AN OVERCONSTRAINED CHAIN OF 16 REGULAR TRIANGLES

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ABSTRACT: We study a particular chain of 16 congruent regular triangles forming two regular tetrahedra and two open pyramids (these pyramids can be folded along their four edges through the vertex). Figure 1 shows the basic parts, figure 2 displays a possible assembly mode. These triangles are interlinked via rotational joints along the edges of the pyramids and two pairs of opposite edges of the tetrahedral. The theoretical degree of freedom of this kinematic chain takes on the value $F \leq 0$. But surprisingly, a physical model of this mechanism seems to admit a one-parametric self-motion! The aim of this paper is to prove the existence of self-motions of this structure.

Keywords: Kinematics, Robotics, Overconstrained Mechanisms, Self-motions, Saturated packings.

1. INTRODUCTION

Harborth and M. Moeller [3] studied an interesting packing of tetrahedra. It consists of 16 regular tetrahedra connected via 32 spherical joints. In this arrangement they define a saturated packing in the following sense: Every vertex of a tetrahedorn is linked to one and only one vertex of another tetrahedron. In addition the tetrahedra are not allowed to interfere with each other. Considering saturated arrangements of regular tetrahedra led to the study of interesting overconstrained kinematic chains (see [2] and [4]). These chains are built up of regular triangles – some of them form regular tetrahedra, some are used as bases for further regular tetrahedra.

A particular chain of this type is presented and studied in this paper. It consists of 16 congruent regular triangles forming the faces of two regular tetrahedra and two ‘open pyramids’ (each with four regular triangular faces). Each pyramid can be folded along its four edges through the vertex and admits a one-parametric self-motion. Figure 1 shows the basic parts, figure 2 displays a possible assembly mode.

The 16 triangles are interlinked via spherical joints at the vertices of the pyramids and the tetrahedra. Pairs of these vertices determine rotational joints along the edges of the pyramids and two pairs of opposite edges of the tetrahedra. We gain four 1R joints interlinking the two tetrahedra and the pyramids, while each pyramid itself has four 1R joints in the edges through the vertex. The theoretical degree of freedom of this kinematic chain (ten rigid bodies and twelve 1R joints) takes on the value F = -6. But surprisingly enough, a physical model of this mechanism seems to admit some one-parametric self-motion!

Figure 3 (with faces of the two tetrahedra) and figure 4 display the framework formed by the edges of the triangles. It consists of 24 stiff rods of unit length which meet in ten points (spherical joints).

In the following we will look into possible self-motions of this structure.

