ON CONVERGENT INTERPOLATORY SUBDIVISION SCHEMES IN RIEMANNIAN GEOMETRY

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Abstract. We show the convergence (for all input data) of refinement rules in Riemannian manifolds which are analogous to the linear four-point scheme and similar univariate interpolatory schemes, and which are generalized to the Riemannian setting by the so-called log/exp analogy. For this purpose we use a lemma on the Hölder regularity of limits of contractive refinement schemes in metric spaces. In combination with earlier results on smoothness of limits we settle the question of existence of interpolatory refinement rules intrinsic to Riemannian geometry which have C^r limits for all input data, for $r \leq 3$. We further establish well-definedness of the reconstruction procedure of "interpolatory" multiscale transforms intrinsic to Riemannian geometry.

1. Introduction

Linear stationary refinement rules (see [1] for a comprehensive overview) have been successfully generalized to data which live in surfaces, Lie groups, Riemannian manifolds and other nonlinear geometries. Their properties regarding existence of limits, smoothness, approximation power, stability, and use for discrete multiscale representations of data have been systematically investigated. This line of research was instigated by [6]; the introduction of the so-called method of proximity by [18] and the proof of conditional convergence and C^1 smoothness of certain univariate subdivision schemes can be considered the first systematic result.

Subsequent work investigated smoothness of the continuous limits produced by refinement schemes for manifold-valued data. [20] showed how to achieve manifold refinement schemes whose smoothness is the same as their linear counterparts, and [12] showed this property in particular for "barycentric" manifold subdivision rules (multivariate case of regular combinatorics). [13] established that a so-called interpolatory wavelet transform in Riemannian geometry lets us characterize Hölder smoothness by the decay of detail coefficients in much the same manner as is possible by the analogous linear construction.

Unfortunately the question of convergence is not as satisfactorily resolved as the question of smoothness. Results which are valid for general classes of schemes apply to "dense enough" input data. This restriction is natural, but the bounds which are known to imply convergence are very far from optimal.

Recently there has been progress concerning refinement schemes which converge for *all* input data. After initial univariate results by [19], it could be shown that for multivariate

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barycentric refinement rules with nonnegative coefficients, convergence in the linear case implies convergence in general Hadamard spaces [10, 11].

It is our aim to show convergence for all input data for Riemannian analogues of several prominent interpolatory schemes, including some of the Deslauriers-Dubuc series [3] and the four-point scheme of [8]. They are constructed via the so-called log/exp analogy. As a corollary we obtain interpolatory convergent schemes which produce C^r limits, up to $r = 3$. Another corollary is the reconstruction of Hölder α functions from discrete detail coefficients which decay with $2^{-\alpha k}$, with k as the level of detail.

Our results apply to complete Riemannian manifolds, which includes surfaces which are closed subsets of vector spaces. They are thus valid for a wide class of geometries. The subdivision rules they apply to are interpolatory and are required to be contractive in a certain manner. The four-point scheme and several Deslauriers-Dubuc schemes have this property and are used as examples.

Linear interpolatory refinement rules. We start our exposition with a refinement rule $\mathcal{S}$ acting on data $x : \mathbb{Z} \to V$, where V is a linear space. A finitely supported mask $a : \mathbb{Z} \to \mathbb{R}$ defines $\mathcal{S}$ as follows. A sequence $(x_i)_{i\in\mathbb{Z}}$ is mapped to the sequence $(\mathcal{S}x_j)_{j\in\mathbb{Z}}$, by

$$(1) \qquad (\mathcal{S}x)_j = \sum_{i\in\mathbb{Z}} a_{j-2i} x_i, \quad j \in \mathbb{Z}, \quad \text{where } \sum_{i\in\mathbb{Z}} a_{j-2i} = 1, \text{ for all } j.$$

The sum condition characterizes invariance of the rule w.r.t. translation of the data, and is known to be necessary for convergence. A synonym for "refinement rule" is the notion "subdivision scheme". $\mathcal{S}$ is *interpolatory*, if generally $(\mathcal{S}x)_{2i} = x_i$, which means the even coefficients of the mask are zero except for $a_0 = 1$. In this case, (1) is equivalent to

$$(\mathcal{S}x)_{2j} = x_j, \quad (\mathcal{S}x)_{2j+1} = \sum_{i\in\mathbb{Z}} a_{1-2i} x_{j+i}, \quad j \in \mathbb{Z}, \quad \text{where } \sum_{i\in\mathbb{Z}} a_{1-2i} = 1.$$

Example 1.1. The well-known $(2d+2)$-point Deslauriers-Dubuc interpolatory rule $\mathcal{D}_d$ acts by inserting a new data item $(\mathcal{S}x)_{2j+1}$ in between x_j, x_{j+1} by polynomial interpolation [7, 3]: We find the unique degree $2d+1$ polynomial, temporarily denoted by $f(t)$, which maps arguments $-d, \ldots, d+1$ to values $x_{j-d}, \ldots, x_{j+d+1}$. Then we let $(\mathcal{S}x)_{2j+1} = f(\frac{1}{2})$. For different values of d we get the following masks:

$$\begin{aligned}
d = 1: &\qquad (a_{\pm1}, a_{\pm3}) = 2^{-4}(9, -1), \\
d = 2: &\qquad (a_{\pm1}, \ldots, a_{\pm5}) = 2^{-8}(150, -25, 3), \\
d = 3: &\qquad (a_{\pm1}, \ldots, a_{\pm7}) = 2^{-12}(2450, -490, 98, -10).
\end{aligned}$$

