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A REMARKABLE CLASS OF OVERCONSTRAINED LINKAGES

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Abstract: Given 3 congruently parametrised axial Darboux motions $\zeta_i = \Sigma_i \setminus \Sigma_0$ (i=1,2,3) with pairwise orthogonal and skew axes, which are gained of ζ_0 by rotation of 120, 240 degrees, resp. round a certain axis r_0. Then we are able to show, that each relative motion $\Sigma_i \setminus \Sigma_j$ $(i \neq j;\, i,\, j = 1,2,3)$ has a two-parametric familiy of (real) points situated on hyperboloids of one sheet $\Phi_{i,j}$, which is moved on spheres centered on an analogue surface $\Phi_{j,i}$. The intersection of two hyperboloids in a system Σ_i splits into a straight line g_i and a cubic circle c_i. It is shown that it is possible to form stiff triangles connecting corresponding points of c_1, c_2, c_3, such that the motions are not disturbed. For the points on the straight lines an analogous result holds: There the stiff triangles degenerate to straight lines, which in Σ_0 determine a regulus on a hyperboloid of rotation at each moment of the motion. Finally as an example two pairs of points on g_i and c_i are linked to get an overconstrained linkage.

1. In the 3-dimensional Euclidean space E_3 we use Cartesian frames $\{O_i;\, x_i,\, y_i,\, z_i\}$ $(i = 0,1,2,3)$ to describe points of the given systems Σ_i $(i = 0,1,2,3)$ by their position vectors $\vec{x}_i := (x_i, y_i, z_i)^t$. In Σ_0 we define 3 not intersecting orthogonal axes a_1, a_2, a_3 given by equations

$$a_1 \ldots x_0 = A,\, y_0 = 0, \quad a_2 \ldots y_0 = A,\, z_0 = 0, \quad a_3 \ldots x_0 = 0,\, y_0 = A \qquad (1)$$

with an arbitrary real $A \neq 0$ (see figure 1). Then we may define 3 congruent (even congruently parametrized) axial DARBOUX-motions $\zeta_i = \Sigma_i \setminus \Sigma_0$ (i=1,2,3) with axes $a_1,\, a_2,\, a_3$. They all shall be parametrized by their angle of rotation $t \in [0,\, 2\pi]$. Then a parametrisation of these motions shall be given by (see O. Bottema-B. Roth (1979) ,p.321)

$$\zeta_1\colon \vec{x}_0(t, \vec{x}_1) := \begin{pmatrix} A + x_1 \cos t - y_1 \sin t \\ x_1 \sin t + y_1 \cos t \\ z_1 + B \sin t \end{pmatrix}, \qquad \zeta_2\colon \vec{x}_0(t, \vec{x}_2) := \begin{pmatrix} z_2 + B \sin t \\ A + x_2 \cos t - y_2 \sin t \\ x_2 \sin t + y_2 \cos t \end{pmatrix},$$

$$\zeta_3\colon \vec{x}_0(t, \vec{x}_3) := \begin{pmatrix} x_3 \sin t + y_3 \cos t \\ z_3 + B \sin t \\ A + x_3 \cos t - y_3 \sin t \end{pmatrix} \qquad (t \in [0, 2\pi]) \qquad (2)$$

with a further real constant $B \neq 0$. The common time parameter $t \in [0, 2\pi]$ links the 3 systems $\Sigma_1, \Sigma_2, \Sigma_3$. This linkage represents a generalisation of a partial motion of the motions studied in Röschel (1995). We want to discuss properties of the relative motions and show that the motion is not disturbed, if certain stiff rods (with spherical links) connect certain points of each pair of system. $\Sigma_1 \setminus \Sigma_0$ may be moved into $\Sigma_2 \setminus \Sigma_0$ ($\Sigma_3 \setminus \Sigma_0$) (including its parametrisation) by a rotation of 120^0 (240^0) round the axis $r_0 = [(0,0,0), (1,1,1)]$ (see figure 1 for $t = 0$). Therefore $\Sigma_2 \setminus \Sigma_0$ and $\Sigma_3 \setminus \Sigma_0$ are conjugate motions to $\Sigma_1 \setminus \Sigma_0$ with respect to these fixed rotations.

