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REMARKS ON CUBIC RULED SURFACES WITH CONSTANT DISTRIBUTION PARAMETER IN E_4

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ABSTRACT: This paper is devoted to considerations on a special first order invariant of two-dimensional ruled surfaces of E_n – the so-called *distribution parameter d* in a generator. It is defined as the limit of the distance/angle ratio of the generator and its neighbour. Ruled surfaces with constant parameter of distribution are of special interest and have been studied within the 3-dimensional Euclidean space E_3 by many authors. H. BRAUNER could prove that the only nontrivial cubic ruled surface with constant distribution parameter in E_3 is a special type of a CAYLEY surface. The aim of this paper is to investigate these problems for higher dimensions. We will in fact determine all cubic ruled surfaces of E_n with constant distribution parameter. There we will follow the paper [13] and will be able to prove the following fundamental statement: *There are twisted cubic ruled surfaces of constant distribution parameter in E_n ($n > 3$). They span a 4-dimensional Euclidean space E_4, are conoidal ruled surfaces and are contained in a quadratic cylinder of revolution Γ with 2-dimensional generators.* We give a standard parametrisation and geometrically discuss further properties of such surfaces. Surprisingly we have got a class of cubic ruled surfaces with constant distribution parameter way beyond the 3-dimensional CAYLEY surface case. We determine the striction curve of these surfaces. It, in general, is a rational curve of degree 4. In an additional part of the paper we will apply isometries of the cylinder Γ into the 3-dimensional Euclidean space E_3. This way the cubic ruled surface Φ embedded into Γ is isometrically mapped into a conoidal ruled surface with constant distribution parameter immersed in E_3. The properties of these image surfaces are also being studied in this paper.

Keywords: Ruled Surfaces, Constant Distribution Parameter, Twisted Cubic Ruled Surfaces in E_4, CAYLEY-surface.

1. RULED SURFACES AND THE DISTRIBUTION PARAMETER

A C^1– immersion $X(t,u):(t,u)\in G\subset \Re^2 \to E^n$ ($n>2$) given by

$$X(t,u) = L(t)+u\,E(t) \quad (t\in T, u\in\Re) \qquad (1)$$

with the C^1–curve $L(t): t\in T\subset\Re\to E_n$ and the C^1– set of direction vectors $E(t): t\in T\to\Re^n$ defines a *two-dimensional ruled C^1– surface* in the n-dimensional Euclidean space E_n.

The curve $L(t)$ is called *basic curve* of X. The corresponding generators are given by t = *const*. The tangential behavior of the surface along a generator $t = t_0$ is determined by $E(t_0)$ and the derivative vectors $\dot E(t_0)$ and $\dot L(t_0)$. The tangent planes along the points of a generator belong to a subspace of E_n with dimension $f(t) := \mathrm{Dim}\,[E(t), \dot E(t), \dot L(t)] < 4$. Generators with $f(t) = 3$ are called *regular*. The tangent planes at the points of a regular generator are contained in a 3-dimensional *tangent space of X at the generator* spanned by the generator and $[E(t),\ \dot{E}(t),\ \dot{L}(t)]$.

We are able to measure the distance $dist(t_0,t)$ and the angle $\varphi\ (t_0,t)$ for a particular generator $t=t_0$ with respect to the generator given by t. As in the case of a ruled surface imbedded into the 3-dimensional Euclidean space E_3 we define the *distribution parameter* $d(t_0)$ (shortly named "DP") of the ruled surface $X(t,u)$ in the generator $t = t_0$ as the limit

$$d(t_0) := \lim_{t \to t_0} \frac{dist(t,t_0)}{\varphi(t,t_0)}. \qquad (2)$$

This yields

$$d(t) := \frac{E^2(t)\ Vol(E(t), \dot{E}(t), \dot{L}(t))}{E^2(t)\,\dot{E}^2(t) - (E(t)\,\dot{E}(t))^2}. \qquad (3)$$

The determinant used in E_3 is replaced by $Vol(A,\ B,\ C)$ here. It denotes the volume of the parallelepiped defined by the 3 vectors A, B, C. Its square is defined via GRAM's determinant

$$Vol^2(A,B,C) = Det\begin{pmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{pmatrix} \qquad (4)$$

Remarks: An overview on line manifolds of E_3 has been given by G. WEISS [15], [16]. For generalized ruled surfaces there are a couple of further distribution parameters (see H. FRANK - O. GIERING [6]).

This paper is devoted to ruled surfaces of constant distribution parameter $d(t) = const. \in \Re - \{0\}$ (cases with $d(t) \equiv 0$ characterize developable surfaces which are excluded here).

We consider the set of two-dimensional algebraic varieties of E_n with a one-parametric set of straight line generators. An element X with the additional property that any arbitrary $(n-1)$-dimensional subspace H of E_n either contains the whole 2-dimensional ruled surface X or intersects X in a cubic curve (in algebraic sense) is called *cubic ruled surface* in E_n $(n > 2)$.

Remarks: The cubic ruled surfaces of the 3-dimensional Euclidean space E_3 are well-known. This is why we restrict the following considerations to the case $n > 3$.

As any irreducible non-degenerate two-dimensional variety of degree k of E_n is contained in a subspace of dimension $\leq k+1$ (for a proof see the textbook by J. HARRIS [9], p. 231) we have: Any non-degenerate cubic ruled surface X is contained in a 4-dimensional Euclidean space. Therefore we can confine the following considerations to $n = 4$, where we follow the publication [13] of the author.

2. RULED SURFACES IN THE REAL PROJECTIVE SPACE OF DIMENSION 4

We embed E_4 into a real projective space P_4 of dimension 4 (if needed with its complex extension). In P_4 there are two different types of cubic ruled surfaces: The so-called *3-dimensional cases* contained in a 3-dimensional subspace and a second type called *twisted*, its span being 4-dimensional.

For results on ruled surfaces with constant DP immersed into a 3- dimensional Euclidean space we refer to results by H. BRAUNER [2] – [4] and J. KRAMES [11]. They demonstrated that a special cubic CAYLEY-surface is the only nontrivial cubic ruled surface with constant parameter of distribution in E_3.

The aim of this paper is to investigate the genuinely 4-dimensional cases which addresses the so-called *twisted cubic ruled surfaces* of E_4 with constant distribution parameter. They span the 4-dimensional space, but are not contained in a 3-dimensional parametrisation and discussed properties of these surfaces. Then we applied isometries of the cylinder Γ into the 3-dimensional Euclidean space E_3. This way the cubic ruled surface X embedded into Γ was isometrically mapped into a conoidal ruled surface X^* with constant distribution parameter immersed in E_3.

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