$\mathcal{D}_0$, the "simplest rule", has mask $(a_{-1}, a_0, a_1) = (\frac{1}{2}, 1, \frac{1}{2})$ and acts by $(\mathcal{D}_0 x)_{2i} = x_i$, $(\mathcal{D}_0 x)_{2i+1} = \frac{x_i + x_{i+1}}{2}$. $\mathcal{D}_1$ coincides with the special case $\omega = \frac{1}{16}$ of the four-point rule proposed by [8]:

$$(\mathcal{F}_\omega x)_{2i} = x_i, \quad (\mathcal{F}_\omega x)_{2i+1} = \Big(\frac{1}{2} + \omega\Big)(x_i + x_{i+1}) - \omega(x_{i-1} + x_{i+2}),$$

whose mask is given by $a_{-3,\ldots,3} = (-\omega, 0, \frac{1}{2} + \omega, 1, \frac{1}{2} + \omega, 0, -\omega)$.

We later refer to the polynomial reproduction degree "$\deg(\mathcal{S})$" of a linear refinement rule $\mathcal{S}$. It is the maximal value of k such that for any degree k polynomial f, applying subdivision to integer samples of f coincides with sampling f over the half integers (symbolically, $\mathcal{S}(f|_{\mathbb{Z}}) = f|_{\mathbb{Z}/2}$). The very definition of $\mathcal{D}_d$ implies that $\deg(\mathcal{D}_d) = 2d+1$.

Manifold refinement rules. It is obvious that the following equation equivalently defines the linear refinement rule $\mathcal{S}$ of (1):

$$(2)\qquad (\mathcal{S}x)_j = b_j + \sum_{i\in\mathbb{Z}} a_{j-2i}(x_i - b_j),\ j \in \mathbb{Z};\quad \text{for any base point sequence } (b_i)_{i\in\mathbb{Z}}.$$

This alternative definition is useful when transferring the definition of refinement rules to situations where a difference vector of points and its inverse construction — the addition of point and vector — may be defined. In a Riemannian manifold M, it is natural to employ the definitions

$$p \oplus v = \exp_p(v), \quad q \ominus p = \exp_p^{-1} q,$$

where $t \mapsto \exp_p(tv)$ is the geodesic emanating from p with initial tangent vector v contained in the tangent space T_pM [4]. The subtraction operation $v = q \ominus p$ results in a tangent vector v attached to p which "points to q" in the Riemannian way: starting in p and traveling along a geodesic emanating from p in direction v will reach the point q, if the distance travelled equals $\|v\|$. Setting aside for a moment the topic of well-definedness of $\ominus$, the operations $\oplus$, $\ominus$ now lead to a definition of subdivision rules in Riemannian manifolds:

Definition 1.2. *For data $x : \mathbb{Z} \to M$ and base points $b : \mathbb{Z} \to M$ the log-exponential version of the refinement rules* (1), (2) *is given by*

$$(\mathcal{S}x)_j = b_j \oplus \sum_{i\in\mathbb{Z}} a_{j-2i}(x_i \ominus b_j), \quad j \in \mathbb{Z}.$$

As to notation, the linear rule (2) *which inspired the definition of $\mathcal{S}$ is denoted by $\mathcal{S}^{\text{lin}}$.*

This idea has been first presented for the Lie group case by [6], and has been published by [17]. A refined version with a clever choice of base points produces limit functions of the same smoothness as those of the linear model rule [20, 12].

Remark. Other expressions equivalent to (1) have been used to define and analyze non-linear analogues of linear refinement rules. These include $(\mathcal{S}x)_j$ as the Riemannian barycenter of masses a_{j-2i} located at points x_i (see e.g. [12] where it occurs as a special log/exp construction). In case of nonnegative mask, $(\mathcal{S}x)_j$ can be seen as the expected value of a random variable which assumes the value x_i with probability a_{j-2i} [10, 11].

Well-definedness and properties of subdivision rules. The aim of this paper is to analyze the limit curve which emerges while iterated application of a subdivision rule produces denser and denser data. For a nonlinear rule $\mathcal{S}$, however, other questions have to be investigated before:

(i) Is $\mathcal{S}$ globally and unambiguously defined?
(ii) Do individual points $(\mathcal{S}x)_j$ depend continuously, even smoothly, on the x_i's?

Obviously, for linear rules the answers are affirmative, and when investigating convergence of manifold subdivision rules for "dense enough" input data (as most previous work does) one can assume an affirmative answer without loss of generality. In this paper we do not restrict ourselves in this way, so we address questions (i), (ii) in the following paragraphs. We summarize our findings in Lemma 1.3.

The right setting to work in is a complete Riemannian manifold M. Since we do not want a lack of smoothness of M to interfere with our analysis of the finite degree of smoothness enjoyed by the limit curves generated by subdivision, we assume that M is C^∞, or "smooth", in differential geometry terminology. In that case $\oplus$ is C^∞ as well. *Completeness* is equivalent to any of the following properties, cf. [4, §7, 2.5]:

(a) each geodesic extends to arbitrary length;
(b) M is complete as a metric space;
(c) For subsets K of M we have: K is bounded and closed $\iff$ K is compact.

