# Approximation order of interpolatory nonlinear subdivision schemes.

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Abstract. Linear interpolatory subdivision schemes of $C^r$ smoothness have approximation order at least $r+1$. The present paper extends this result to nonlinear univariate schemes which are *in proximity* with linear schemes in a certain specific sense. The results apply to nonlinear subdivision schemes in Lie groups and in surfaces which are obtained from linear subdivision schemes. We indicate how to extend the results to the multivariate case.

*Keywords:* nonlinear subdivision, interpolatory subdivision, approximation order, proximity inequalities

## 1. Introduction and notation

It is known that $C^r$ smoothness of an interpolatory linear subdivision scheme $S$ implies approximation order $r+1$. This means that applying an interpolatory subdivision scheme $S$ to samples $f(ih)$ of a $C^{r+1}$ function $f$ yields a limit $S^\infty f$ which is close to the original function: $\|S^\infty f - f\| \leq \text{const} \cdot h^{r+1}$.

A univariate nonlinear subdivision scheme $T$ which fulfills the *proximity inequality* $\|Sp - Tp\| \leq C \cdot \|\Delta p\|^2$ introduced in [**8**] fulfills the inequality $\|S^\infty f - T^\infty f\| \leq \text{const} \cdot h^2$, as shown in that paper. This statement applies e.g. to the geodesic analogues, log-exponential analogues, and projection analogues of linear schemes, as discussed in [**8, 11, 6**]. We can therefore conclude that in every case where $S$ has approximation order 2, also $T$ has this property. It is the aim of the present paper to show a better result, namely approximation order $r+1$ for such interpolatory nonlinear schemes which are in proximity of order $r$ with $C^r$ interpolatory linear ones. As we deal with approximation order via smoothness, we cannot expect our results to be optimal. This is most noticeable in schemes such as the interpolatory four-point scheme constructed by interpolation with cubics [**1**]: It enjoys only $C^1$ smoothness, so only the case $r=1$ applies and we show approximation order two for nonlinear schemes which are in proximity with the four-point scheme. In fact this approximation order should be four, as indicated by numerical evidence [**13**].

**1.1. Linear schemes and their smoothness.** Sampling a vector valued function $f : \mathbb{R} \to V$ at parameter values $h \cdot i$ ($i \in \mathbb{Z}$) yields a sequence $(p_i)_{i\in\mathbb{Z}}$ of data points:

$$p_i := f(ih) \quad (i \in \mathbb{Z}). \tag{1}$$

Applying a subdivision scheme $S$ with dilation factor $N > 1$ to these samples produces finer and finer sequences $Sp, S^2p, \dots$ of data. We consider linear stationary subdivision schemes with finite mask $(\alpha_j)_{j\in\mathbb{Z}}$, i.e.,

$$(Sp)_i = \sum\nolimits_{j\in\mathbb{Z}} \alpha_{i-Nj} p_j, \tag{2}$$

and nonlinear schemes derived from them. When both letters $S$ and $T$ are used to describe subdivision schemes, $S$ usually refers to a scheme of the form (2) which is affinely invariant, i.e., $\sum_{j\in\mathbb{Z}} \alpha_{i-Nj} = 1$ for all $i$. The letter $T$ refers to a possibly nonlinear scheme which is related to $S$ and always has the same dilation factor. This means that

$$q_i = p_{i+j} \implies (Tq)_i = (Tp)_{i+Nj}. \tag{3}$$

We assume that $V$ is endowed with a Euclidean norm $\|\cdot\|$ and define

$$\|p\| := \sup\nolimits_{i\in\mathbb{Z}} \|p_i\|. \tag{4}$$

A scheme of type (2) then has the norm

$$\|S\| = \max\nolimits_{0\le i<N} \sum\nolimits_{j\in\mathbb{Z}} |\alpha_{i-jN}|. \tag{5}$$

Each of the sequences $(S^k p_i)_{i\in\mathbb{Z}}$ is associated, by piecewise linear interpolation, with a piecewise linear function $S^k f$ approximating $f$. Their limit function, if it exists, is denoted by $S^\infty f$:

$$S^k f := \mathcal{F}_k(S^k p), \quad S^\infty f := \lim_{k\to\infty} S^k f, \tag{6}$$

where

$$\mathcal{F}_k(q)\Big(\frac{h}{N^k}(i+\beta)\Big) := (1-\beta)q_i + \beta q_{i+1} \quad (0 \le \beta \le 1,\ i \in \mathbb{Z}). \tag{7}$$