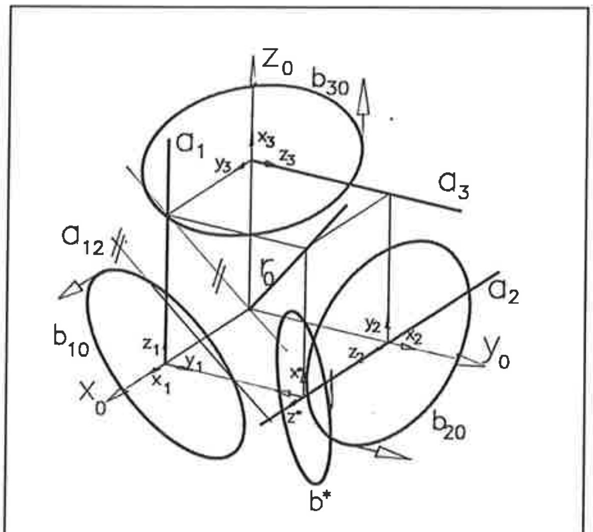

Figure 1. The given axial Darboux motions

2. The relative motions $\Sigma_i \setminus \Sigma_j$ $(i \neq j)$. Our definition of the motions guarantees that for each moment t the positions $\Sigma_j(t)$ in Σ_0 may be gained by reflecting $\Sigma_i(t)$ at a straight line a_{ij} followed by a fixed displacement in Σ_j. Figure 1 shows the situation for i=1, j=2: We have drawn a point path b_{i0} under $\Sigma_i \setminus \Sigma_0$ (i=1,2), which is gained from b_{10} by rotation round r_0. But it may be generated by reflecting the positions of Σ_1 with respect to the straight line a_{12} (results denoted by "*") followed by a fixed screw in Σ_2 (with axis a_2, angle $-\pi/2$ and translation distance $-A$). The situation for $(i, j) \neq (1,2)$ is similar.
Therefore $\Sigma_i \setminus \Sigma_j$ is line-symmetric in the sense of J.Krames with respect to a basic surface Γ_{ji}, which is the path surface of the straight line a_{ij} under the inverse motion Σ_0 / Σ_j. Such motions have been studied by J.Krames (1937b).

Γ_{ji} is a ruled surface of degree 4 (J.Krames (1937b)). In this case in Σ_i there exists a two-parametric family of points with spherical paths under $\Sigma_i \setminus \Sigma_j$. These points in general are situated on a pair of complex conjugate planes (a special singular case of that described by J.Krames (1937b)) and on an orthogonal hyperboloid of one sheet Φ_{ij}, the corresponding centers lie on congruent surfaces, the corresponding hyperboloid will be denoted by Φ_{ji}. We restrict ourselves to the case $i = 1$ - the other cases result from permutations and appropriate changes of some signs. Computation yields

$$\begin{aligned} \Phi_{1,2} \quad &\ldots\ 2B\,x_1\,z_1 = A\,(-B^2 + x_1^2 + y_1^2), \\ \Phi_{1,3} \quad &\ldots\ 2B\,y_1\,z_1 = A\,(-B^2 + x_1^2 + 2By_1 + y_1^2) \end{aligned} \qquad (3)$$

After some algebra the relations V_{1j}: $\Phi_{1j} \subset \Sigma_1 \to \Phi_{j1} \subset \Sigma_j$ $(j = 2,3)$ between the points with spherical paths and the centers of their spheres may be written in the following typical cases:

$V_{1,2}$: $\Phi_{1,2} \to \Phi_{2,1}$

$$x_2 = \frac{B\,(B^2 - x_1^2 - y_1^2)}{x_1^2 + (y_1 - B)^2},\ y_2 = \frac{-2B^2 x_1}{x_1^2 + (y_1 - B)^2},\ z_2 = \frac{A\,(x_1 - y_1)}{x_1} \qquad (4)$$

and

$V_{1,3}$: $\Phi_{1,3} \to \Phi_{3,1}$

$$x_3 = \frac{-2B^2 y_1}{(x_1 + B)^2 + y_1^2},\ y_3 = \frac{B(-B^2 + x_1^2 + y_1^2)}{(x_1 + B)^2 + y_1^2},\ z_3 = \frac{A\,x_1}{y_1} \qquad (5)$$