Property (a) in particular says that $p \oplus v$ is defined for all p, v. The Hopf-Rinow theorem [4, §7, 2.5] states that for a complete Riemannian manifold M, and all $p, q \in M$ there is a shortest curve $c(t)$ in M connecting $p = c(0)$ with $q = c(1)$, and that curve is an unbroken smooth geodesic of the form $\exp_p(t \cdot v)$, so that $v = q \ominus p$. We conclude that $\ominus$ is globally definable. To resolve ambiguities we require that $\ominus$ is made a mapping

$$(p, q) \in M \times M \stackrel{\ominus}{\longmapsto} q \ominus p \in T_p M,$$

by choosing $v = q \ominus p$ once and for all, for all instances (p, q) where the Hopf-Rinow theorem provides more than one solution v.

The Riemannian injectivity radius $\rho_{\mathrm{inj}}(p)$ is the maximal radius of an open ball for which $\exp_p$ is a diffeomorphism. It encodes the local topology and geometry of M, is always positive, and might be infinite. Further it is known that for radii $\epsilon \leq \rho_{\mathrm{inj}}(p)$, the open ϵ-ball around 0 in $T_p M$ is mapped by $\exp_p$ to the ball $B(p, \epsilon)$ of p in the metric space M. By the inverse function theorem, the mapping $q \mapsto q \ominus p$ is as smooth as $\exp_p$, when restricted to $B(p, \rho_{\mathrm{inj}}(p))$. We summarize:

Lemma 1.3. *In a complete Riemannian manifold M, any log/exp rule $\mathcal{S}$ is globally definable. In the notation of Def. 1.2, the point $(\mathcal{S}x)_j$ depends on those data points x_i for which $a_{j-2i} \neq 0$, and this dependence is as smooth as the Riemannian exponential mapping, provided the distance of these data points x_i from the base point b_j does not exceed $\rho_{inj}(b_j)$.*

If the shortest geodesic which joins p and q is unique, then the geodesic midpoint "$m_{p,q}$" of p, q is unique. It can be written equivalently as

$$m_{p,q} = p \oplus \frac{1}{2}(q \ominus p) = q \oplus \frac{1}{2}(p \ominus q).$$

Then also the shortest geodesic joining $m_{p,q}$ with either p, q is unique. Even without uniqueness, we have equality up to sign of the two vectors "endpoint minus midpoint", both of which are contained in the tangent space $T_{m_{p,q}} M$:

$$(p \ominus m_{p,q}) + (q \ominus m_{p,q}) = 0. \tag{3}$$

Otherwise there would be shortest path connecting p, q which is broken (being not C^1) in m_{pq}, an obvious contradiction. As explained in more detail below, our conclusions do not require uniqueness of shortest geodesics.

Example 1.4. For certain kinds of refinement rules, it is natural how to choose the base point sequence. For interpolatory rules, the choice of b_{2i} is arbitrary, since $(\mathcal{S}x)_{2i} = b_{2i} \oplus (x_i \ominus b_{2i}) = x_i$ anyway. A natural choice of base point for the computation of x_{2i+1} is the edge midpoint $m_{x_i,x_{i+1}}$. We compute

$$(\mathcal{S}x)_{2i+1} = m_{x_i,x_{i+1}} \oplus \sum\nolimits_{j\in\mathbb{Z}} a_{1-2j}(x_{i+j} \ominus m_{x_i,x_{i+1}}) \implies$$

$$(\mathcal{S}x)_{2i+1} = m_{x_i,x_{i+1}} \oplus \Big((a_1 - a_{-1})(x_i \ominus m_{x_i,x_{i+1}}) + \sum_{j\neq 0,1} a_{1-2j}(x_{i+j} \ominus m_{x_i,x_{i+1}})\Big). \tag{4}$$

We have used (3) for grouping the terms involving a_1, a_{-1}. Since the mask of the Deslauriers-Dubuc schemes and the four-point scheme is symmetric, we get the following Riemannian versions of these schemes:

$$(\mathcal{D}_d x)_{2i+1} = m_{x_i,x_{i+1}} \oplus \Big(\sum\nolimits_{j\neq 0,1} a_{1-2j}(x_{i+j} \ominus m_{x_i,x_{i+1}})\Big), \qquad (i \in \mathbb{Z}).$$

$$(\mathcal{F}_\omega x)_{2i+1} = m_{x_i,x_{i+1}} \oplus \Big(-\omega\big(x_{i-1} \ominus m_{x_i,x_{i+1}} + x_{i+2} \ominus m_{x_i,x_{i+1}}\big)\Big), \qquad (i \in \mathbb{Z}).$$

The "simplest scheme" reads $(\mathcal{D}_0 x)_{2i} = x_i$, $(\mathcal{D}_0 x)_{2i+1} = m_{x_i,x_{i+1}}$.

