Spectral Processing

Misha Kazhdan

[Taubin, 1995] A Signal Processing Approach to Fair Surface Design
[Desbrun, *et al.*, 1999] Implicit Fairing of Arbitrary Meshes…
[Vallet and Levy, 2008] Spectral Geometry Processing with Manifold Harmonics
[Bhat *et al.*, 2008] Fourier Analysis of the 2D Screened Poisson Equation…
And much, much, much, more…

Outline

- Motivation
- Laplacian Spectrum
- Applications
- Conclusion

Motivation

Recall:

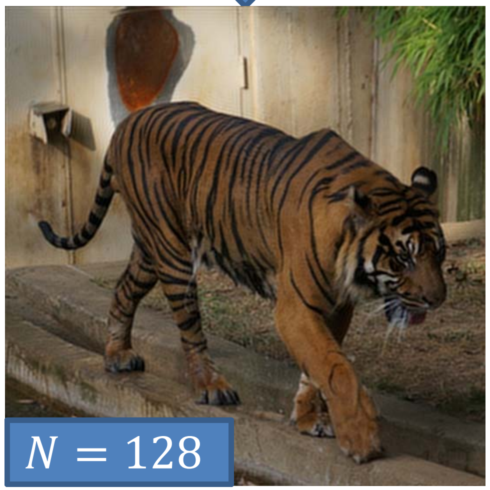
Given a signal, $f: [0,2\pi) \to \mathbb{R}$, we can write it out in terms of its *Fourier decomposition*:

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}[k] \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$

$\hat{f}[k] \in \mathbb{C}$ is the k-th *Fourier coefficients* of f.

Motivation

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}[k] \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$



Frequency Decomposition:

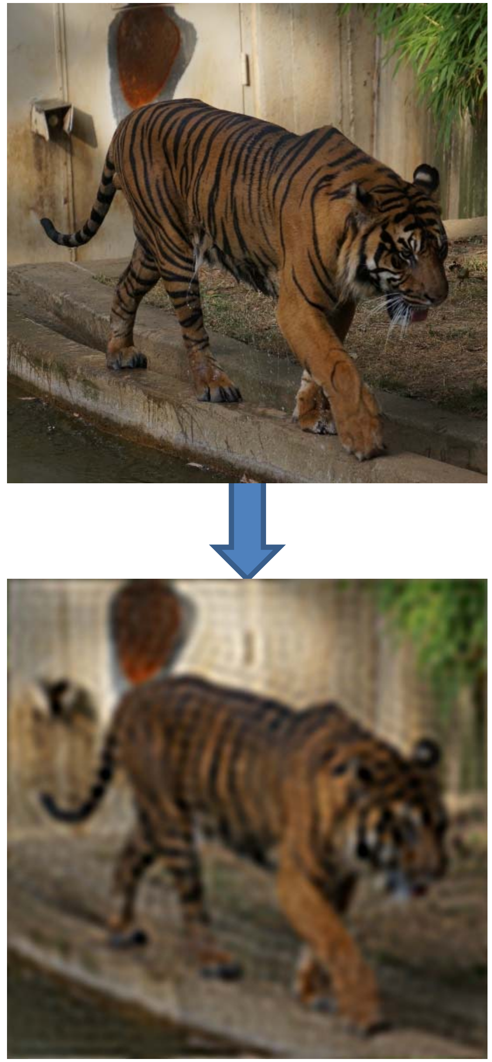
For smaller $N \in \mathbb{Z}$, the finite sum:

$$f_N(\theta) = \sum_{k=-N}^{N} \hat{f}[k] \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$

represents the lower frequency components of f.

Motivation

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}[k] \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$



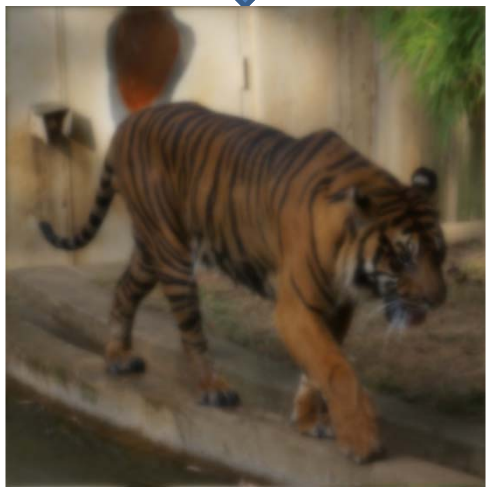
Filtering:

By modulating the values of $\hat{f}[k]$ as a function of frequency, we can realize different signal filters:

$$\hat{f}[k] \leftarrow \begin{cases} \hat{f}[k] & \text{if } |k| < N \\ 0 & \text{otherwise} \end{cases}$$

Motivation

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}[k] \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$



<u>Filtering</u>:

By modulating the values of $\hat{f}[k]$ as a function of frequency, we can realize different signal filters:

$$\hat{f}[k] \leftarrow \hat{f}[k] \cdot e^{-k^2}$$

Motivation

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}[k] \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$

Filtering:

By modulating the values of $\hat{f}[k]$ as a function of frequency, we can realize different signal filters:

$$\hat{f}[k] \leftarrow \hat{f}[k] \cdot \left(2 - e^{-k^2}\right)$$

Motivation

Goal:

We would like to extend this type of processing to the context of signals defined on surfaces*:

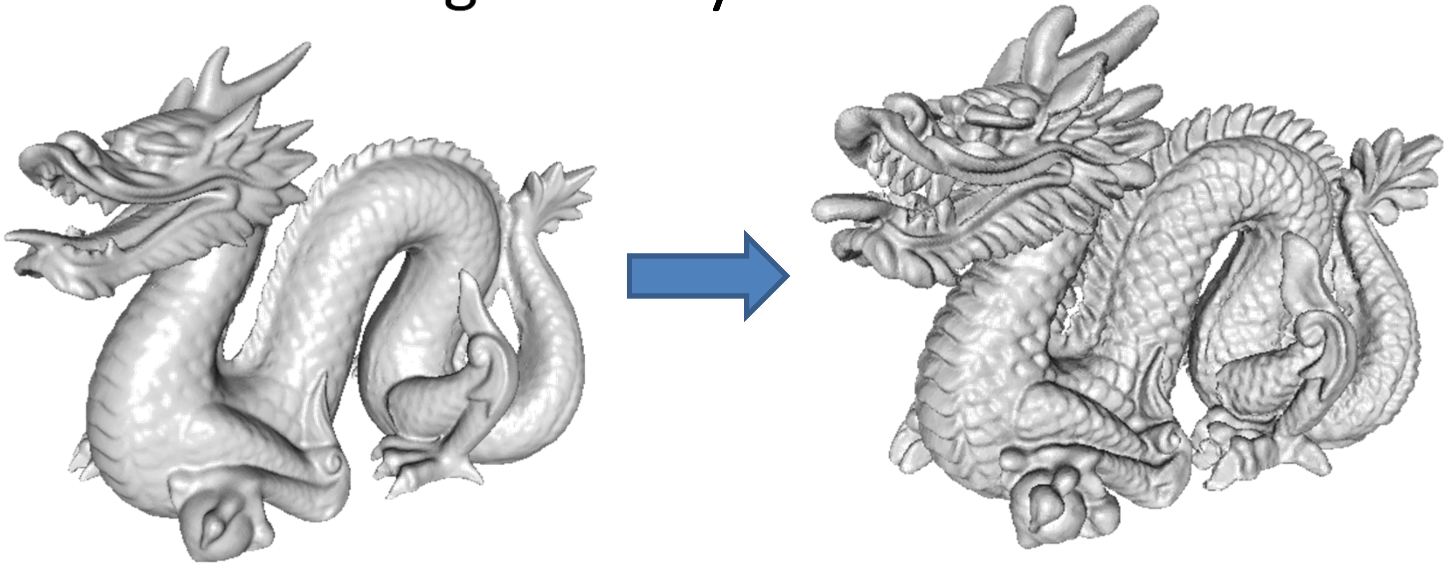
$$\hat{f}[k] \leftarrow \hat{f}[k] \cdot \left(2 - e^{-k^2}\right)$$

*For simplicity, we will assume all surfaces are w/o boundary.