We now show how to find these equations for the example $i = 1$, $j = 2$ (for the other choices of indices it may be done in a similar way):
Two points $\vec{x}_1 = (x_1, y_1, z_1)^t$, $\vec{x}_2 = (x_2, y_2, z_2)^t$ fixed in Σ_1, Σ_2, resp. keep constant distances during the motions ζ_1, ζ_2, iff the difference vector

$$\vec{d}_{1,2}(t, \vec{x}_1, \vec{x}_2) := \vec{x}_0(t, \vec{x}_2) - \vec{x}_0(t, \vec{x}_1) = \begin{pmatrix} z_2 - A \\ A \\ -z_1 \end{pmatrix} + \begin{pmatrix} -x_1 \\ x_2 - y_1 \\ y_2 \end{pmatrix} \cos t + \begin{pmatrix} B + y_1 \\ -x_1 - y_2 \\ x_2 - B \end{pmatrix} \sin t :=$$

$$:= \vec{a} + \vec{b}\,\cos t + \vec{c}\,\sin t \qquad (6)$$

has constant length. This yields

$$0 = \vec{d}_{1,2}(t, \vec{x}_1, \vec{x}_2)\frac{\partial \vec{d}_{1,2}(t, \vec{x}_1, \vec{x}_2)}{\partial t} = \vec{a}\,\vec{c}\,\cos t - \vec{a}\,\vec{b}\,\sin t + \vec{b}\,\vec{c}\,\cos 2t + 0.5\,(\vec{c}^{\,2} - \vec{b}^{\,2})\sin 2t$$

$\forall\ t \in [0, 2\pi]$. This condition is valid, iff

$$0 = \vec{a}\,\vec{b} = A\,x_2 - z_1\,y_2 - x_1\,z_2 + A\,(x_1 - y_1),$$
$$0 = \vec{a}\,\vec{c} = -z_1\,x_2 - A\,y_2 + (B + y_1)\,z_2 + B\,z_1 - A\,(B + x_1 + y_1),$$
$$0 = \vec{b}\,\vec{c} = -x_1\,x_2 + (y_1 - B)\,y_2 - B\,x_1,$$
$$0 = \vec{b}^2 - \vec{c}^2 = 2\,[(B - y_1)\,x_2 - x_1\,y_2 - B\,(B + y_1)].$$

This is a system of 4 linear equations for x_2, y_2, z_2. It has solutions, if the condition

$$B\,[x_1^2 + y_1^2 + B^2]\,[2\,B\,x_1\,z_1 + A\,(B^2 - x_1^2 - y_1^2)] = 0 \qquad (7)$$

holds. This equation determines the points of Σ_1 which may hold constant distances to certain points of Σ_2. Therefore for $A\,B \neq 0$ these points of Σ_1 belong to a complex conjugate pair of planes and the hyperboloid of one sheet $\Phi_{1,2}$ (3). For the points of $\Phi_{1,2}$ we get the corresponding points in Σ_2 via the map $V_{1,2}$ (4). □

The correspondence $V_{1,2}$ (4) between the x- and y- coordinates (the ground projections $(x_i, y_i, 0), (x_j, y_j, 0)$) *does not depend on the constant* A (therefore for the first two coordinates we may use the results of Röschel (1995)). The equations show, that the ground projection of our relationships **may be gained by an inversion with respect to a circle, followed by a displacement.** The centers of inversion have the coordinates $(0, B, 0)$, $(-B, 0, 0)$ resp. for $V_{1,3}$. The radius of the circles of inversion is $B\sqrt{2}$. We sum up in

Theorem 1: *Given the 3 congruent special axial Darboux-motions* $\Sigma_i \setminus \Sigma_0$ $(i = 1,2,3)$ *in a fixed space* Σ_0 *represented by* (2). *Then each relative motion* $\Sigma_i \setminus \Sigma_j$ *is line-symmetric in the sense of J. Krames and moves a two-parametric family of points of* Σ_i *on spheres centered in* Σ_j. *The corresponding (real) points are (in general) situated on special hyperboloids of one sheet in both systems.*