The cancellation of terms involving a_1, a_{-1} is very welcome for later convergence analysis. Thus we give the following definition of a Riemannian version of an interpolatory rule $\mathcal{S}^{\text{lin}}$ which coincides with Definition 1.2, if the geodesic midpoints $m_{x_i,x_{i+1}}$ of successive data items are uniquely defined and are chosen as base points. By abuse of language we still refer to "the" Riemannian version of $\mathcal{S}^{\text{lin}}$, even if there is ambiguity in the definition of $\ominus$ and consequently in the definition of $\mathcal{S}$.

Definition 1.5. *The Riemannian version $\mathcal{S}$ of an interpolatory rule $\mathcal{S}^{\text{lin}}$ with mask (a_i) is defined by Equation (4).*

2. Convergence of Riemannian interpolatory subdivision rules

Convergence of interpolatory rules. For interpolatory rules, where half of $\mathcal{S}x$ coincides with x, the definition of convergence is not difficult:

Definition 2.1. *Consider data $x = (x_i)_{i\in\mathbb{Z}}$ and the sequences $\mathcal{S}x, \mathcal{S}^2x, \ldots$ created by iteratively applying the subdivision rule $\mathcal{S}$. Formally $\mathcal{S}^0 x = x$. Since $\mathcal{S}$ is interpolatory and $(\mathcal{S}^{k_1}x)_{i_1} = (\mathcal{S}^{k_2}x)_{i_2}$ whenever $i_1/2^{k_1} = i_2/2^{k_2}$, we may unambiguously define the limit function $\mathcal{S}^\infty x$ for any dyadic rational number by letting*

$$(\mathcal{S}^\infty x)(\frac{i}{2^k}) = (\mathcal{S}^k x)_i, \quad i \in \mathbb{Z},\ k = 0, 1, 2, \ldots$$

If $\mathcal{S}^\infty x$ extends to a continuous function on the reals, then $\mathcal{S}$ is said to converge, for the given input data x, to this continuous limit function, again denoted by $\mathcal{S}^\infty x$.

Example 2.2. The linear Deslauriers-Dubuc rules $\mathcal{D}_d$ of Example 1.1 converge to a continuous limit for all $d \geq 0$ when acting on data in vector spaces. This limit for $d \geq 1$ is r times continuously differentiable (C^r) with r increasing as d does, see e.g. [2, Table

2]. In particular the critical Hölder smoothness $\alpha_{\max}(\mathcal{D}_d)$, $d = 1, 2, 3$ of limit functions has the values 2, ≈ 2.83 and ≈ 3.55, respectively. This implies that the limits of $\mathcal{D}_d$ enjoy C^d smoothness for $d = 1, 2, 3$.

The four-point rule $\mathcal{F}_\omega$ produces C^1 limits for $\omega \in (0, \omega_{\max})$, with $32\omega_{\max}^3 + 4\omega_{\max} = 1$, i.e., $\omega_{\max} \approx 0.19273$ [15]. We therefore have $\alpha_{\max}(\mathcal{F}_\omega) > 1$ for $\omega \in (0, \omega_{\max})$. No scheme $\mathcal{F}_\omega$ has C^2 limits, not even $\mathcal{F}_{1/16} = \mathcal{D}_1$, where limits are $C^{2-\epsilon}$.

Contractivity. It has been shown by several authors that certain contractivity properties of refinement rules — linear and nonlinear — imply convergence to a continuous limit. One particular such property is to require that edgelengths contract by some factor $\gamma < 1$ when subdivision is applied:

$$\sup_{i\in\mathbb{Z}} \delta_i(\mathcal{S}x) \le \gamma \sup_{j\in\mathbb{Z}} \delta_j(x) \quad \text{where} \quad \gamma < 1, \quad \delta_i(x) = \operatorname{dist}(x_i, x_{i+1}). \tag{5}$$

The metric "dist" is the one of the complete Riemannian manifold under consideration.

Condition (5) is known to imply convergence to a continuous limit for univariate interpolatory schemes in Euclidean spaces [9] and also for the "barycentric" Riemannian analogues of univariate schemes with nonnegative coefficients in Hadamard manifolds [19]. A general result which applies to multivariate refinement in complete metric spaces states that a more general kind of contractivity together with proximity of $\mathcal{S}$ to a convergent rule with certain properties implies convergence of $\mathcal{S}$ [10].

In this paper we verify that the result of [9] is valid for complete metric spaces. We start by testing some examples of Riemannian refinement rules.

Lemma 2.3. *Consider a linear interpolatory subdivision rule $\mathcal{S}^{\text{lin}}$ with mask (a_i). Let*

$$\gamma = \frac{1 + |a_1 - a_{-1}|}{2} + \sum_{j\ge 2} \big(|a_{1-2j}| + |a_{2j-1}|\big)\big(j - \frac{1}{2}\big).$$

Then the Riemannian version $\mathcal{S}$ of $\mathcal{S}^{\text{lin}}$, in a complete Riemannian manifold obeys the inequality

$$\delta_{2i}(\mathcal{S}x) \le \gamma \max_j \delta_j(x) \quad \delta_{2i+1}(\mathcal{S}x) \le \gamma \max_j \delta_j(x), \quad \text{for all } i,$$

where the maximum is taken over those finitely many values $\delta_j(x)$ which involve at least one data point contributing to the computation of $(\mathcal{S}x)_{2i+1}$.