The factor $h$ in the definition of $\mathcal{F}_k$ comes from the understanding that the initial data originate from sampling a function $f$ as described above. If for functions we employ the sup norm, then we have the equality $\|q\| = \|\mathcal{F}_k(q)\|$ for all sequences $q$. Affinely invariant schemes of type (2) have a unique derived scheme $S^{[1]}$ defined by $N\Delta S = S^{[1]}\Delta$, where $\Delta$ is the forward difference operator $(\Delta p)_i = p_{i+1} - p_i$. In this paper all the linear schemes are $C^r$ schemes (generating $C^r$ limits) for some

$r \geq 2$, and have derived schemes up to order $r+1$. The latter obey the following inequalities:

$$\mu_0 := \frac{1}{N}\|S^{[1]}\| < 1, \ \ldots, \ \mu_r := \frac{1}{N}\|S^{[r+1]}\| < 1. \tag{8}$$

Here $S^{[j]}$ denotes the derived scheme of order $j$ of $S$, which satisfies

$$S^{[j]}\Delta^j = N^j \Delta^j S. \tag{9}$$

Inequalities (8) guarantee that $S$ converges and produces $C^r$ limits, and that also $S^{[1]}, \ldots, S^{[r]}$ are convergent subdivision schemes [**2**]. Hence there exist constants $A_0, \ldots, A_r$ such that for all positive integers $k$,

$$\|S^k\| \leq A_0, \ \ldots, \ \|(S^{[r]})^k\| \leq A_r. \tag{10}$$

Equations (8) and (9) imply that the derivatives of $S^\infty f$ are approximated by the piecewise linear interpolants of the difference sequences $(N^k/h)\Delta S^k p$, $(N^k/h)^2 \Delta^2 S^k p$ and so on:

$$(S^\infty f)' = \lim_{k\to\infty} \mathcal{F}_k\Big(\frac{N^k}{h}\Delta S^k p\Big), \quad \ldots, \quad (S^\infty f)^{(r)} = \lim_{k\to\infty} \mathcal{F}_k\Big(\frac{N^{rk}}{h^r}\Delta^r S^k p\Big) \tag{11}$$

(a proof can be found e.g. in [**8**, Lemma B.1, p. 617]).

*Remark:* Any $C^r$ interpolatory linear scheme has derived schemes up to order $r+1$ satisfying (8) [**2**].

### 1.2. Nonlinear schemes and their smoothness.

We now consider a subdivision scheme $T$ which is allowed to be nonlinear, but such that there is a linear scheme $S$ in proximity with $T$. The lower order proximity conditions read

$$\|(T-S)p\| \leq C\|\Delta p\|^2, \quad \|\Delta(T-S)p\| \leq C(\|\Delta p\|^3 + \|\Delta p\|\|\Delta^2 p\|), \tag{12}$$

and the general proximity condition of order $r$ has the form

$$\|\Delta^{r-1}(T-S)p\| \leq C \sum\nolimits_{(\alpha_1,\ldots,\alpha_r)\in\Omega_r} \|\Delta p\|^{\alpha_1} \cdots \|\Delta^r p\|^{\alpha_r}, \tag{13}$$

with

$$\Omega_r = \{(\alpha_1, \ldots, \alpha_r) : \alpha_1, \ldots, \alpha_r \in \mathbb{N}_0, \ \textstyle\sum_{j=1}^r j\alpha_j = r+1\}.$$

These conditions are supposed to hold for input data with $\|\Delta p\|$ small enough. It is easy to demonstrate that (12) indeed consists of the cases $r = 1$ and $r = 2$ of (13): In the case $r = 1$ the sum in (13) has only one summand, namely $\alpha_1 = 2$, while in the case $r = 2$ we have $\Omega_2 = \{(3, 0), (1, 1)\}$.