Motivation

Goal:

We would like to extend this type of processing to the context of signals defined on surfaces* and even to the geometry of the surface itself:



$$\hat{f}[k] \leftarrow \hat{f}[k] \cdot \left(2 - e^{-k^2}\right)$$

*For simplicity, we will assume all surfaces are w/o boundary.

Motivation

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}[k] \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$

[WARNING]:

– In Euclidean space we can use the FFT to obtain the Fourier decomposition efficiently.

– For signals on surfaces, this is more challenging.

Outline

- Motivation
- Laplacian Spectrum
  - Fourier ↔ Laplacian
  - FEM discretization
- Applications
- Conclusion

How do we obtain the Fourier decomposition?

Fourier $\leftrightarrow$ Laplacian

Recall:

In Euclidean space, the *Laplacian*, is the operator that takes a function and returns the sum of (unmixed) second partial derivatives:

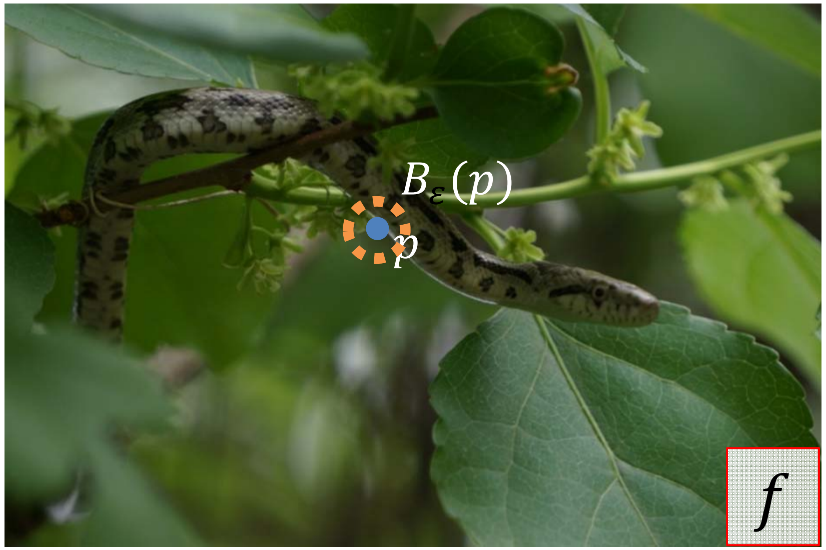
$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \cdots$$

Fourier ↔ Laplacian

Informally:

The Laplacian gives the difference between the value at a point and the average in the vicinity:

$$\Delta f(p) \sim \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\left(\mathrm{Avg}_{B_\varepsilon(p)}(f) - f(p)\right)$$



Fourier $\leftrightarrow$ Laplacian

Note:

The complex exponential $f(\theta) = \frac{e^{ik\theta}}{\sqrt{2\pi}}$ has Laplacian:

$$\frac{\partial^2}{\partial\theta^2}\left(\frac{e^{ik\theta}}{\sqrt{2\pi}}\right)$$

$$\Downarrow$$

$f(\theta) = \frac{e^{ik\theta}}{\sqrt{2\pi}}$ is an eigenfunction of the Laplacian with eigenvalue $-k^2$.

Fourier $\leftrightarrow$ Laplacian

<u>Note</u>:

Similarly, $f(\theta,\phi)=\frac{e^{ik\theta}\cdot e^{il\phi}}{2\pi}$ has Laplacian:

$$\left(\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial\phi^2}\right)\left(\frac{e^{ik\theta}\cdot e^{il\phi}}{2\pi}\right)=-(k^2+l^2)\cdot\left(\frac{e^{ik\theta}\cdot e^{il\phi}}{2\pi}\right)$$

$$\Downarrow$$

$f(\theta,\phi)=\frac{e^{ik\theta}\cdot e^{il\phi}}{2\pi}$ is an eigenfunction of the Laplacian with eigenvalue $-(k^2+l^2)$.

Fourier $\leftrightarrow$ Laplacian

Approach:

- Though we cannot compute the FFT for signals on general surfaces, we can define a Laplacian.
- To compute the Fourier decomposition of a signal, f, on a mesh we decompose f as the linear combination of eigenvectors of the Laplacian:

$$f(x) = \sum_{i=1}^{n} \hat{f}[i] \cdot \phi_i(x) \quad \text{with} \quad \Delta\phi_i = \lambda_i \cdot \phi_i.$$

This is called the *harmonic decomposition* of f.

Fourier $\leftrightarrow$ Laplacian

How do we know the eigenvectors of the Laplacian form a basis?

Claims:

1. The Laplacian is a symmetric operator.
2. The eigenvectors of a symmetric operator form an orthogonal basis (and have real eigenvalues).

Fourier $\leftrightarrow$ Laplacian

Preliminaries (1):

- [Definition of the Laplacian]

$$\Delta f = \mathrm{div}(\nabla f)$$

- [Product Rule]

$$\mathrm{div}(f \cdot \vec{v}) = f \cdot \mathrm{div}(\vec{v}) + \langle \nabla f, \vec{v} \rangle$$

- [Inner Product on Functions]
  Given a surface $S \subset \mathbb{R}^3$:

$$\langle f, g \rangle_S = \int_S f(x) \cdot g(x)\, dx$$

- [Divergence Theorem*]

$$\int_S [\mathrm{div}(\vec{v})](p) = \int_{\partial S} \langle \vec{v}(s), \vec{n}(s) \rangle\, ds$$

Fourier $\leftrightarrow$ Laplacian

<u>Preliminaries (2)</u>:

– [Lagrange Multipliers]
The constrained maximizer:
$$\arg\max_x E(x) \ \text{ s.t. } f(x) = c$$
is obtained when the gradient of E is perpendicular to the contour lines of f:
$$\nabla E = \lambda \nabla g$$

Symmetry of The Laplacian

1. <u>The Laplacian is a symmetric operator</u>

Given a surface $S \subset \mathbb{R}^3$, we want to show that for any functions $f, g: S \to \mathbb{R}$ we have:

$$\langle \Delta f, g \rangle_S = \langle f, \Delta g \rangle_S$$

$$\Updownarrow$$

$$\int_S \Delta f \cdot g \, dx = \int_S f \cdot \Delta g \, dx$$

Symmetry of The Laplacian

Proof:

By the definition of the Laplacian:

$$\Delta f = \mathrm{div}(\nabla f)$$

$$\begin{aligned}
\langle \Delta f, g \rangle_S &= \int_S \Delta f \cdot g \, dx \\
&= \int_S \mathrm{div}(\nabla f) \cdot g \, dx \\
&= \int_S \left(\mathrm{div}(g \cdot \nabla f) - \langle \nabla f, \nabla g \rangle\right) dx \\
&= -\int_S \langle \nabla f, \nabla g \rangle \, dx
\end{aligned}$$

Symmetry of The Laplacian

Proof:

By the product rule:

$$\begin{aligned}
\langle \Delta f, g\rangle_S &= \int_S \Delta f \cdot g \, dx \\
&= \int_S \mathrm{div}(\nabla f) \cdot g \, dx \\
&= \int_S \left(\mathrm{div}(g \cdot \nabla f) - \langle \nabla f, \nabla g\rangle\right) dx \\
&= -\int_S \langle \nabla f, \nabla g\rangle \, dx
\end{aligned}$$

Symmetry of The Laplacian

<u>Proof</u>:

By the Divergence Theorem*:

$$\int_S [\mathrm{div}(\vec{v})](p) = \int_{\partial S} \langle \vec{v}(s), \vec{n}(s) \rangle \, ds = 0$$

$$\begin{aligned}
\langle \Delta f, g \rangle_S &= \int_S \Delta f \cdot g \, dx \\
&= \int_S \mathrm{div}(\nabla f) \cdot g \, dx \\
&= \int_S \left(\mathrm{div}(g \cdot \nabla f) - \langle \nabla f, \nabla g \rangle \right) dx \\
&= -\int_S \langle \nabla f, \nabla g \rangle \, dx
\end{aligned}$$

Spectra of Symmetric Operators

2. <u>The e.vectors of a symmetric operator form an orthogonal basis (and have real e.values)</u>

We show this in two steps:

I. There always exists at least one real eigenvector.

II. If v is an eigenvector, then the space of vectors perpendicular to v is fixed by the operator.

$$\Downarrow$$

We can restrict to the subspace perpendicular to the found eigenvectors and recurse.