3. In the case $A = 0$ Röschel (1995) gives a way to generate overconstrained linkages consisting of rods of constant length (with spherical links in each system Σ_i). We now want to extend this procedure to the case $A \neq 0$:
Interesting cases occour, if we start with points P_1 on the intersection of the hyperboloids $\Phi_{1,2}$ and $\Phi_{1,3}$. The curve of intersection is an algebraic curve of order 4 containing the absolute points of the planes $z_1 =$ const. and the point of infinity of the

z_1 – axis. The correspondences $V_{1,i}$ $(i = 2,3)$ map $P_1 \in k_{1,2,3} := \Phi_{1,2} \cap \Phi_{1,3}$ into points $P_i := V_{1,i}(P_1)$. By simple computations it is possible to proof the surprising

Theorem 2: *Given the 3 congruent special axial Darboux-motions* $\Sigma_i \setminus \Sigma_0$ *in a fixed space* Σ_0 *represented by* (2). *Starting with points* P_1 *on the intersection curve* $k_{1,2,3} := \Phi_{1,2} \cap \Phi_{1,3}$ *the maps* $V_{1,i}$ $(i = 2,3)$ *determine corresponding points* $P_i := V_{1,i}(P_1)$ $(i=2,3)$. *Then for each point* $P_1 \in k_{1,2,3} := \Phi_{1,2} \cap \Phi_{1,3}$ *the relation* $P_i = V_{k,i}(V_{j,k}(V_{i,j}(P_i)))$ *is true.*

This means, that for all starting points P_1 on $k_{1,2,3} := \Phi_{1,2} \cap \Phi_{1,3}$ the corresponding points $P_i := V_{1,i}(P_1)$ $(i = 2,3)$ determine a triangle P_1, P_2, P_3 of fixed shape, which does not disturb our special one- parameter motion, if the triangles and the systems $\Sigma_1, \Sigma_2, \Sigma_3$ have spherical links at the vertices P_1, P_2, P_3.

4. Now we are interested in the shape of the **curve of intersection** $k_{1,2,3} := \Phi_{1,2} \cap \Phi_{1,3}$: Suitable geometrical considerations show, that it splits into two parts; a **straight line** g_1 and a **cubic circle** c_1, which may be parametrized by

$$g_1 \;\ldots\ldots\; (u,\; B + u,\; \frac{A(u+B)}{B})^t,$$

$$c_1 \;\ldots\ldots\; (-\frac{u\,B^2}{u^2 + (u+B)^2},\; -\frac{B^2(u+B)}{u^2+(u+B)^2},\; \frac{A(u+B)}{B})^t \quad (u \in R) \tag{8}$$

Then the maps $V_{1,2}, V_{1,3}$ transform g_1 and c_1 into image curves with the following parametrisations

$$g_2 := V_{1,2}(g_1) \ldots (-\frac{B(u+B)}{u},\; -\frac{B^2}{u},\; -\frac{AB}{u})^t,$$

$$g_3 := V_{1,3}(g_1) \ldots (-\frac{B^2}{u+B},\; \frac{Bu}{u+B},\; \frac{Au}{u+B})^t,$$

$$c_2 := V_{1,2}(c_1) \ldots (\frac{Bu(u+B)}{(u+B)^2+B^2},\; \frac{B^2\,u}{(u+B)^2+B^2},\; -\frac{AB}{u})^t, \tag{9}$$

$$c_3 := V_{1,3}(c_1) \ldots (\frac{B^2(u+B)}{u^2+B^2},\; -\frac{B\,u\,(u+B)}{u^2+B^2},\; \frac{Au}{u+B})^t$$

It is remarkable, that the straight line g_1 is mapped into straight lines g_2, g_3, the cubic circle c_1 into cubic circles c_2, c_3. We sum up in