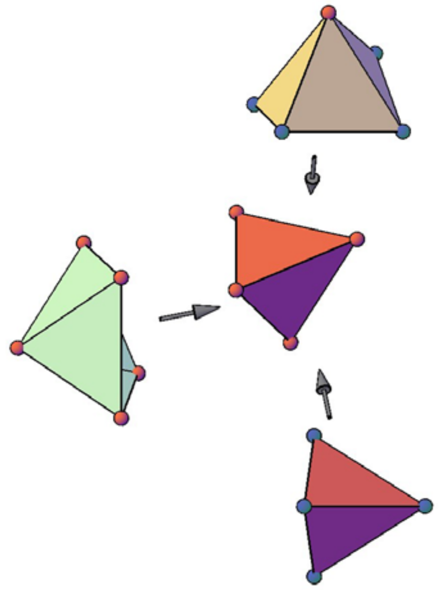

Figure 1

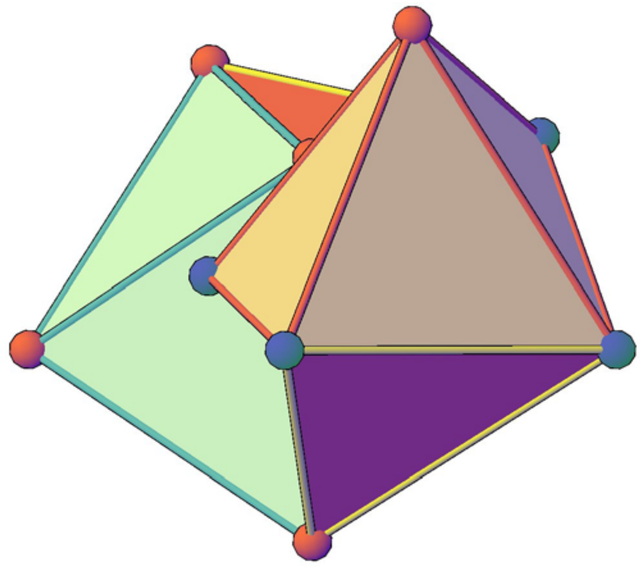

Figure 2

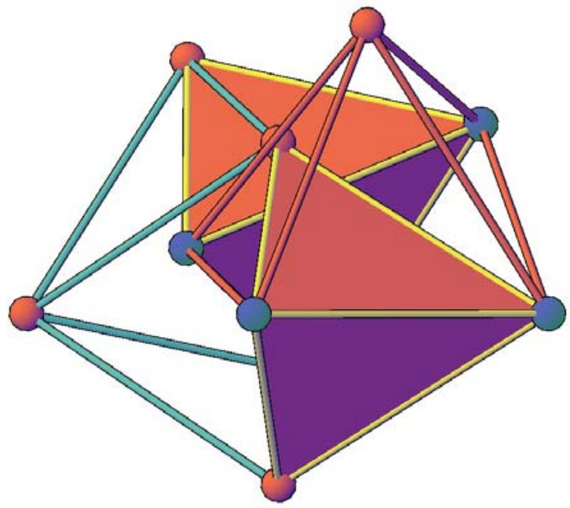

Figure 3

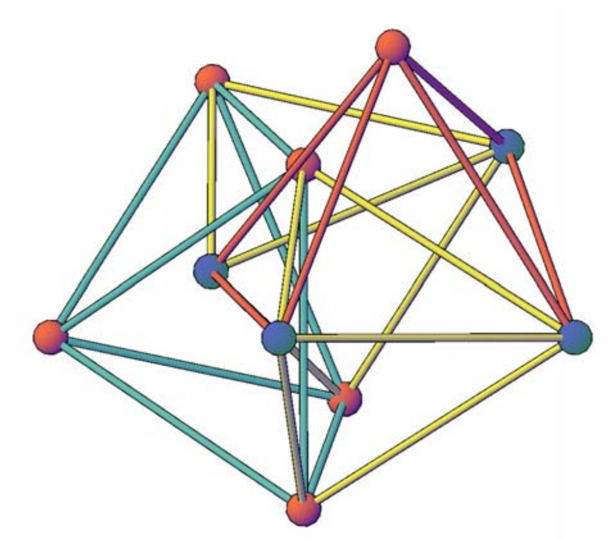

Figure 4

2. THE BASIC PYRAMID

Our mechanism consists of a number of parts. Firstly, we consider the part forming an open pyramid with 4 facets (congruent regular triangles - see figure 5). We use a Cartesian frame {*O; x, y, z*}. It is natural to use coordinate planes of a 'world coordinate system' in the planes of symmetry of the possible folded versions of the pyramid with its center in the origin. Then the one-parametric self-motion of this pyramid can be displayed in this world coordinate system. The points *1, 2, 3* and *4* describe circles with center *O* in coordinate planes. Its radius shall be normed to 1. Thus we can parametrize these point paths by

$$1\,(\cos\alpha, 0, \sin\alpha),\ 2(0, \cos\beta, \sin\beta),\ 3(-\cos\alpha, 0, \sin\alpha),\ 4(0, -\cos\beta, \sin\beta). \qquad (1)$$

There $\alpha \in [\pi/6, \pi/2]$ and $\beta \in [\pi/6, \pi/2]$ are linked via

$$1 = 2\sin\alpha\,\sin\beta. \tag{2}$$

Remark: The two diagonals of the spatial rhombic quadrangle *1,2,3,4* have lengths $d_1 := \overline{13} = 2\,|\cos\alpha|$ and $d_2 = \overline{24} = 2\,|\cos\beta|$. Formula (2) yields the linking condition

$$4 = (4 - d_1^2)(4 - d_2^2). \tag{3}$$

We consider the spatial rhombic quadrangles with side length 1 and diagonals d_1 and d_2 complying with condition (3). This is exactly what it takes to form the base of a four-sided pyramid with regular triangular facets of side length 1.