Spectra of Symmetric Operators

Proof (II):

Suppose that v is an eigenvector of a symmetric operator M and w is orthogonal to v:

$$\langle v, w \rangle = 0$$

Since v is an eigenvector, this implies that:

$$\langle Mv, w \rangle = \langle \lambda v, w \rangle = \lambda \langle v, w \rangle = 0$$

Since M is symmetric, we have:

$$\langle v, Mw \rangle = \langle Mv, w \rangle = 0$$

$\Rightarrow$ The space of vectors orthogonal to v stays orthogonal after applying M.

Spectra of Symmetric Operators

<u>Proof (I)</u>:

Consider the constrained maximization

$$\arg\max_v E(v) = \langle v, Mv \rangle \quad \text{s.t.} \quad \|v\|^2 = 1$$

The sphere is compact so a maximizer v_0 exists.

At the maximizer, we have:

$$\nabla E = \lambda \nabla \|v_0\|^2$$

$$\Downarrow$$

$$2Mv_0 = 2\lambda v_0$$

$\Rightarrow v_0$ is an eigenvector with (real) eigenvalue λ.

What happens in the discrete setting?

FEM Discretization

1. To enable computation, we restrict ourselves to a *finite-dimensional* space of functions, spanned by basis functions $\{B_i : S \to \mathbb{R}\}_{i=1}^{n}$.

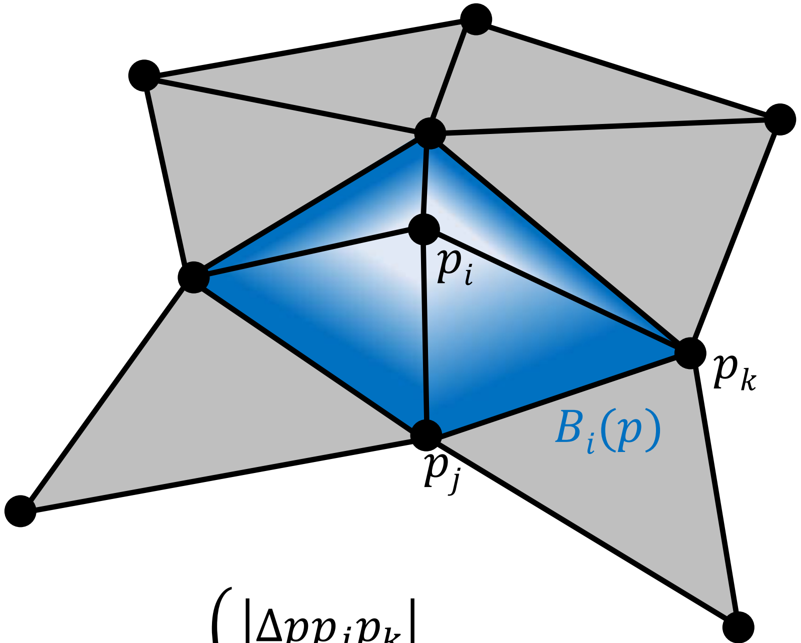
Often these are defined to be the hat functions centered at vertices.

– Piecewise linear

$\Rightarrow$ Gradients are constant within each triangle

– Interpolatory

$\Rightarrow B_i(p_j) = \delta_{ij}$



$$B_i(p) = \begin{cases} \dfrac{|\Delta p p_j p_k|}{|\Delta p_i p_j p_k|} & \text{if } p \in \Delta p_i p_j p_k \\ 0 & \text{otherwise} \end{cases}$$

FEM Discretization

1. To enable computation, we restrict ourselves to a *finite-dimensional* space of functions, spanned by basis functions $\{B_i : S \to \mathbb{R}\}_{i=1}^n$.

Having chosen a basis, we can think of a vector $\vec{f} \in \mathbb{R}^n$ as a "discrete" function:

$$\vec{f} \leftrightarrow f(p) = \sum_{i=1}^{n} f_i \cdot B_i(p)$$

If we use the hat functions as a basis, then:

$$f(p_j) = \sum_{i=1}^{n} f_i \cdot B_i(p_j)$$

FEM Discretization

1. To enable computation, we restrict ourselves to a *finite-dimensional* space of functions, spanned by basis functions $\{B_i : S \to \mathbb{R}\}_{i=1}^{n}$.

[WARNING]:

In general, given:

- $\mathcal{L}$: A continuous linear operator
- $\vec{f} \in \mathbb{R}^n \leftrightarrow f(p)$: A discrete function

The function $\mathcal{L}(f)$ will *not* be in the space of functions spanned by $\{B_i : S \to \mathbb{R}\}_{i=1}^{n}$.

FEM Discretization

1. To enable computation, we restrict ourselves to a *finite-dimensional* space of functions, spanned by basis functions $\{B_i : S \to \mathbb{R}\}_{i=1}^{n}$.
2. Given a continuous linear operator $\mathcal{L}$, we discretize the operator by *projecting*:

$$g = \mathcal{L}(f)$$
$$\Downarrow$$
$$\langle g, B_j \rangle_S = \langle \mathcal{L}(f), B_j \rangle_S \quad \forall j$$

FEM Discretization

$$\langle g, B_j \rangle_S = \langle \mathcal{L}(f), B_j \rangle_S \quad \forall j$$

Writing out the discrete functions:

$$g(p) = \sum_{i=1}^{n} g_i \cdot B_i(p) \quad \text{and} \quad f(p) = \sum_{i=1}^{n} f_i \cdot B_i(p)$$

$$\Downarrow$$

$$\sum_{i=1}^{n} g_i \cdot \langle B_i, B_j \rangle_S = \sum_{i=1}^{n} f_i \cdot \langle \mathcal{L}(B_i), B_j \rangle_S \quad \forall j$$

FEM Discretization

$$\sum_{i=1}^{n} g_i \cdot \langle B_i, B_j \rangle_S = \sum_{i=1}^{n} f_i \cdot \langle \mathcal{L}(B_i), B_j \rangle_S \qquad \forall j$$

Setting M and L to be the matrices:

$$M_{ij} = \langle B_i, B_j \rangle_S \quad \text{and} \quad L_{ij} = \langle \mathcal{L}(B_i), B_j \rangle_S$$

$$\Downarrow$$

$$\sum_{i=1}^{n} M_{ij} \cdot g_i = \sum_{i=1}^{n} L_{ij} \cdot f_i \qquad \forall j$$

$$\Downarrow$$

$$M\vec{g} = L\vec{f}$$

FEM Discretization

$$M_{ij} = \langle B_i, B_j \rangle_S \quad \text{and} \quad L_{ij} = \langle \mathcal{L}(B_i), B_j \rangle_S$$

Both the mass and stiffness matrices are symmetric and positive (semi)-definite.

When $\mathcal{L} = \Delta$, we have:

$$L_{ij} = \langle \Delta B_i, B_j \rangle_S$$

<u>Definition</u>:

The matrix M is called the *mass matrix*.

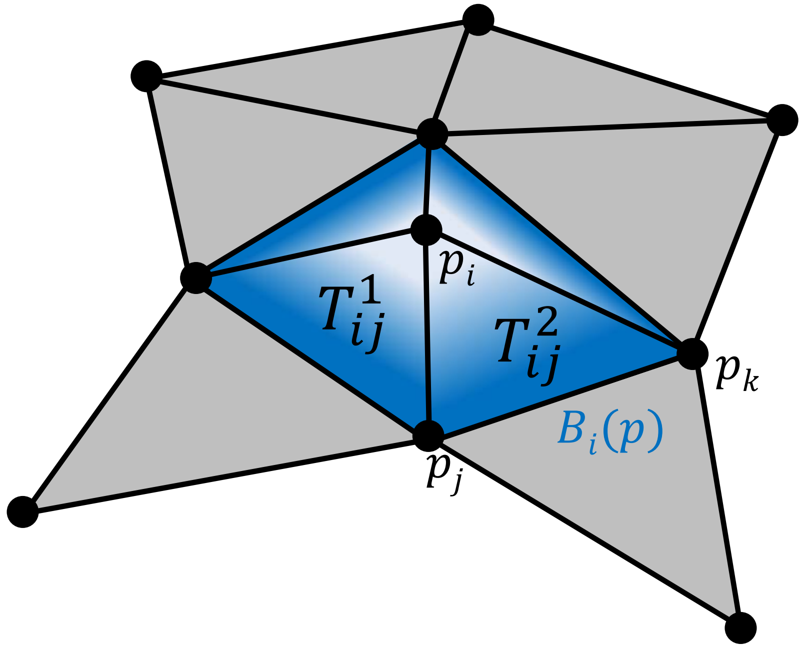
The matrix $-L$ is called the *stiffness matrix*.

