of the coordinate system. For any point $P \in \zeta$ given by $\mathbf{X} = (x_0, x_1, x_2, x_3)^t$ the normal to ζ contains the point $\mathbf{N} = (x_0, 0, 0, x_3)^t$.

All projective collineations keeping ζ invariant are given by the infinitesimal transformations

$$\mathbf{Y} \in \zeta \longrightarrow \dot{\mathbf{Y}} = Q_4 \mathbf{Y}$$

with

$$Q_4 = \begin{pmatrix} \lambda & a & b & 0 \\ R^2 a & \lambda & c & 0 \\ R^2 b & -c & \lambda & 0 \\ d & e & f & g+\lambda \end{pmatrix} \tag{19}$$

with arbitrary real constants. Additionally we start with an arbitrary infinitesimal transformation of a Euclidean motion given by (6). A computation similar to that for the cone of revolution shows that just the Euclidean infinitesimal transformations (6) belonging to screws around the axis of ζ induce automorphic infinitesimal projective transformations of the cylinder ζ.

Any other Euclidean infinitesimal transformation induces nonlinear infinitesimal transformations of ζ. In figure 4 we start with the infinitesimal transformation of a Euclidean rotation round the displayed axis a. The induced tangent field is nonlinear - some integral curves are displayed. The two common points of the axis a and the cylinder ζ are singular points of this tangent field.

So we can state

Theorem 3: *Be given the cylinder of revolution γ in the Euclidean 3-space embedded into the projective space. In terms of intrinsic geometry the infinitesimal transformations Φ of Euclidean motions induce tangent fields Φ^* on γ. These tangent fields Φ^* belong to infinitesimal Non-Euclidean transformations with the absolute γ iff Φ belongs to the screws round the axis of γ, including the pure rotations round this axis and pure translations in the direction of this axis.*

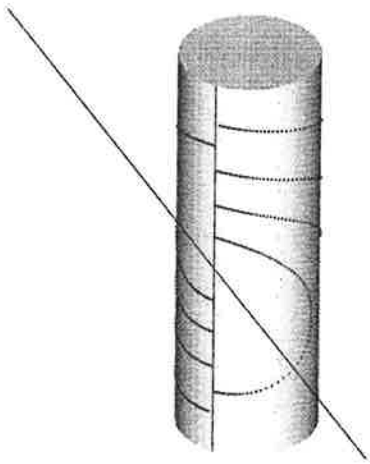

Figure 4: Some integral curves of the non-projective transformation induced by a Euclidean rotation round the displayed axis

7. CONCLUSIONS

We started with a given velocity-distribution of a one-parametric projective motion and restricted ourselves to the points of a sphere. According to the fundamental notions of differential geometry the velocity distribution was split into a normal and a tangent component. Theorem 1 gives a characterisation of those one-parametric projective motions which induce infinitesimal auto-collineations of the sphere. Further on we studied cones and cylinders of revolution with respect to given infinitesimal Euclidean motions. In general, they do not induce auto-collineations of the cone or the cylinder.

REFERENCES

Giering, O. (1982): Vorlesungen über höhere Geometrie. Vieweg, Braunschweig - Wiesbaden.

Roeschel, O. (1995): Raumkinematik und innere Geometrie der Kugel I. Sber. d. Österr. Akad. Wiss. **203**, 149-161.

Strubecker, K. (1931): Über nichteuklidische Schraubungen. Monatsh. Math. u. Phys. **38**, 63 - 84.

Toelke, J. (1975): Projektive kinematische Geometrie. Berichte der Math.-Stat. Sektion im Forschungsz. Graz, Nr. **39**, 1 - 51.

ABOUT THE AUTHORS

Dr. Sybille Mick is a member of the Institute of Geometry at the University of Technology in Graz. Her research interests are Isotropic Geometry and Kinematics. The e-mail is: mick@geometrie.tu-graz.ac.at and the postal address: Institute of Geometry, Graz University of Technology, Kopernikusgasse 24, A-8010 Graz, Austria, Europa.

Otto Roeschel, Univ.-Prof. Dr., is the head of the Institute of Geometry at the University of Technology in Graz, Austria, Europe. His research interests are Kinematics and Computer Aided Geometric Design. He can be reached by e-mail: roeschel@tugraz.at. The postal address is: Institute of Geometry, Graz University of Technology, Kopernikusgasse 24, A-8010 Graz, Austria, Europa.