Theorem 3: *Given the 3 congruent special axial Darboux-motions $\Sigma_i \setminus \Sigma_0$ in a fixed space Σ_0 represented by* (2). *Then the intersection curve $k_{1,2,3} := \Phi_{1,2} \cap \Phi_{1,3}$ of the hyperboloids with (real) spherical paths splits into a straight line g_1 and a cubic circle c_1. Without disturbation of the motions the points P_1 on g_1 may be linked by sticks with points $P_i := V_{1,i}(P_1)$ (i=2,3) on straight lines g_2, g_3, the points on c_1 with points on cubic circles c_2, c_3.*

Now we start with a point $P_1 \in g_1$: These points are mapped into points $P_2 \in g_2$ and $P_3 \in g_3$, whose paths may be studied in Σ_0. Our difference vectors $\vec{d}_{1,2}(t, u) := \vec{x}_0(t, P_2) - \vec{x}_0(t, P_1)$ (6) and $\vec{d}_{1,3}(t, u) := \vec{x}_0(t, P_3) - \vec{x}_0(t, P_1)$ read

$$\vec{d}_{1,2}(t, u) = \frac{1}{u}\,[\frac{A}{B}\begin{pmatrix} -B\,(u+B) \\ u\,B \\ -u\,(u+B) \end{pmatrix} - \begin{pmatrix} u^2 \\ (u+B)^2 \\ B^2 \end{pmatrix}\cos t + \begin{pmatrix} u^2 + 2uB \\ B^2 - u^2 \\ -B(2u+B) \end{pmatrix}\sin t], \quad (10)$$

$$\vec{d}_{1,3}(t, u) = \frac{1}{u+B}\,[\frac{A}{B}\begin{pmatrix} -B(u+B) \\ u\,B \\ -u\,(u+B) \end{pmatrix} - \begin{pmatrix} u^2 \\ (u+B)^2 \\ B^2 \end{pmatrix}\cos t + \begin{pmatrix} u^2 + 2uB \\ B^2 - u^2 \\ -B(2u+B) \end{pmatrix}\sin t].$$

Therefore we have $u\,\vec{d}_{1,2}(t, u) = (u+B)\,\vec{d}_{1,3}(t, u)$ for all $u \in R$. Thus the corresponding points $P_1 \in g_1$, $P_2 := V_{1,2}(P_1)$, $P_3 := V_{1,3}(P_1)$ are collinear during the whole motion! *At each moment t in Σ_0 the triple of straight lines g_1, g_2, g_3 determines a regulus of intersecting straight lines. These straight lines may be materialized as sticks connecting corresponding points on g_1, g_2, g_3!* Computation shows that the regulus of sticks for each $t \in [0, 2\pi]$ in Σ_0 is part of a rotational hyperboloid of one sheet with common axis $r_0 = [(0, 0, 0), (1, 1, 1)]$, which may degenerate. Of course the shape of the hyperboloids changes with t. In this case we have got a model of a moveable hyperboloid of one sheet, where the generators are formed by sticks linked together by spherical links. This special case is known - a model may be found at institutes of geometry at Vienna and Dresden University of Technology.

5. An example. Now we give an example of a simple overconstrained linkage following the results of chapter 4: Starting with points $u = -0.5B$ and $u = B$ in (8) we gain points

$$\begin{aligned} &P_1 \in g_1 \;....\; (-0.5\,B, 0.5\,B, 0.5\,A)^t, \quad R_1 \in g_1 \;....\; (B, 2\,B, 2\,A)^t, \\ &Q_1 \in c_1 \;....\; (B, -B, 0.5\,A)^t, \quad S_1 \in c_1 \;....\; (-0.2\,B, -0.4\,B, 2\,A)^t \end{aligned} \quad (11)$$