FEM Discretization

$$M_{ij} = \langle B_i, B_j \rangle_S \quad \text{and} \quad L_{ij} = -\langle \nabla B_i, \nabla B_j \rangle_S$$

Setting $\{B_i : \to \mathbb{R}\}$ to the hat functions, the matrix M is:

$$M_{ij} = \begin{cases} \dfrac{\left|T_{ij}^1\right| + \left|T_{ij}^2\right|}{12} & \text{if } j \in N(i) \\ \displaystyle\sum_{k \in N(i)} M_{ik} & \text{if } i = j \end{cases}$$

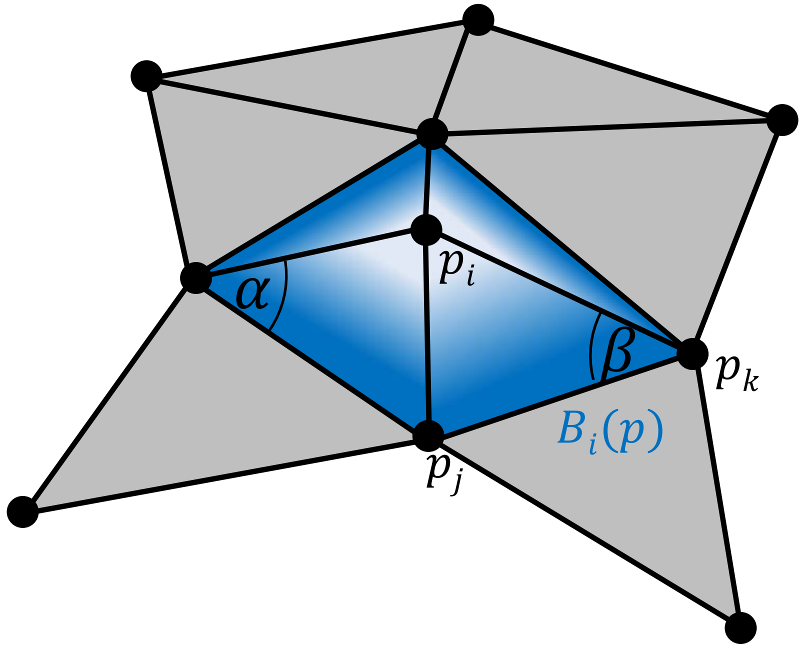


FEM Discretization

$$M_{ij} = \langle B_i, B_j \rangle_S \quad \text{and} \quad L_{ij} = -\langle \nabla B_i, \nabla B_j \rangle_S$$

Setting $\{B_i : \to \mathbb{R}\}$ to the hat functions, the matrix L is the cotangent-Laplacian:

$$L_{ij} = \begin{cases} \cot\alpha + \cot\beta & \text{if } j \in N(i) \\ -\sum_{k \in N(i)} L_{ik} & \text{if } i = j \end{cases}$$

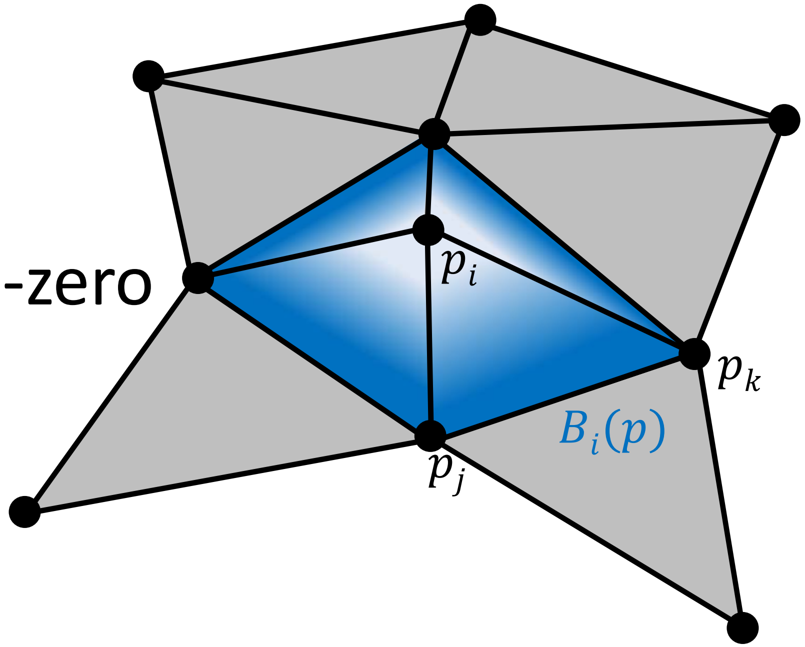


FEM Discretization

$$M_{ij} = \begin{cases} \dfrac{|T_{ij}^1| + |T_{ij}^2|}{12} & \boxed{\text{if } j \in N(i)} \\ \displaystyle\sum_{k \in N(i)} M_{ik} & \text{if } i = j \end{cases} \quad \text{and} \quad L_{ij} = \begin{cases} \cot\alpha + \cot\beta & \boxed{\text{if } j \in N(i)} \\ -\displaystyle\sum_{k \in N(i)} L_{ik} & \text{if } i = j \end{cases}$$

<u>Observations</u>:

– [Sparsity]
Entry (i, j) can only be non-zero if vertex i and vertex j are neighbors in the mesh.

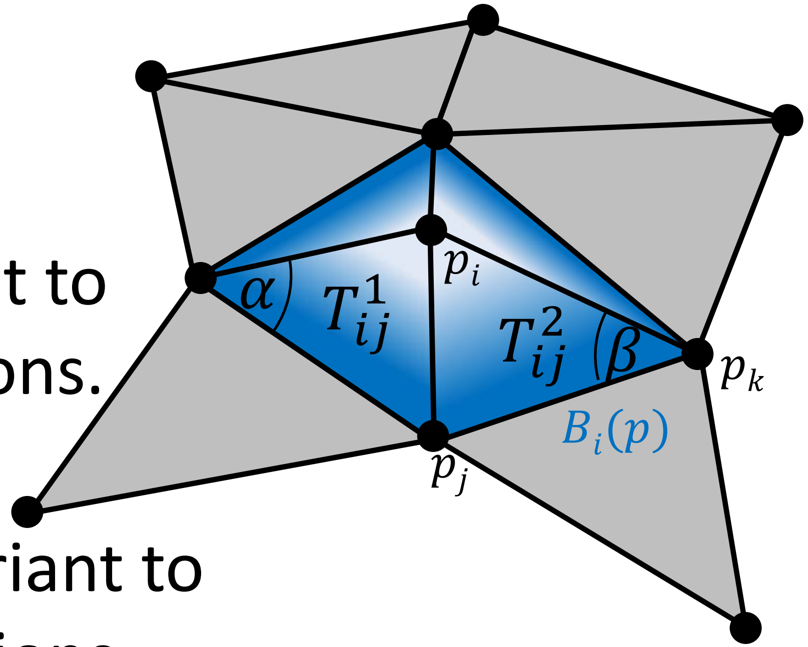


FEM Discretization

$$M_{ij} = \begin{cases} \dfrac{|T_{ij}^1| + |T_{ij}^2|}{12} & \text{if } j \in N(i) \\ \displaystyle\sum_{k \in N(i)} M_{ik} & \text{if } i = j \end{cases} \quad \text{and} \quad L_{ij} = \begin{cases} \cot\alpha + \cot\beta & \text{if } j \in N(i) \\ -\displaystyle\sum_{k \in N(i)} L_{ik} & \text{if } i = j \end{cases}$$

<u>Observations</u>:

- [Authalicity]
  The mass matrix is invariant to area-preserving deformations.
- [Conformality]
  The stiffness matrix is invariant to angle-preserving deformations.