We further require that $S$ satisfies a stronger version of (8), namely (14) below. We shall see presently that they are always fulfilled in the cases which are interesting to us. The first of these inequalities are

$$\nu_0 := \mu_0^2 N < 1, \quad \nu_1 := \max(\mu_0^3 N^2, \mu_0\mu_1 N) < 1,$$

while the general inequality is given by

$$\nu_r := \max_{(\alpha_1,\ldots,\alpha_r)\in\Omega_r} \mu_0^{\alpha_1}\cdots\mu_{r-1}^{\alpha_r} N^{\alpha_1+\cdots+\alpha_r-1} < 1. \tag{14}$$

EXAMPLE 1. The B-spline schemes of degree $r$ have $\mu_0 = \cdots = \mu_{r-1} = 1/N$, so (14) is fulfilled. The Dubuc-Deslauriers scheme "$D$" derived from interpolation with quintic polynomials and dilation factor $N = 2$, with symbol

$$a(z) = \frac{1}{256}(x+1)^6(3x^4 - 18x^3 + 38x^2 - 18x + 3)$$

satisfies neither (8) nor (14). However, the iterated scheme $S = D^2$ with dilation factor $N = 4$ satisfies

$$\mu_0 \approx 0.3718,\ \mu_1 \approx 0.6584,\ \mu_2 \approx 0.7109,\ \nu_0 \approx 0.8223,\ \nu_1 \approx 0.9792.$$

Therefore $S = D^2$ satisfies (8) and (14) up to order $r = 2$.

It is not difficult to show that the inequality (14) is in fact no restriction for the linear schemes considered in this paper.

LEMMA 1. *For any linear subdivision scheme $S$ which produces $C^r$ limits there is an iterate $S^k$ which obeys the inequalities* (14)*.*

PROOF. We have to show that there is some $k$ such that for $(\alpha_1,\ldots,\alpha_r) \in \Omega_r$,

$$\Big(\frac{\|(S^{[1]})^k\|}{N^k}\Big)^{\alpha_1}\cdots\Big(\frac{\|(S^{[r]})^k\|}{N^k}\Big)^{\alpha_r}(N^k)^{\alpha_1+\cdots+\alpha_r-1} < 1.$$

This is done with the help of (10): The left hand side in the above inequality is bounded by

$$\Big(\frac{A_1}{N^k}\Big)^{\alpha_1}\cdots\Big(\frac{A_r}{N^k}\Big)^{\alpha_r}(N^k)^{\alpha_1+\cdots+\alpha_r-1} = \frac{1}{N^k}A_1^{\alpha_1}\cdots A_r^{\alpha_r},$$

which does not exceed 1 if $k$ is chosen appropriately. □

It has been shown in [**8, 7**] that the limit curves $T^\infty f$ are $C^r$, if conditions (13) and (14) are met. In this case, (11) holds not only for $S$, but also for $T$ (cf. [**7**, Th. 6]).

Without going into details we mention that there is a variety of constructions of nonlinear schemes $T$ which are based on linear schemes $S$, and for which either the proximity conditions (12) or even the general ones given in (13) have been verified (see [**7**] for the geodesic analogue of some linear schemes, which work in surfaces and Riemannian manifolds, [**7, 4**] for two kinds of projection analogue which operate in surfaces, [**10, 6, 9**] for two different kinds of log-exponential analogue which operate in matrix groups and symmetric spaces). For interpolatory schemes, results which do not depend on (14) but only on (8) can be obtained (see [**12, 3**] for subdivision in Lie groups).

It is shown in [**7**] that the inequalities (13) and (14) up to order $r-1$ imply the following contractivity inequalities

$$\|\Delta T^k p\| \le \Big(\mu_0+\epsilon\Big)^k \|\Delta p\|, \ \ldots \ \|\Delta^r T^k p\| \le C\Big(\frac{\mu_{r-1}+\epsilon}{N^{r-1}}\Big)^k \|\Delta p\|, \tag{15}$$

for any $\epsilon > 0$, provided $\|\Delta p\|$ is small enough. The inequalities in (15) for $r=1$ and $r=2$ will be used later in the proof of Lemma 3 and are key arguments in the proof of smoothness of $T^\infty f$ according to [**8, 7**].

## 2. Auxiliary inequalities for finite differences

The main result of this paper, Theorem 9 below, depends on auxiliary inequalities concerning finite differences. These are shown by induction. First comes a technical inequality which is used in several places:

Lemma 2. *For $S, T$, satisfying* (13) *for some $r \ge 2$, and $S$ a linear $C^r$ scheme,*

$$\|\Delta^r(S^k - T^k)p\| \le 2CA_r N^{r(1-k)} \sum_{n=0}^{k-1} N^{rn} \sum_{(\alpha_1,\ldots,\alpha_r)\in\Omega_r} \|\Delta^1 T^n p\|^{\alpha_1} \ldots \|\Delta^r T^n p\|^{\alpha_r}.$$