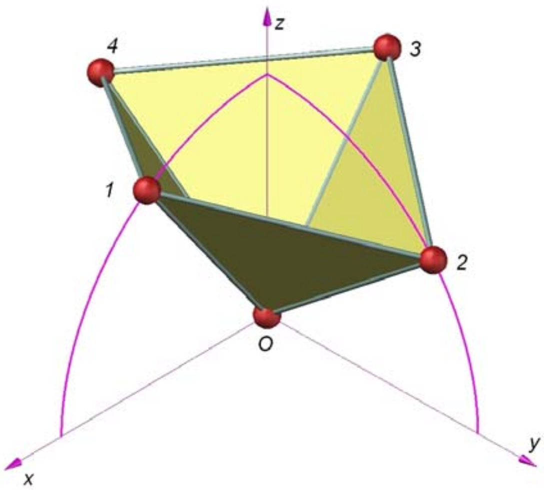

Figure 5

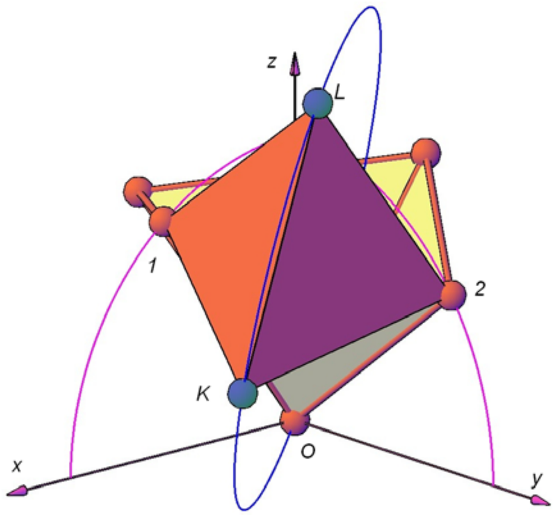

Figure 6

3. THE LINKED TETRAHEDRA

Now we put a regular tetrahedron with two vertices in the points 1 and 2 onto the pyramid (see figure 6). Its other vertices will be denoted *K* and *L*. This tetrahedron admits a one-parametric rotation about the line [1,2]. The points *K* and *L* are situated on a circle $k(\alpha)$ containing the origin O. They can be parametrized by

$$\vec{k}(u) = \frac{(1-\cos u)}{2}\begin{pmatrix}\cos\alpha\\ \cos\beta\\ \sin\alpha+\sin\beta\end{pmatrix} + \sin u\begin{pmatrix}-\sin\alpha\cos\beta\\ -\sin\beta\cos\alpha\\ \cos\alpha\cos\beta\end{pmatrix}$$

$$:= \vec{m}_1(1-\cos u) + \vec{n}_1\sin u \qquad (u\in[0,2\pi)). \tag{4}$$

The angle s between two faces of a regular tetrahedron is given by $\cos s = 1/3$ and $\sin s = \pm 2\sqrt{2}/3$. L be the point on this circle relating to the parameter $u+s$. This gives the parametrization of the positions of L:

$$\vec{l}(u) := \vec{k}(u+s) = \vec{m}_1\frac{(3-\cos u + 2\sqrt{2}\sin u)}{3} + \vec{n}_1\frac{(\sin u + 2\sqrt{2}\cos u)}{3}. \tag{5}$$

The same procedure delivers a regular tetrahedron linked to the edge 2 and 3 (see figure 7). Its new vertices are denoted by K^* and L^*. Parametrisation yields

$$\vec{k}^*(v) = \frac{(1-\cos v)}{2}\begin{pmatrix} -\cos\alpha \\ -\cos\beta \\ \sin\alpha+\sin\beta \end{pmatrix} + \sin v \begin{pmatrix} \sin\alpha\cos\beta \\ \sin\beta\cos\alpha \\ \cos\alpha\cos\beta \end{pmatrix} = \tag{6}$$

$$:= \vec{m}_2(1-\cos v) + \vec{n}_2 \sin v \qquad\qquad (v\in[0,2\pi))$$

and

$$\vec{l}^*(v) := \vec{k}^*(v+\varepsilon s) = \frac{(3-\cos v+\varepsilon 2\sqrt{2}\sin v)}{3}\vec{m}_2 + \frac{(\sin v+\varepsilon 2\sqrt{2}\cos v)}{3}\vec{n}_2. \tag{7}$$

The variable $\varepsilon\in\{-1,+1\}$ determines the orientation of the second tetrahedron (the mechanism has two corresponding assembly modes). Figure 7 refers to the case for $\varepsilon=+1$.