FEM Discretization

[WARNING]:

Given a discrete function $\vec{f} \leftrightarrow f(p)$, the vector:

$$\vec{g} = L\vec{f}$$

Often, the matrix M is approximated by the diagonal *lumped mass matrix*, making inversion simple.

In the literature, the operator $M^{-1}L$ is sometimes called the *normalized* Laplacian.

$$\vec{g} = M^{-1}L\vec{f}$$

FEM Discretization

<u>Laplacian Spectrum</u>:

In the continuous setting, the spectrum of the Laplacian, $\{(\phi_i : S \to \mathbb{R}, -\lambda_i \in \mathbb{R}^{\geq 0})\}$, satisfies:

$$\Delta\phi_i = -\lambda_i \cdot \phi_i$$

And the $\{\phi_i\}$ form an orthonormal basis:

$$\langle \phi_i, \phi_j \rangle_s = \int_S \phi_i(p) \cdot \phi_j(p)\, dp = \delta_{ij}$$

FEM Discretization

Laplacian Spectrum:

In the discrete setting, the spectrum of the Laplacian, $\left\{\left(\vec{\phi}_i \in \mathbb{R}^n, -\lambda_i \in \mathbb{R}^{\geq 0}\right)\right\}$, satisfies:

$$L\vec{\phi}_i = -\lambda_i \cdot M\vec{\phi}_i$$

And the $\left\{\vec{\phi}_i\right\}$ form an orthonormal basis:

$$\langle \vec{\phi}_i, \vec{\phi}_j \rangle_s = \vec{\phi}_i^T \cdot M \cdot \vec{\phi}_j = \delta_{ij}$$

Finding the $\left\{\left(\vec{\phi}_i, \lambda_i\right)\right\}$ is called the *generalized eigenvalue problem.*

The Spectrum of the Laplacian

Interpreting the Eigenvalues:

If ϕ is a (unit-norm) eigenfunction of the Laplacian, with eigenvalue λ:

$$\Delta\phi = \lambda \cdot \phi$$
$$\Downarrow$$
$$\langle \Delta\phi, \phi \rangle_S = \lambda \cdot \langle \phi, \phi \rangle_S$$
$$\Downarrow$$
$$-\langle \nabla\phi, \nabla\phi \rangle_S = \lambda$$

$\Rightarrow$ The eigenvalue λ is a measure of how much ϕ changes, i.e. the frequency of ϕ.

How do we compute the spectral decomposition?

Getting the Dominant Eigenvector

Assume that a matrix A is diagonalizable, with (unit-norm) spectrum $\{(\vec{\phi}_i, \lambda_i)\}$.

Given $\vec{v}$, we have the harmonic decomposition:

$$\vec{v} = \sum_{i=1}^{n} \hat{v}[i] \cdot \vec{\phi}_i$$

$$\Rightarrow \qquad A\vec{v} = \sum_{i=1}^{n} \hat{v}[i] \cdot \lambda_i \cdot \vec{\phi}_i$$

$$\Rightarrow \qquad A^k\vec{v} = \sum_{i=1}^{n} \hat{v}[i] \cdot \lambda_i^k \cdot \vec{\phi}_i$$

Getting the Dominant Eigenvector

$$A^k \vec{v} = \sum_{i=1}^{n} \hat{v}[i] \cdot \lambda_i^k \cdot \vec{\phi}_i$$

Without loss of too much generality, assume λ_n is the largest eigenvalue, $|\lambda_i / \lambda_n| < 1$ for $i \neq n$.

$$A^k \vec{v} = \lambda_n^k \cdot \sum_{i=1}^{n} \hat{v}[i] \cdot \left(\frac{\lambda_i}{\lambda_n}\right)^k \cdot \vec{\phi}_i$$

Then $(\lambda_i / \lambda_n)^k \to 0$ as $k \to \infty$, for $i \neq n$.

$$\Rightarrow \qquad \frac{A^k \vec{v}}{|A^k \vec{v}|} \to \vec{\phi}_n \quad \text{as} \quad k \to \infty$$

Getting the Dominant Eigenvector

<u>ArnoldiDominant</u>($A \in \mathbb{R}^n \times \mathbb{R}^n$)

1. $\vec{v} \leftarrow$ RandomVector( )
2. while(…)
3. $\quad \vec{v} \leftarrow A\vec{v}$
4. $\quad \vec{v} \leftarrow \vec{v}/|\vec{v}|$
5. $\lambda \leftarrow \langle A\vec{v}, \vec{v} \rangle$
6. return ($\vec{v}$, λ)

Getting the Sub-Dominant Eigenvector

If the matrix A is symmetric, the eigenvectors will be orthogonal:

<u>ArnoldiSubDominant</u>($A \in \mathbb{R}^n \times \mathbb{R}^n$)

1. $(\vec{v}_0, \lambda_0) \leftarrow$ ArnoldiDominant(A)
2. $\vec{v}_1 \leftarrow$ RandomVector( )
3. while(…)
4. $\quad\quad \vec{v}_1 \leftarrow A\vec{v}_1$
5. $\quad\quad \vec{v}_1 \leftarrow \vec{v}_1 - \langle \vec{v}_1, \vec{v}_0 \rangle \cdot \vec{v}_0$

A similar approach can be applied to:

- Solving the generalized eigenvalue problem
- Finding the eigenvectors with smallest eigenvalues
- Finding the eigenvectors with eigenvalues closest to λ

Outline

- Motivation
- Laplacian Spectrum
- Applications
  - Signal/Geometry Filtering
  - Partial Differential Equations
  - Complexity and Approximation
- Conclusion

Signal/Geometry Filtering

HarmonicDecomposition($S \subset \mathbb{R}^3$, $\vec{f} \in \mathbb{R}^n$)

1. $(M, L) \leftarrow$ MassAndStiffness (S)
2. $\left\{\left(\vec{\phi}_i, -\lambda_i\right)\right\}_{i=1}^{n} \leftarrow$ GeneralizedEigen(M , L)
3. For each $i \in [1, n]$:
4. $\quad \hat{f}[i] \leftarrow \langle \vec{f}, \vec{\phi}_i \rangle_S$

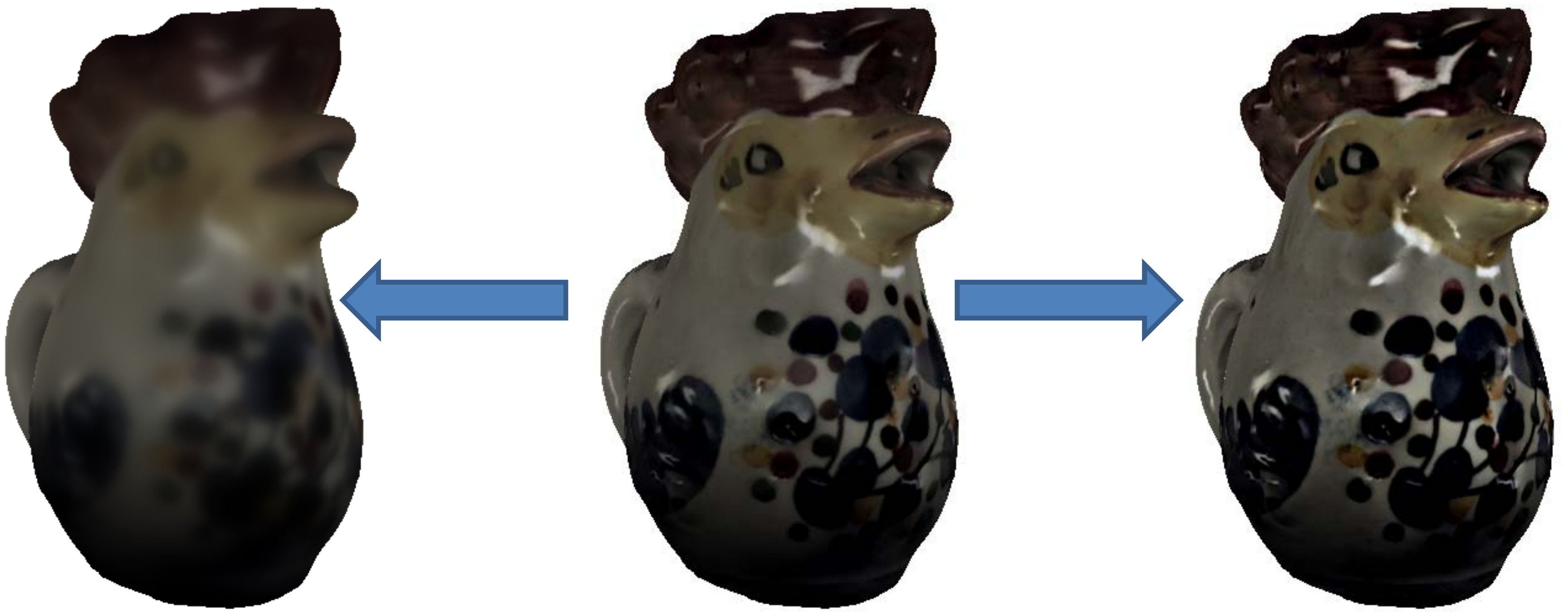
$$\boxed{\vec{f}^T \cdot M \cdot \vec{\phi}_i}$$