Proof. We obtain an upper bound for $\widetilde{D}_k := \|\Delta^r(S^k - T^k)p\|$ by using a telescopic sum and the defining property (9) of derived schemes.

$$\begin{aligned}\widetilde{D}_k &\le \sum_{n=0}^{k-1} \big\|\Delta^r(S^{k-n} - S^{k-n-1}T)T^n p\big\| \\ &\le \sum_{n=0}^{k-1} \big\|(N^{n+1-k})^r (S^{[r]})^{k-n-1}\Delta^r(S-T)T^n p\big\| \\ &\le A_r N^{r(1-k)} \sum_{n=0}^{k-1} N^{rn} \|\Delta^r(S-T)T^n p\|.\end{aligned}$$

The general relation $\|\Delta q\| \le 2\|q\|$ together with (13) now implies that

$$\begin{aligned}\widetilde{D}_k &\le 2A_r N^{r(1-k)} \sum_{n=0}^{k-1} N^{rn} \|\Delta^{r-1}(S-T)T^n p\| \\ &\le 2CA_r N^{r(1-k)} \sum_{n=0}^{k-1} N^{rn} \sum\nolimits_{(\alpha_1,\ldots,\alpha_r)\in\Omega_r} \|\Delta T^n p\|^{\alpha_1} \cdots \|\Delta^r T^n p\|^{\alpha_r},\end{aligned}$$

which completes the proof. □

First we prove a lemma which is the basis of our induction argument.

LEMMA 3. *Let $S, T$ be as in Lemma 2 with $r = 2$. Assume that a $C^2$ vector-valued function $f : \mathbb{R} \to V$ with bounded first and second derivatives is sampled at density $h$. Assume further that the sampling density $h$ does not exceed $h_0$, where $h_0$ is chosen such that* (12) *is satisfied, and that in addition the contractivity inequality* (15) *up to order $r = 2$ is valid. Then*

$$D_k := \left\| \frac{\Delta^2 (S^k - T^k) p}{(h/N^k)^2} \right\| \quad (p_i = f(ih), \ k = 1, 2, 3, \dots) \tag{16}$$

*is bounded by a constant $\delta_2$ which depends only on $f$ and $S, T$, but not on $h$ or $k$. Moreover,*

$$\|\Delta^2 T^k p\| \le \delta_2' N^{-2k} h^2, \tag{17}$$

*with $\delta_2' = \delta_2 + A_2 \|f''\|$.*

PROOF. By Lemma 2,

$$D_k \le \frac{2A_2 C}{h^2} \sum_{n=0}^{k-1} N^{2+2n} \left( \|\Delta T^n p\|^3 + \|\Delta T^n p\| \|\Delta^2 T^n p\| \right).$$

We now employ the convergence conditions (15) and the estimate for $\nu_1$ in (14), writing $\widetilde{\mu}_i$ instead of $\mu_i + \epsilon$:

$$\begin{aligned} D_k &\le \frac{2A_2 C N^2}{h^2} \sum_{n=0}^{k-1} N^{2n} \left( (\widetilde{\mu}_0^n \|\Delta p\|)^3 + \widetilde{\mu}_0^n \|\Delta p\| \left( \frac{\widetilde{\mu}_1}{N} \right)^n \|\Delta p\| \right) \\ &\le \frac{2A_2 C N^2}{h^2} \sum_{n=0}^{k-1} \left( (\widetilde{\mu}_0^3 N^2)^n \|\Delta p\|^3 + (\widetilde{\mu}_0 \widetilde{\mu}_1 N)^n \|\Delta p\|^2 \right) \\ &\le 2A_2 C N^2 \frac{\|\Delta p\|^2}{h^2} \sum_{n=0}^{k-1} (\|\Delta p\| \nu_1^n + \nu_1^n) \le 2A_2 C N^2 \frac{\|\Delta p\|^2}{h^2} \frac{1 + \|\Delta p\|}{1 - \nu_1}. \end{aligned}$$

With $p_i = f(ih)$ we have

$$\|\Delta p\| \le h \|f'\|_\infty, \quad \|\Delta^2 p\| \le h^2 \|f''\|_\infty. \tag{18}$$