Proof. Let δ equal the maximum which occurs on the right hand side of the inequalities above. Then the distance of the midpoint $m_{x_i,x_{i+1}}$ from points $x_i, x_{i-1}, \dots$ is bounded by $\frac{\delta}{2}, \frac{3\delta}{2}, \dots$ and similar for the points $x_{i+1}, x_{i+2}, \dots$. Using the inequality

$$\operatorname{dist}(x, x \oplus v) \le \|v\|,$$

we conclude that the distance of x_i from the point $(\mathcal{S}x)_{2i+1}$ created by subdivision is bounded by

$$\begin{aligned} &\operatorname{dist}\big(x_i, m_{x_i,x_{i+1}}\big) + \Big\|(a_1 - a_{-1})x_i \ominus m_{x_i,x_{i+1}} + \textstyle\sum_{j\ne 0,1} a_{1-2j}(x_{i+j} \ominus m_{x_i,x_{i+1}})\Big\| \\ &\le \frac{\delta}{2} + |a_1 - a_{-1}|\frac{\delta}{2} + \textstyle\sum_{j\ge 2} \big(|a_{1-2j}| + |a_{2j-1}|\big)\Big(j - \dfrac{1}{2}\Big)\delta. \end{aligned}$$

The same applies to the distance of x_{i+1} from $(\mathcal{S}x)_{2i+1}$. $\square$

Example 2.4. We compute the constant γ mentioned in the previous lemma for the Deslauriers-Dubuc schemes $\mathcal{D}_d$ and the four-point rule $\mathcal{F}_\omega$ of Ex. 1.4. We get

$$\mathcal{D}_0 : \gamma = \frac{1}{2}, \quad \mathcal{D}_1 : \gamma = \frac{11}{16}, \quad \mathcal{D}_2 : \gamma = \frac{109}{128}, \quad \mathcal{D}_3 : \gamma = \frac{2039}{2048}, \quad \mathcal{F}_\omega : \gamma = \frac{1}{2} + 3|\omega|.$$

Convergence from contractivity. If the constant γ which occurs in Lemma 2.3 happens to be smaller than 1, then the corresponding Riemannian subdivision rule enjoys edgelength contractivity. In the case of subdivision acting on vector space data, this property is known to imply convergence [9]. We formulate a version of this result which applies to complete metric spaces:

Proposition 2.5. *The limit $\mathcal{S}^\infty x$ of an interpolatory univariate scheme $\mathcal{S}$ acting on data in a complete metric space is continuous, if $\mathcal{S}$ is contractive in the sense that there is $\gamma < 1$ such that for all j,*

$$\delta_j(\mathcal{S}x) \leq \gamma \sup_i \delta_i(x),$$

where the supremum is taken over all data points which influence the left hand expression. The limit is uniformly continuous, if $\delta_i(x)$ is globally bounded.

Proposition 2.6. *Under the same assumptions, $\mathcal{S}^\infty x$ has Hölder continuity*

$$-\log \gamma / \log 2.$$

Proof of Propositions 2.5 and 2.6. Recalling Def. 2.1, we define the limit function $f = \mathcal{S}^\infty x$ on the dyadic rationals and show that an inequality of the kind $\operatorname{dist}(f(\alpha), f(\beta)) \leq C|\alpha - \beta|^r$ holds for all dyadic rationals α, β.

Assume w.l.o.g. $\alpha < \beta$ and choose k such that the length of the interval $[\alpha, \beta]$ is contained in $[\frac{1}{2^{k+1}}, \frac{1}{2^k}]$. The interval $[\alpha, \beta]$ contains a common element with $\frac{1}{2^{k+1}}\mathbb{Z}$, which is called $\alpha_{k+1} = \beta_{k+1}$. Now there is a decreasing sequence of dyadic rationals $(\alpha_j)_{j>k}$ with $\alpha_j \in \frac{1}{2^j}\mathbb{Z}$ approaching α from above (note that there exists an index j_0 such that for all $j \geq j_0$, we have $\alpha_j = \alpha$). Analogously one finds an increasing sequence of $(\beta_j)_{j>k}$ with $\beta_j \in \frac{1}{2^j}\mathbb{Z}$ approaching β from below. We estimate

$$\begin{aligned}\operatorname{dist}\big(f(\alpha), f(\beta)\big) &\leq \operatorname{dist}\big(f(\alpha), f(\alpha_{k+1})\big) + \operatorname{dist}\big(f(\beta_{k+1}), f(\beta)\big) \\ &\leq \sum_{j>k} \operatorname{dist}\big(f(\alpha_j), f(\alpha_{j+1})\big) + \sum_{j>k} \operatorname{dist}\big(f(\beta_j), f(\beta_{j+1})\big).\end{aligned}$$

If $\alpha_j \neq \alpha_{j+1}$, the corresponding contribution to the first sum is an edge length of $\mathcal{S}^{j+1}x$; otherwise that contribution is zero. A similar argument applies to the second sum. In any case the respective summand is bounded by $\delta_i(\mathcal{S}^{j+1}x) \leq \gamma^{j+1} \sup_i \delta_i(x)$:

$$\operatorname{dist}\big(f(\alpha), f(\beta)\big) \leq \sum_{j=k+2}^{\infty} 2\gamma^j \sup_i \delta_i(x) \leq \frac{2\gamma^{k+2}}{1-\gamma} \sup_i \delta_i(x).$$