Remark: The vectors $\vec{m}_2,\vec{n}_2$ from (7) are gained form $\vec{m}_1,\vec{n}_1$ (4) by a half-turn about the z-axis. According to our definitions and (2) we have

$$\begin{aligned} &\vec{m}_1\vec{m}_1 = \vec{n}_1\vec{n}_1 = \vec{m}_2\vec{m}_2 = \vec{n}_2\vec{n}_2 = 3/4, \\ &\vec{m}_1\vec{n}_1 = \vec{m}_2\vec{n}_2 = 0, \\ &A := \vec{m}_1\vec{m}_2 = \frac{1}{4}(\sin^2\alpha - \cos^2\alpha + \sin^2\beta - \cos^2\beta + 2\sin\alpha\sin\beta), \\ &B := \vec{n}_1\vec{m}_2 = \vec{m}_1\vec{n}_2 = \cos\alpha\cos\beta(\sin\alpha+\sin\beta), \\ &C := \vec{n}_1\vec{n}_2 = \cos^2\alpha\cos^2\beta - \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta. \end{aligned} \tag{8}$$

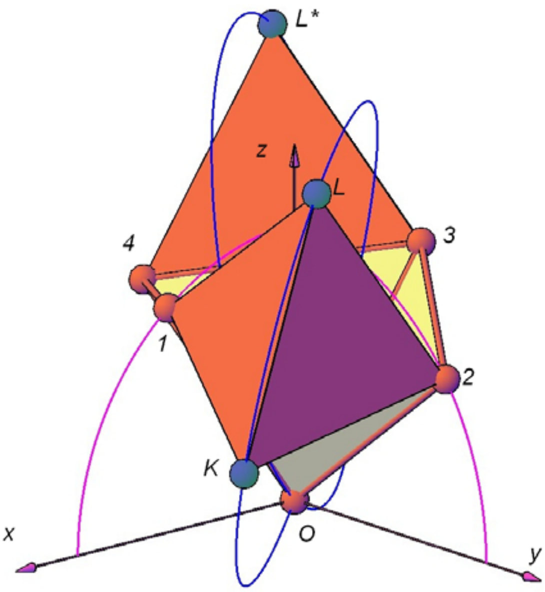

Figure 7

Figure 8

4. INTERLINKING THE TWO TETRAHEDRA

Now we want to interlink the 'free vertices' of the two tetrahedra by using a pyramid like the first one. The free ends *K, L, L*, K** of the configuration have to belong to the 'open base' of a further pyramid (just like the points *1,2,3,4* before). This base has to form a spatial quadrangle with equal side lengths. Two of its sides are the sides *KL* and *L*K**. Our construction guarantees their length being equal to 1. The other two sides of this spatial rhombic quadrangle yield the following conditions of equal length

$$\overline{KK^*}^2 = 1,\ \overline{LL^*}^2 = 1. \tag{9}$$

But not all spatial rhombic quadrangles are valid as 'open base' of such a pyramid. Our pyramid has four regular triangles as facets. The diagonals of the open base have lengths $l_1 := \overline{KL^*}$ and $l_2 := \overline{K^*L}$. According to formula (3) the base fits to a suitable pyramid exactly if the following condition holds:

$$4 = (4 - l_1^2)(4 - l_2^2) = (4 - \overline{KL^*}^2)(4 - \overline{K^*L}^2). \tag{10}$$

The two equations (9) and equation (10) together with (2) describe the possible positions of the mechanism. Our description of the mechanism uses the four variables α, β, u, v. So we would expect only single solutions of these four equations. But, surprisingly, we will see that the four equations have at least a one-parametric family of solutions. Thus, the mechanism enjoys some self-motion which is at least one-parametric.