with corresponding points

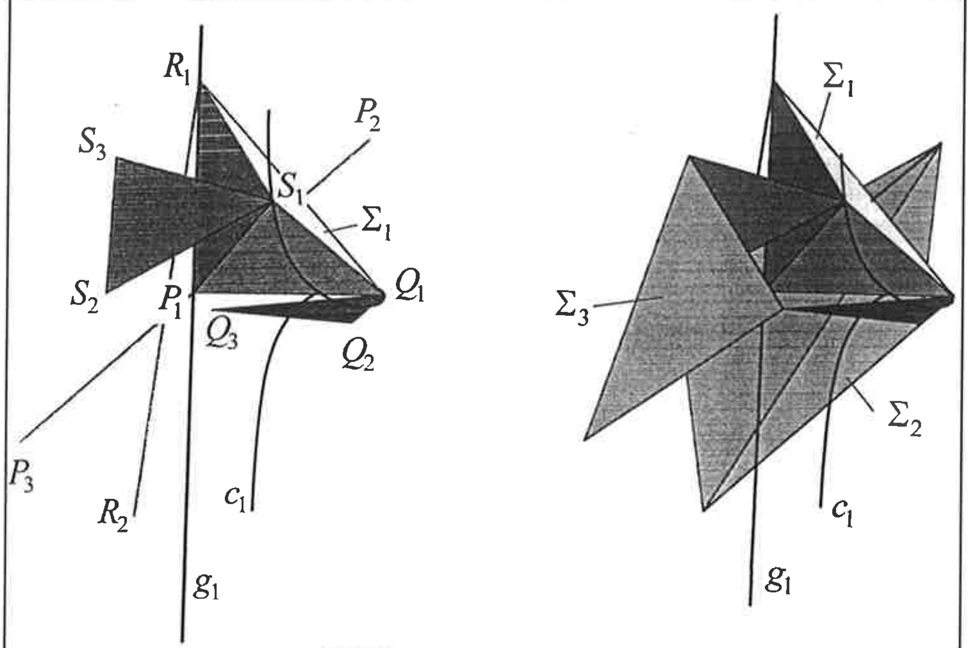

Figure 2: The straight line and the cubic circle with some rods

$$P_2 \in g_2 \ldots (B, 2\,B, 2\,A)^t, \qquad R_2 \in g_2 \ldots (-2\,B, -B, -A)^t, \qquad (12)$$
$$Q_2 \in c_2 \ldots (-0.2\,B, -0.4\,B, 2\,A)^t, \quad S_2 \in c_2 \ldots (0.4\,B, 0.2\,B, -A)^t$$

$$P_3 \in g_3 \ldots (-2\,B, -B, -A)^t, \quad R_3 \in g_3 \ldots (-0.5\,B, 0.5\,B, 0.5\,A)^t, \qquad (13)$$
$$Q_3 \in c_3 \ldots (0.4\,B, 0.2\,B, -A)^t, \quad S_3 \in c_3 \ldots (B, -B, 0.5\,A)^t.$$

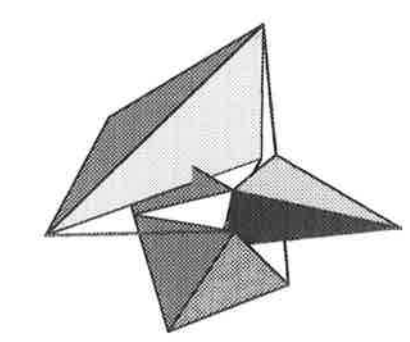

Figure 3: The 3 tetrahedra and their rods

We have got a configuration consisting of 3 tetrahedra, two triangular and two straight line links. The general degree of freedom would be $F = 0$. Adding more sticks according to our rules we would get a linkage with general degree of freedom $F \leq -1$. Figure 2 shows a very simple configuration of this type consisting of the points (11), (12), (13) for $A = B = 100$ and $t = \pi/6$. As many lines are hidden by the tetrahedra the left side only shows the first tetrahedron, the straight line g_1, the cubic c_1, the triangular and the straight line links. The right side gives the image of the whole configuration. It has to be mentioned that only the edges of the tetrahedra should be materialized - but in order to gain more instructive pictures the author has chosen the materialized version.

Figure 3 shows the situation for the same model for $t = 19\,\pi / 24$. Here the straight line links are situated outside of the 3 tetrahedra. Therefore the two positions only may be gained by a coninous motion, if intersections of some sticks are allowed. But both models built of rods have a one- parametric motion with timeparameter t taken from two different parts of the allowed time - interval.

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