The inequality $2^{-k} \geq |\beta - \alpha|$ implies $k \leq -\lg|\beta - \alpha|$, so

$$\gamma^k \leq \gamma^{-\lg|\alpha - \beta|}$$

(here $\lg x = \log x / \log 2$ is the dyadic logarithm). We can thus continue to estimate

$$\operatorname{dist}\big(f(\alpha), f(\beta)\big) \leq C\gamma^k \leq C(2^{\lg \gamma})^{\lg|\alpha-\beta|} = C|\alpha - \beta|^{\lg \gamma}.$$

If $\delta_i(x)$ is bounded, so is the constant C. This shows Hölder continuity (locally) for the function f on the dyadic rationals. By completion we extend f to a continuous function on $\mathbb{R}$ which enjoys the same Hölder regularity. Since for any compact interval I, $(S^\infty x)|_I$ depends only on finitely many data points x_i, the statement follows. □

Combining Ex. 2.4 with Proposition 2.5, we get the following result:

Theorem 2.7. *The Deslauriers-Dubuc refinement rules $\mathcal{D}_0, \dots \mathcal{D}_3$, acting on data in a complete Riemannian manifold according to Ex. 1.4 and Def. 1.5, possess continuous limits for all input data. So does the four-point rule $\mathcal{F}_\omega$, if the weight ω is chosen in the interval $(-\frac{1}{6}, \frac{1}{6})$.*

3. Riemannian multiresolution analysis

Multiresolution analysis of functions, meaning a representation by a discrete coarse level description and a hierarchy of detail coefficients, has been transferred to the setting of Lie groups and similar geometries by [6, 17]. A general version of such constructions has been analyzed by [13]. It has also been shown that under certain assumptions on the nature of the decomposition (no redundancy, perfect reconstruction) the greater part of the constructions available in the linear case do not transfer to general Riemannian manifolds [14]. One of the few which do is *interpolatory wavelets*, defined by sampling and an interpolatory subdivision rule $\mathcal{S}$ as predictor [5]. This construction, described below, turns out to be globally defined in complete Riemannian manifolds. Its construction uses restriction $\mathcal{R}^k f$ of a function to $\frac{1}{2^k}\mathbb{Z}$:

$$(\mathcal{R}^k f)_i = f(2^{-k} i), \quad i \in \mathbb{Z}.$$

Now a continuous function $f : \mathbb{R} \to M$ is discretely represented by coarse samples $x^0 : \mathbb{Z} \to M$ and detail coefficients $w^k : \mathbb{Z} \to TM$ which are tangent vectors:

$$x^0 = f|_\mathbb{Z} = \mathcal{R}^0 f, \quad w^k = \mathcal{R}^k f \ominus \mathcal{S}\mathcal{R}^{k-1} f \qquad (k > 0). \tag{6}$$

Here the $\ominus$ operator is applied element-wise. Reconstruction of f from $x^0, w^1, w^2, \dots$ works via iteratively reversing the decomposition. We define f on the dyadic rationals via

$$x^k = \mathcal{S} x^{k-1} \oplus w^k, \qquad f(\frac{i}{2^k}) = x^k_i. \tag{7}$$

If f extends continuously to the reals, then this extension, again denoted by f, is unique and is called the limit of the reconstruction procedure.

Remark. Any interpolatory rule $\mathcal{S}$ can be used here. In any case all even coefficients w^k_{2i} which occur in (6) are zero. When speaking about reconstructing an unknown function from its detail coefficients, this property of the detail coefficients is therefore always assumed.

Lemma 3.1. *In complete Riemannian manifolds, decomposition and reconstruction according to Equations* (6), (7) *have the following properties:*

(*i*) *Both decomposition and reconstruction are globally defined.*
(*ii*) *For all continuous $f : \mathbb{R} \to M$, reconstruction after decomposition restores f.*

(*iii*) *Each coefficient* w_i^k *is computed from finitely many values of* f *on the grid* $2^{-k}\mathbb{Z}$. *As* k *increases, this dependence is smooth. More precisely, for every interval* $[\alpha, \beta]$ *there is* $k_0 > 0$ *such that for every integer* $k \geq k_0$*, and for all* $i \in \mathbb{Z}$ *with* $\frac{i}{2^k} \in [\alpha, \beta]$*, this dependence is as smooth as the exponential mapping.*

Proof. Global definability of $\oplus$, $\ominus$, $\mathcal{S}$ directly implies (i). For (ii) we observe that $p \oplus \big(q \ominus p\big) = q$ and that $x = y \implies \mathcal{S}x = \mathcal{S}y$ because we required $\ominus$ to be a mapping. This implies that $f(t)$ is perfectly reconstructed for all dyadic rationals t. By continuity, this follows for all $t \in \mathbb{R}$.