Therefore, $D_k \le \delta_2 := 2A_2 C N^2 \|f'\|^2 \frac{1}{1-\nu_1} (1 + h_0 \|f'\|)$, and we have found the desired upper bound for (16). As to (17), we first note that by (9), $\Delta^2 S^k p = N^{-2k} (S^{[2]})^k \Delta^2 p$. Consequently,

$$\|\Delta^2 T^k p\| \le \|\Delta^2 (S^k - T^k) p\| + \|\Delta^2 S^k p\| \le \frac{\delta_2 h^2}{N^{2k}} + \frac{A_2}{N^{2k}} h^2 \|f^{(2)}\| = \delta_2' \frac{h^2}{N^{2k}}.$$

□

COROLLARY 4. *Under the same conditions as in Lemma 3, $(S^\infty f)''$ and $(T^\infty f)''$ exist and are bounded, independently of $h$, for all functions $f$ with bounded first and second derivatives.*

PROOF. Lemma 3 implies $\|(S^\infty f - T^\infty f)''\| < \delta_2$. Since $S$ is a linear $C^2$ scheme,

$$\|(S^\infty f)''\| \leq \frac{N^{2k}}{h^2}\|\Delta^2 S^k p\| \leq \frac{N^{2k}}{h^2}\|\frac{1}{N^{2k}}(S^{[2]})^k\Delta^2 p\| \leq A_2\|f''\|. \tag{19}$$

Thus

$$(T^\infty f)'' \leq \delta_2 + A_2\|f''\|. \qquad \square$$

We now extend Lemma 3 to higher order finite differences.

LEMMA 6. *Let $S, T$ be as in Lemma 2. Assume that a $C^r$ vector-valued function $f : \mathbb{R} \to V$ with bounded derivatives up to order $r$ ($r \geq 2$) is sampled at density $h \leq h_0$, where $h_0$ is chosen such that* (13) *and* (15) *are satisfied up to order $r$. Then for any $\epsilon > 0$,*

$$\left\|\frac{N^{k(r-\epsilon)}}{h^r}\Delta^r(S^k - T^k)p\right\|, \quad p_i = f(ih),\ k = 1, 2, 3, \dots \tag{21}$$

*is bounded by a constant $\delta_{r,\epsilon}$ which depends on $\epsilon, r$, the schemes $S, T$ and $f$, but does not depend on $h$ or $k$. Moreover,*

$$\|\Delta^r T^k p\| \leq \delta'_{r,\epsilon} N^{-k(r-\epsilon)} h^r, \tag{22}$$

*with $\delta'_{r,\epsilon} = A_r\|f^{(r)}\| + \delta_{r,\epsilon}$.*

PROOF. According to Lemma 3, the result is true in the case $r = 2$, even for the boundary case $\epsilon = 0$ which implies the case of general $\epsilon > 0$. For the purpose of induction we assume that the result is true for all values of $r$ smaller than the one under consideration. By Lemma 2, Equation (22) for smaller values of $r$, and using $\|\Delta^r q\| \leq 2\|\Delta^{r-1} q\|$, we obtain for any $\widetilde{\epsilon} > 0$:

$$\begin{aligned}
\widetilde{D}_k &\leq \frac{2CA_r N^r}{N^{rk}}\sum_{n=0}^{k-1} N^{rn} \sum_{(\alpha_1,\dots,\alpha_r)\in\Omega_r} \|\Delta T^n p\|^{\alpha_1} \cdots \|\Delta^{r-1}T^n p\|^{\alpha_{r-1}} \cdot 2^{\alpha_r}\|\Delta^{r-1}T^n p\|^{\alpha_r} \\
&\leq \frac{C'}{N^{rk}}\sum_{n=0}^{k-1} N^{rn} \sum_{(\alpha_1,\dots,\alpha_r)\in\Omega_r} \left(\frac{\delta'_{1,0}h}{N^n}\right)^{\alpha_1}\left(\frac{\delta'_{2,0}h^2}{N^{2n}}\right)^{\alpha_2}\left(\frac{\delta'_{3,\widetilde{\epsilon}}h^3}{N^{(3-\widetilde{\epsilon})n}}\right)^{\alpha_3}\cdots\left(\frac{\delta'_{r-1,\widetilde{\epsilon}}h^{r-1}}{N^{(r-1-\widetilde{\epsilon})n}}\right)^{\alpha_{r-1}+\alpha_r} \\
&\leq \frac{C'}{N^{rk}}\sum_{n=0}^{k-1} N^{rn} \sum_{(\alpha_1,\dots,\alpha_r)\in\Omega_r} \left(\frac{h}{N^n}\right)^{\alpha_1}\left(\frac{h^2}{N^{2n}}\right)^{\alpha_2}\cdots\left(\frac{h^{r-1}}{N^{(r-1-\widetilde{\epsilon})n}}\right)^{\alpha_{r-1}+\alpha_r} \\
&\leq \frac{C'}{N^{rk}}\sum_{n=0}^{k-1} N^{rn} \sum_{(\alpha_1,\dots,\alpha_r)\in\Omega_r} \frac{h^{\alpha_1+2\alpha_2+\cdots+(r-1)\alpha_{r-1}+(r-1)\alpha_r}}{(N^{\alpha_1+2\alpha_2+(3-\widetilde{\epsilon})\alpha_3+\cdots+(r-1-\widetilde{\epsilon})\alpha_{r-1}+(r-1-\widetilde{\epsilon})\alpha_r})^n}.
\end{aligned}$$