5. POSSIBLE POSES

Now we are looking for possible poses of the second 'open pyramid': This means to determine admissible values of the four variables α, β, u, v such that conditions (2), (9) and (10) hold. We compute

$$\begin{aligned} &0 = \overline{KK^*}^2 - 1 = D(v)(1-\cos u) + E(v)\sin u + F(v) \qquad \text{with} \\ &D(v) := (3/2) - 2A(1-\cos v) - 2B\sin v, \\ &E(v) := -2B(1-\cos v) - 2C\sin v, \\ &F(v) := (1 - 3\cos v)/2. \end{aligned} \tag{11}$$

Substituting $u \to u + s$ and $v \to v + \varepsilon s$ into equation (11) yields a similar expression

$$\begin{aligned} 0 = \overline{LL^*}^2 - 1 &= D(v+\varepsilon s)\,(1-\cos(u+s)) + E(v+\varepsilon s)\,\sin(u+s) + F(v+\varepsilon s) =. \\ &= D^*(v)(1-\cos u) + E^*(v)\sin u + F^*(v) \qquad \text{with} \\ D^*(v) &:= [D(v+\varepsilon s) - 2\sqrt{2}\,E(v+\varepsilon s)]/3, \\ E^*(v) &:= [2\sqrt{2}\,D(v+\varepsilon s) + E(v+\varepsilon s)]/3, \\ F^*(v) &:= [2D(v+\varepsilon s) + 2\sqrt{2}\,E(v+\varepsilon s) + 3F(v+\varepsilon s)]/3. \end{aligned} \tag{12}$$

The equations (11) and (12) are linear in $\{1, \cos u, \sin u\}$ and $\{1, \cos v, \sin v\}$, resp. They can be solved for $\{1-\cos u, \sin u\}$ and give

$$\tan(u/2) = \frac{1-\cos u}{\sin u} = \frac{E^*(v+\varepsilon s)F(v) - E(v)F^*(v+\varepsilon s)}{F^*(v+\varepsilon s)D(v) - F(v)D^*(v+\varepsilon s)}. \tag{13}$$

This allows to eliminate the variable *u*. The remaining equations are (2), (13) and (10) with the substitutions in place. We further put $t := \tan(v/2)$. Then (11) and (13) yield two polynomial equations in *t* with degrees 8 and 12, respectively. As the result is lengthly we don not display it here. A possible self-motion of the mechanism belongs to a common (non-constant) zero of the two polynomial equations. There we have to use equation (2) as linking condition for α and β.

Now we take a look at such common zeros of the two polynomial equations. At this point our considerations split into two cases depending on the assembly mode of the mechanism:

CASE A - $\varepsilon = +1$: The polynomial equations have one common real root depending on the variable α. Figure 8 displays a graph of this zero interpreted as a function $t = t(\alpha)$ of the angle α. Together with the solution $\beta = \arcsin(1/(2\sin\alpha))$ of (2) and *u* from (13) we have a one-parametric self-motion to this mechanism assembly mode.

CASE B - $\varepsilon = -1$: The use a computer algebra system allows to show that the two polynomial equations do not share any real root (depending on the variable α). For this assembly mode there are no self-motions of the mechanism. The mechanism is stiff at any of its position.

We sum up:

Theorem 1: *The described kinematic chain of 16 regular triangles has two different assembly modes. Only one of them admits a one-parametric self-motion. The second assembly mode is fixed at any of its positions.*

Remark: It is quite remarkable, that the mechanism behaves differently in each of its two assembly modes.

CONCLUSIONS

We presented and studied a particular chain of 16 regular triangles. The triangles formed the faces of two regular tetrahedra and two ‘open pyramids’. We studied kinematic properties of this chain where some of the triangles were interlinked by rotational joints. We were able to prove that this mechanism admits a one-parametric self-motion.

Firstly we studied the behavior of the parts and its two different assembly modes. Analytic methods led to the description of the poses by several equations. Surprisingly, this overconstrained system of equations led to different behavior for the different assembly modes of the kinematic chain. For one assembly mode we were able to detect a non-constant common solution of these equations. For the second assembly mode such a non-constant solution does not exist. This common solution implies a one-parametric self-motion for the corresponding assembly mode. The movability of this overconstrained mechanism is all the more astonishing, as the second assembly mode behaves differently.

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