The idea of proving (iii) is to increase k such that all $\ominus$ operations needed for w_i^k involve points which are close enough for $\ominus$ to be smooth. The technical details are as follows. First choose a positive integer k_0 and find a compact interval $I \supset [\alpha, \beta]$ such that all data needed to compute $w_i^{k_0}$, $\frac{i}{2^{k_0}} \in [\alpha, \beta]$, are samples of $f|_I$. This property then holds for all $k \geq k_0$. Secondly observe that there is an upper bound δ_{k_0} for the distance of samples $f(\frac{i}{2^k})$, $f(\frac{i+1}{2^k})$ in I, for all $k \geq k_0$. By finiteness of the mask and Lemma 2.3, there is $l > 0$ such that all $\ominus$ operations needed to compute w_i^k, $\frac{i}{2^k} \in [\alpha, \beta]$ involve points whose distance does not exceed $l\delta_{k_0}$. Now the set $\{p \in M \mid \operatorname{dist}(p, f(I)) \leq l\delta_{k_0}\}$ is compact, and there is a lower bound $\rho > 0$ for the injectivity radius in this set [16, 2.1.10]. We now increase k_0 such that $l\delta_{k_0} < \rho$, and all $\ominus$ operations are smooth. □

Remark. Note that the detail coefficient w_{2i+1}^k is contained in the tangent space of the point $(\mathcal{S}x^{k-1})_{2i+1}$ which depends on the entire data hierarchy below level k. Only if reconstruction up to level k is already performed, it makes sense to choose the detail coefficients w_i^k. This property implies that usual operations on detail coefficients like thresholding, quantizing, etc. are restricted. However it is possible to overcome this problem by attaching detail coefficients to other data points — say, the nearest point of x^0 — by Riemannian parallel transport which could be along the broken geodesic defined by data x^k or along the shortest unbroken geodesic. We do not care about this here, since parallel transport is an isometry of tangent spaces and we are only concerned with the norms of detail coefficients. The interested reader is referred to [13].

It has been shown by [13], for slightly different decomposition and reconstruction procedures, that the Hölder regularity of a function is characterized by the decay rate of its detail coefficients. The following statement, to be improved in the next section, is a weaker version of this statement.

Theorem 3.2. *Consider a complete Riemannian manifold* M*, samples* $x^0 : \mathbb{Z} \to M$ *and detail coefficients* $w^k : \mathbb{Z} \to TM$*,* $k \geq 0$ *which via the reconstruction procedure* (7) *are to define a function* $f : \mathbb{R} \to M$*. Assume that the detail coefficients contract by* $\|w_i^k\| \leq C\mu^k$ *with* $\mu < 1$*, and that the interpolatory refinement rule* $\mathcal{S}$ *in* (7) *has edge length contractivity constant* $\gamma < 1$*. Then reconstruction recovers a continuous function* f*. For any* $\tilde{\gamma}$ *with* $\max(\gamma, \mu) < \tilde{\gamma} < 1$*, the function* f *has Hölder regularity*

$$-\log \tilde{\gamma} / \log 2 \in (0, 1).$$

Proof. We use the notation $D_k = \sup_i \delta_i(x^k)$ for the supremum of edge lengths of the discrete reconstruction at level k. Because of $x_{2i}^k = x_i^{k-1} = (\mathcal{S}x^{k-1})_{2i}$,

$$\delta_{2i}(x^k) \leq \operatorname{dist}\big(x_{2i}^k, (\mathcal{S}x^{k-1})_{2i+1}\big) + \|w_{2i+1}^k\| \leq \gamma D_{k-1} + C\mu^k.$$

The same holds for $\delta_{2i+1}(x^k)$. We choose an auxiliary constant $\tilde\gamma$ which obeys $\max(\gamma,\mu) < \tilde\gamma < 1$ and find a constant C' such that $k\max(\gamma,\mu)^k \le C'\tilde\gamma^k$ for all k. We now recursively estimate D_k:

$$D_k \le \gamma D_{k-1} + C\mu^k \le \gamma\big(\gamma D_{k-2} + C\mu^{k-1}\big) + C\mu^k \le \cdots \le \gamma^k D_0 + C\sum\nolimits_{j=0}^{k-1} \gamma^j \mu^{k-j}$$
$$\le \gamma^k D_0 + Ck\max(\gamma,\mu)^k \le (D_0 + CC')\tilde\gamma^k.$$

Analogous to Proposition 2.6 this implies that the limit is continuous with Hölder regularity $-\log\tilde\gamma/\log 2$. □

Apparently the conclusions of Theorem 3.2 are valid for interpolatory wavelets defined by the Dubuc-Deslauriers schemes $\mathcal{D}_0,\dots,\mathcal{D}_3$, or by the four-point rule $\mathcal{F}_\omega$, with $\omega \in (-\frac16,\frac16)$.

4. Smoothness

Once a subdivision scheme $\mathcal{S}$ is known to possess continuous limits (cf. Def. 2.1), it is natural to ask how smooth these limits are. More precisely we ask for the degree of smoothness of the function $t \mapsto (\mathcal{S}^\infty x)(t)$, which can mean either being C^r, for some $r > 0$, or having a certain Hölder regularity. For simplicity, we still assume that the intrinsic smoothness of the underlying manifold is C^∞.