Process($F: \mathbb{R} \to \mathbb{R}$)

1. $\vec{g} \leftarrow 0$
2. For each $i \in [1, n]$:
3. $\quad \hat{g}[i] \leftarrow \hat{f}[i] \cdot F(\lambda_i)$
4. $\quad \vec{g} \leftarrow \vec{g} + \hat{g}[i] \cdot \vec{\phi}_i$
5. return $\vec{g}$

Signal Filtering

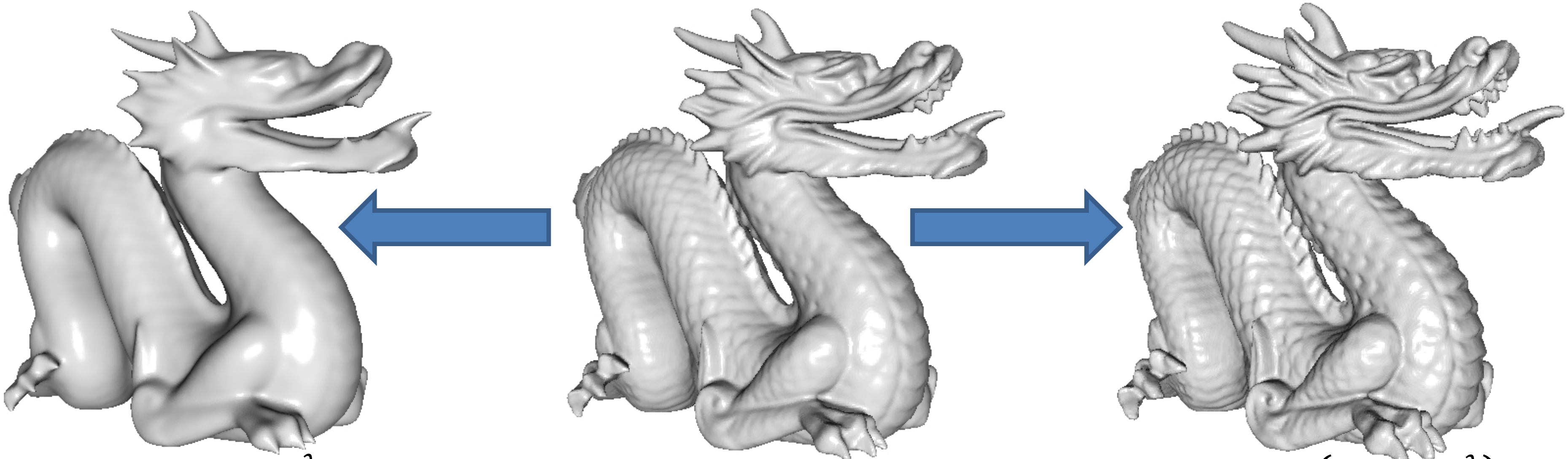
Given a color at each vertex, we can modulate the frequency coefficients of each channel to smooth/sharpen the colors.



$F(\lambda) = e^{-\lambda}$ Input $F(\lambda) = \left(2 - e^{-\lambda}\right)$

Geometry Filtering

Using the position of the vertices as the signal, we can modulate the frequency coefficients of each coordinate to smooth/sharpen the shape.



$F(\lambda) = e^{-\lambda}$ Input $F(\lambda) = \left(2 - e^{-\lambda}\right)$

Partial Differential Equations

Recall:

The Laplacian of a function at a point $p \in S$ is the difference between the value at p and the average value of its neighbors.

Heat Diffusion

<u>Newton's Law of Cooling</u>:

The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.

Translating this into the PDE, if $H(p,t)$ is the heat at position $p \in S$ at time t, then:

$$\frac{\partial H}{\partial t} = \eta \cdot \Delta H$$

Heat Diffusion

Newton's Law of Cooling:

The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.

Goal:

Given an initial heat distribution $h: S \to \mathbb{R}$, find the solution to the PDE:

$$\frac{\partial H}{\partial t} = \eta \cdot \Delta H$$

such that:

$$H(p, 0) = h(p)$$

Heat Diffusion

$$\boxed{\frac{\partial H}{\partial t} = \eta \cdot \Delta H}$$

Note:

Let $\{(\phi_i, -\lambda_i)\}_{i=1}^{n}$ be the Laplacian spectrum.

$\Rightarrow$ The functions:

$$H_i(p,t) = e^{-\eta \cdot \lambda_i \cdot t} \cdot \phi_i(p)$$

are solutions to the PDE.

$\Rightarrow$ Any linear sum of the $H_i(p,t)$ is a solution.

Heat Diffusion

$$\boxed{\frac{\partial H}{\partial t} = \eta \cdot \Delta H}$$

Compute the harmonic decomposition of h:

$$h(p) = \sum_{i=1}^{n} \hat{h}[i] \cdot \phi_i(p)$$

Then consider the function:

$$H(p,t) = \sum_{i=0}^{n} \hat{h}[i] \cdot H_i(p,t) = \sum_{i=0}^{n} \hat{h}[i] \cdot e^{-\eta \cdot \lambda_i \cdot t} \cdot \phi_i(p)$$

– It is a solution to the heat equation.

– It satisfies $H(p,0) = h(p)$.

Heat Diffusion (Colors)

HarmonicDecomposition($S \subset \mathbb{R}^3$, $h: S \to \mathbb{R}$)

1. …

Process($t \in [0, \infty)$)

1. $\vec{g} \leftarrow 0$
2. For each $i \in [1, n]$:
3. $\quad \hat{g}[i] \leftarrow \hat{h}[i] \cdot e^{-\eta \cdot \lambda_i \cdot t}$
4. $\quad \vec{g} \leftarrow \vec{g} + \hat{g}[i] \cdot \vec{\phi}_i$
5. return $\vec{g}$

Heat Diffusion (Geometry)

<u>HarmonicDecomposition</u>($S \subset \mathbb{R}^3$, $h: S \to \mathbb{R}$)

1. …

<u>Process</u>($t \in [0, \infty)$)

1. $\vec{g} \leftarrow 0$
2. For each $i \in [1, n]$:
3. $\quad \hat{g}[i] \leftarrow \hat{h}[i] \cdot e^{-\eta \cdot \lambda_i \cdot t}$
4. $\quad \vec{g} \leftarrow \vec{g} + \hat{g}[i] \cdot \vec{\phi}_i$
5. return $\vec{g}$

Heat Diffusion (Geometry)

[WARNING]:

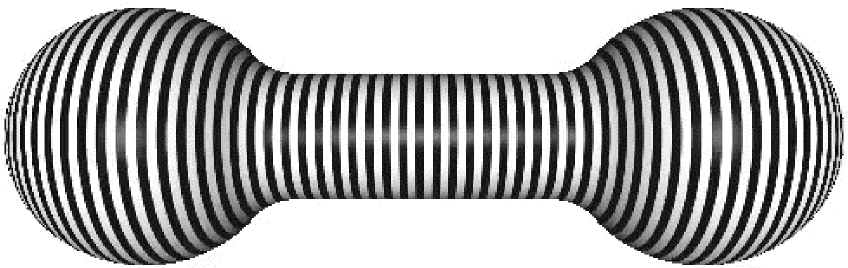
1. As the geometry diffuses, the areas and angles of the triangles change.