Fortunately, general results of the form "dense enough input data $\Longrightarrow$ smoothness" are available for many subdivision schemes $\mathcal{S}$, and of course convergence implies that eventually we have dense enough input data. In many cases it has been possible to show that the limit functions produced by $\mathcal{S}$ are at least as smooth as the limits of the linear rule $\mathcal{S}^{\mathrm{lin}}$ it is derived from. [21] show this for interpolatory rules which are transferred to the Lie group setting by the log/exp construction (using base point x_i for computing x_{2i+1}). In the non-interpolatory case and with smoothness higher than C^2, the choice of base point sequence is crucial for the smoothness of limits, see the detailed study by [20, 12].

We are going to derive results on the smoothness of limits curves after a detour where we study the properties of the "interpolatory wavelets" decomposition and reconstruction procedures associated with an interpolatory subdivision rule. These properties, stated below, essentially say that the degree of Hölder regularity of a function is characterized by the decay rate of its detail coefficients. We measure the decay and generally the magnitude $\|w_i^k\|$ of detail coefficients w_i^k in the canonical way, by using the Riemannian scalar product on the tangent vectors w_i^k. We have the following result:

Theorem 4.1. *Consider an interpolatory subdivision rule $\mathcal{S}$ according to Definition 1.5, which operates in a complete Riemannian manifold and which has edgelength contractivity $\gamma < 1$. Then the reconstruction and composition procedures associated with $\mathcal{S}$ have the following properties:*

- (i) *For any $\alpha < \alpha_{\max}(\mathcal{S}^{\mathrm{lin}})$, the limit function f reconstructed from samples x^0 and a hierarchy $w^0, w^1, \dots$ of detail coefficients exists and enjoys Hölder continuity α, if the detail coefficients decay with $\|w_i^k\| \le C2^{-k\alpha}$.*
- (ii) *For any $\alpha < \deg(\mathcal{S}^{\mathrm{lin}})$, the detail coefficients $w^0, w^1, \dots$ of a Hölder α function $f : \mathbb{R} \to M$ decay with $\|w_i^k\| \le C2^{-\alpha k}$.*

The constants C are understood to be uniform for data in compact intervals.

Proof. A proof of this statement from scratch would be rather long. Fortunately a proof can also be given by indicating in what way the proof of a similar statement can be modified, but this causes this paper to be no longer self-contained.

Conclusions (i), (ii) have been shown for subdivision rules analogous to ours by [13, Theorem 8], under the stronger assumptions that all procedures are defined and that the continuous limit f in (i) exists. They consider subdivision rules according to Definition 1.2, but with a base point sequence different from ours.

If [13, Theorem 8] holds also for our rules, the proof would be complete, since its assumptions are fulfilled thanks to Lemma 3.1 and Theorem 3.2. A look at the proof now reveals that the choice of the base point sequence is relevant only in so far as it affects "proximity" of subdivision rules $\mathcal{S}$ and $\mathcal{S}^{\text{lin}}$.

Using local coordinates in the Riemannian manifold, this proximity is expressed as $\|(\mathcal{S}x)_i - (\mathcal{S}^{\text{lin}}x)_i\| \leq C\Omega_n(x)$, where $n = \deg(\mathcal{S}^{\text{lin}})$, C is a constant valid for all input data in a compact set, the norm refers to the usual Euclidean norm in the coordinate chart, and the symbol $\Omega_n(x)$, which is used by [20, 12], means a rather involved combination of iterated differences of the sequence $(x_i)_{i\in\mathbb{Z}}$. Since our rules satisfy the proximity inequality by [12, Theorem 3.3], we conclude that (i), (ii) hold. □

Remark. Theorem 4.1 is obviously stronger than Theorem 3.2 which claims only low Hölder regularity. This is consistent with the fact that the proof of [13, Theorem 8] is rather involved. The important point however is that Theorem 3.2 establishes well-definedness of decomposition and reconstruction for all input data $(x_i)_{i\in\mathbb{Z}}$ and for all detail coefficients enjoying a certain mild decay condition. In contrast, the results of [13] refer to input data which are "dense enough" in an unspecified way. So the strength of Theorem 4.1 rests on Theorem 3.2.

Remark. It is not difficult to see that the statements in Theorem 4.1 concerning Hölder regularity α hold true not only for C^∞ manifolds, but also for manifolds where the exponential mapping enjoys smoothness higher than α.

Corollary 4.2. *The critical Hölder regularity $\alpha_{\max}$ of limit functions produced by the refinement schemes $\mathcal{D}_d$, $d = 1, 2, 3$, in Riemannian manifolds has the following values:*

$$d = 1 : \alpha_{\max} = 2, \quad d = 2 : \alpha_{\max} \approx 2.83, \quad d = 3 : \alpha_{\max} \approx 3.55.$$

In particular, limit curves produced by $\mathcal{D}_d$ are d times continuously differentiable, for $d = 0, \dots, 3$. For the four-point scheme $\mathcal{F}_\omega$ with $\omega \in (0, \frac{1}{6})$, limits are C^1.

Proof. The limit functions in question, denoted by $\mathcal{S}^\infty x$, are continuous by Theorem 2.7. Since $\mathcal{S}^\infty x$ is also constructed via the reconstruction procedure (7) if we let $x^0 = x$ and $w^0 = w^1 = \cdots = 0$, the conclusion follows from Theorem 4.1. The values of $\alpha_{\max}$ are taken from Ex. 2.2. □

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