   ⇒ The mass and stiffness matrices change.

   ⇒ The harmonic decomposition changes.

   If we take this into account, we get a non-linear PDE called *mean curvature flow*.

2. Mean curvature flow can create singularities.



Wave Equation

The acceleration of a wave's height is proportional to the difference in height of the surrounding.

Translating this into the PDE, if $H(p,t)$ is the height at position $p \in S$ at time t, then:

$$\frac{\partial^2 H}{\partial t^2} = \eta \cdot \Delta H$$

Wave Equation

The acceleration of a wave's height is proportional to the difference in height of the surrounding.

Goal:

Given an initial height distribution $h: S \to \mathbb{R}$, find the solution to the PDE:

$$\frac{\partial^2 H}{\partial t^2} = \eta \cdot \Delta H$$

such that:

$$H(p,0) = h(p) \qquad \text{and} \qquad \frac{\partial H}{\partial t}(p,0) = 0$$

Wave Equation $\boxed{\frac{\partial^2 H}{\partial t^2} = \eta \cdot \Delta H}$

Note:

If $\{(\phi_i, -\lambda_i)\}_{i=1}^{n}$ are the eigenfunctions/values of the Laplacian, then:

$$H_i^c(p,t) = \cos\left(\sqrt{\eta \cdot \lambda_i} \cdot t\right) \cdot \phi_i(p)$$
$$H_i^s(p,t) = \sin\left(\sqrt{\eta \cdot \lambda_i} \cdot t\right) \cdot \phi_i(p)$$

are solutions to the PDE.

$\Rightarrow$ Any linear sum of the $H_i^c(p,t)$ and $H_i^s(p,t)$ is a solution to the PDE.

Wave Equation $\boxed{\frac{\partial^2 H}{\partial t^2} = \eta \cdot \Delta H}$

Compute the harmonic decomposition of h:

$$h(p) = \sum_{i=1}^{n} \hat{h}[i] \cdot \phi_i(p)$$

Then consider the function:

$$H(p,t) = \sum_{i=0}^{n} \hat{h}[i] \cdot H_i^c(p,t) = \sum_{i=0}^{n} \hat{h}[i] \cdot \cos\left(\sqrt{\eta \cdot \lambda_i} \cdot t\right) \cdot \phi_i(p)$$

– It is a solution to the wave equation.

– It satisfies $H(p,0) = h(p)$.

– It satisfies $\frac{\partial H}{\partial t}(p,0) = 0$.

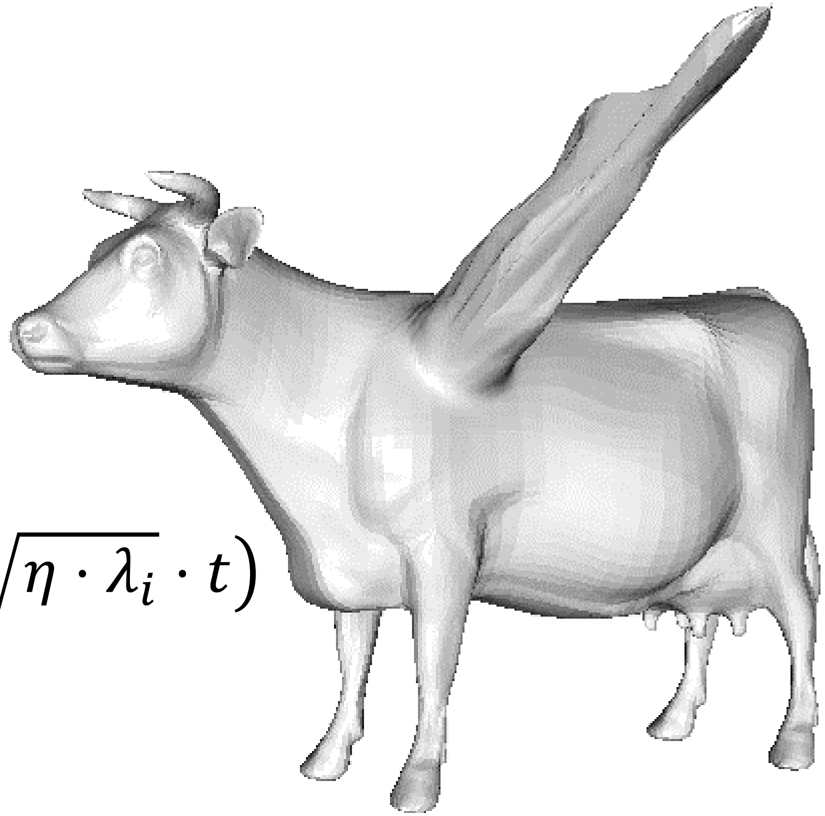
Wave Equation

HarmonicDecomposition($S \subset \mathbb{R}^3$, $h: S \to \mathbb{R}$)

1. …

Process($t \in [0, \infty)$)

1. $\vec{g} \leftarrow 0$
2. For each $i \in [1, n]$:
3. $\quad \hat{g}[i] \leftarrow \hat{h}[i] \cdot \cos\left(\sqrt{\eta \cdot \lambda_i} \cdot t\right)$
4. $\quad \vec{g} \leftarrow \vec{g} + \hat{g}[i] \cdot \vec{\phi}_i$
5. return $\vec{g}$



How practical is it to use the spectral decomposition?

Complexity

<u>Challenge</u>:

If we have a mesh with n vertices we get n generalized eigenvectors.

✘ $O(n^2)$ storage / $O(> n^2)$ computation.

<u>Approximate</u>:

Sometimes a low-frequency solution will do.

✓ $O(k \cdot n)$ storage

Sometimes a numerically inaccurate solution will do.

✓ $O(n)$ storage / $O(?)$ computation

Approximate Spectral Processing

<u>Preliminaries</u>:

Let M be the mass matrix, $-L$ the stiffness matrix, and $\{(\vec{\phi}_i, -\lambda_i)\}$ the spectrum, we have:

$$M\vec{\phi}_i = M\vec{\phi}_i \qquad\qquad -L\phi_i = \lambda_i \cdot M\phi_i$$

Taking α times the 1st equation plus β times the 2nd:

$$(\alpha \cdot M - \beta \cdot L)\vec{\phi}_i = (\alpha + \beta \cdot \lambda_i) \cdot M\vec{\phi}_i$$

Multiplying by $(\alpha \cdot M - \beta \cdot L)^{-1}$:

$$\vec{\phi}_i = (\alpha + \beta \cdot \lambda_i) \cdot \left((\alpha \cdot M - \beta \cdot L)^{-1} \circ M\right)\vec{\phi}_i$$

$$\frac{1}{(\alpha + \beta \cdot \lambda_i)} \cdot \vec{\phi}_i = \left((\alpha \cdot M - \beta \cdot L)^{-1} \circ M\right)\vec{\phi}_i$$

Approximate Spectral Processing

Preliminaries:

$$(\gamma + \delta \cdot \lambda_i) \cdot M\vec{\phi}_i = (\gamma \cdot M - \delta \cdot L)\vec{\phi}_i$$

$$\frac{1}{(\alpha + \beta \cdot \lambda_i)} \cdot \vec{\phi}_i = \left((\alpha \cdot M - \beta \cdot L)^{-1} \circ M\right)\vec{\phi}_i$$

Combining these, we get:

$$\left((\alpha \cdot M - \beta \cdot L)^{-1} \circ (\gamma \cdot M - \delta \cdot L)\right)\vec{\phi}_i =$$

$$= (\gamma + \delta \cdot \lambda_i) \cdot \left((\alpha \cdot M - \beta \cdot L)^{-1} \circ M\right)\vec{\phi}_i$$

$$= \frac{\gamma + \delta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i} \cdot \vec{\phi}_i$$

Approximate Spectral Processing

Example (Signal Smoothing):

The goal is to obtain a smoothed signal:

$$\hat{f}[i] \leftarrow \hat{f}[i] \cdot F(\lambda_i)$$

We can relax the condition that $F(\lambda) = e^{-\lambda}$ and use a different filter $F\colon \mathbb{R} \to \mathbb{R}$.

The new filter should:

– preserve the low frequencies

– decay at higher frequencies

Approximate Spectral Processing

$$\left((\alpha \cdot M - \beta \cdot L)^{-1} \circ (\gamma \cdot M - \delta \cdot L)\right)\vec{\phi}_i = \left(\frac{\gamma + \delta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i}\right) \cdot \vec{\phi}_i$$

<u>Example (Signal Smoothing)</u>:

Consider the solution to the linear system:

$$\vec{g} = \left((M - \beta \cdot L)^{-1} \circ M\right)\vec{f}$$

Solving this linear system is equivalent to filtering with:

$$F(\lambda) = \frac{1}{1 + \beta \cdot \lambda}$$

with β the rate of decay of higher frequencies.

$$= \sum \hat{f}[i] \cdot \frac{1}{1 + \beta \cdot \lambda_i} \cdot \vec{\phi}_i$$

Approximate Spectral Processing

$$\left((\alpha \cdot M - \beta \cdot L)^{-1} \circ (\gamma \cdot M - \delta \cdot L)\right)\vec{\phi}_i = \left(\frac{\gamma + \delta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i}\right) \cdot \vec{\phi}_i$$

<u>Example (Signal Sharpening)</u>:

Consider the solution to the linear system:

$$\vec{g} = \left((M - \beta \cdot L)^{-1} \circ (M - \beta \cdot \sigma \cdot L)\right)\vec{f}$$

Taking the spectral decomposition of $\vec{f}$:

$$\vec{g} = \sum \hat{f}[i] \cdot \frac{1 + \sigma \cdot \beta \cdot \lambda_i}{1 + \beta \cdot \lambda_i} \cdot \vec{\phi}_i$$

Approximate Spectral Processing

$$\left((\alpha \cdot M - \beta \cdot L)^{-1} \circ (\gamma \cdot M - \delta \cdot L)\right)\vec{\phi}_i = \left(\frac{\gamma + \delta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i}\right) \cdot \vec{\phi}_i$$

Example (Signal Sharpening):

Consider the solution to the linear system:

$$\vec{g} = \left((M - \beta \cdot L)^{-1} \circ (M - \beta \cdot \sigma \cdot L)\right)\vec{f}$$

$$\Rightarrow \qquad F(\lambda) = \frac{1 + \sigma \cdot \beta \cdot \lambda}{1 + \beta \cdot \lambda}$$

Signal smoothing is a special instance, with $\sigma = 0$.

- $\lim_{\lambda \to 0} F(\lambda) = 1$: Low-frequencies preserved
- $\lim_{\lambda \to \infty} F(\lambda) = \sigma$: High frequencies scaled by σ.

Approximate Spectral Processing

<u>Process</u>($S \subset \mathbb{R}^3$, $\vec{f} \in \mathbb{R}^n$, $\sigma \in \mathbb{R}$, $\beta \in \mathbb{R}$)

1. $(M, L) \leftarrow$ MassAndStiffness (S)
2. $\vec{g} \leftarrow (M - \beta \cdot \sigma \cdot L)\vec{f}$
3. $A \leftarrow (M - \beta \cdot L)$
4. return Solve(A , $\vec{g}$)

By approximating, we replace the computational complexity of storing/computing the spectral decomposition with the complexity of solving a sparse linear system.

Heat Diffusion (Revisited)

$$\frac{\partial H}{\partial t} = \Delta H \quad \text{s.t.} \quad H(p,0) = h(p)$$

Discretization (Temporal):

Letting $H_t: S \to \mathbb{R}$ be the solution at time t, we can (temporally) discretize the PDE in two ways:

Explicit	Implicit
$\frac{H_{t+\delta} - H_t}{\delta} \approx \Delta H_t$	$\frac{H_{t+\delta} - H_t}{\delta} \approx \Delta H_{t+\delta}$
$\Downarrow$	$\Downarrow$
$H_{t+\delta} = H_t + \delta \cdot \Delta H_t$	$(1 - \delta \cdot \Delta) H_{t+\delta} = H_t$

Heat Diffusion (Revisited)

Explicit

$$H_{t+\delta} = H_t + \delta \cdot \Delta H_t$$

Implicit

$$(1 - \delta \cdot \Delta) H_{t+\delta} = H_t$$

Discretization (Spatial):

Projecting onto the discrete function basis gives:

Explicit

$$\Downarrow$$

$$M\vec{H}_{t+\delta} = M\vec{H}_t + \delta \cdot L\vec{H}_t$$

$$\Downarrow$$

$$\vec{H}_{t+\delta} = \left(M^{-1} \circ (M + \delta \cdot L)\right)\vec{H}_t$$

Implicit

$$\Downarrow$$

$$(M - \delta \cdot L)\vec{H}_{t+\delta} = M\vec{H}_t$$

$$\Downarrow$$

$$\vec{H}_{t+\delta} = \left((M - \delta \cdot L)^{-1} \circ M\right)\vec{H}_t$$

Heat Diffusion (Revisited)

Explicit	Implicit
$\vec{H}_{t+\delta} = \left(M^{-1} \circ (M + \delta \cdot L)\right)\vec{H}_t$	$\vec{H}_{t+\delta} = \left((M - \delta \cdot L)^{-1} \circ M\right)\vec{H}_t$
$\Downarrow$	$\Downarrow$
$\hat{H}_{t+\delta}[i] = (1 - \delta \cdot \lambda_i) \cdot \hat{H}_t[i]$	$\hat{H}_{t+\delta}[i] = \frac{1}{1 + \delta \cdot \lambda_i} \cdot \hat{H}_t[i]$

Discretization:

Both methods give an inaccurate answer when a large time-step, δ, is used.

But…

$$\vec{g} = \left((\alpha \cdot M - \beta \cdot L)^{-1} \circ (\gamma \cdot M - \delta \cdot L)\right)\vec{f}$$

$$\Rightarrow \qquad F(\lambda) = \frac{\gamma + \delta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i}$$

Heat Diffusion (Revisited)

<u>Explicit</u>	<u>Implicit</u>

Though neither approximation gives an accurate answer at large times-steps, implicit integration is guaranteed to be (*unconditionally*) *stable.*

D

A similar approach can be used to approximate the solution to the wave equation without a harmonic decomposition.

B

$$\lim_{\lambda\to 0}(1-\delta\cdot\lambda_i)=1 \qquad \Bigg| \qquad \lim_{\lambda\to 0}\left(\frac{1}{1+\delta\cdot\lambda_i}\right)=1$$

But at high frequencies (and large time-steps):

$$\lim_{\lambda\to \infty}(1-\delta\cdot\lambda_i)=-\infty \qquad \Bigg| \qquad \lim_{\lambda\to 0}\left(\frac{1}{1+\delta\cdot\lambda_i}\right)=0$$

Outline

- Motivation
- Laplacian Spectrum
- Applications
- Conclusion

Conclusion

Though there is no Fourier Transform for general surfaces, we can use the spectrum of the Laplacian to get a frequency decomposition.

This enables:

- Filtering of signals
- Solving PDEs

by modulating the frequency coefficients.

Conclusion

Though computing a full spectral decomposition is not space/time efficient, we can often:

– Use the lower frequencies.

– Design linear operators whose solution has the desired frequency modulation.

Using the theory of spectral decomposition:

– We can design stable simulations, without explicitly computing the decomposition.

Future

We have seen that the mass and stiffness matrices are invariant to (respectively) authalic and conformal deformations.

→ Eigenvalues as isometry-invariant signatures
[Reuter, 2006] Laplace-Beltrami Spectra as `Shape-DNA'…

→ Isometric embedding for intrinsic symmetry detection
[Ovsjanikov *et al.*, 2008] Global Intrinsic Symmetries of Shapes

→ Heat diffusion for computing isometry-invariant distances.
[Sun *et al.*, 2009] A Concise and Provably Informative Multi-Scale…

→ Authalicity/Conformality constraints for correspondence
[Rustamov *et al.*, 2013] Map-Based Exploration of Intrinsic Shape…

→ And much, much, much more…

Thank You!