For the next estimates we employ the fact that $(\alpha_1, \dots, \alpha_r) \in \Omega_r \implies \alpha_j \leq \lfloor\frac{r+1}{j}\rfloor$, which in turn implies $\alpha_r \leq 1$, $h^{-\alpha_r} \leq h^{-1}$, $N^{\alpha_r} < N$, as well as $\sum_{j=3}^r \alpha_j \leq$

$(r+1)\sum_{j=3}^{r+1}\frac{1}{j} =: \eta_r$, and we choose $\widetilde{\epsilon}$ such that $\eta_r\widetilde{\epsilon} < \epsilon/2$:

$$\begin{aligned}\widetilde{D}_k &\le \frac{C'}{N^{(r-\epsilon)k}}\sum_{n=0}^{k-1} N^{rn-k\epsilon}\sum_{\alpha_1,\dots,\alpha_r}\frac{h^{r+1-\alpha_r}}{(N^{r+1-\sum_{j=3}^r\alpha_j\widetilde{\epsilon}-\alpha_r})^n} \le \frac{C''}{N^{(r-\epsilon)k}}\sum_{n=0}^{k-1}\frac{N^{-k\epsilon}h^r}{(N^{-\sum_{j=3}^r\alpha_j\widetilde{\epsilon}})^n}\\ &\le \frac{C''h^r}{N^{(r-\epsilon)k}}\sum_{n=0}^{k-1}N^{-k\epsilon+n\eta_r\widetilde{\epsilon}} \le \frac{C''h^r}{N^{(r-\epsilon)k}}\sum_{n=0}^{k-1}N^{(n/2-k)\epsilon}\\ &= \frac{C''h^r}{N^{(r-\epsilon)k}}\sum_{j=k+1}^{2k}N^{-j\epsilon/2} \le \frac{C''h^r}{N^{(r-\epsilon)k}}\sum_{j=0}^{\infty}N^{-j\epsilon/2} \le \frac{C''h^r}{N^{(r-\epsilon)k}}\cdot\frac{1}{1-N^{-\epsilon/2}}.\end{aligned}$$

Thus,

$$\widetilde{D}_k = \|\Delta^r(S^k-T^k)p\| \le \delta_{r,\epsilon}\frac{h^r}{N^{(r-\epsilon)k}}$$

with $\delta_{r,\epsilon}$ depending on $r,\epsilon,S,T,f$ but not on $h$ or $k$, and the proof of (21) is complete. Next, we show (22). From (9) and $\|\Delta^r p\| \le h^r\|f^{(r)}\|$, we get

$$\|\Delta^r T^k p\| \le \|\Delta^r(S-T)^k p\| + \|\Delta^r S^k p\| \le \frac{\delta_{r,\epsilon}h^r}{N^{(r-\epsilon)k}} + \frac{A_r}{N^{rk}}h^r\|f^{(r)}\| \le \delta'_{r,\epsilon}\frac{h^r}{N^{k(r-\epsilon)}}.$$

This concludes the proof. □

Lemma 7. *Under the same conditions as in Lemma 6,*

$$\left\|\frac{N^{(r-1)k}}{h^{r-1}}\Delta^{r-1}(S^k-T^k)p\right\| \le \text{const}\cdot h^2. \tag{23}$$

Proof. The first part of the proof is analogous to the proof of Lemma 2. The difference is that we do not use the relation $\|\Delta q\| \le 2\|q\|$ and use the proximity condition (13) directly. We thus obtain the estimate

$$\begin{aligned}\widetilde{F}_k &:= \|\Delta^{r-1}(S^k-T^k)p\|\\ &\le \frac{CA_{r-1}N^{r-1}}{N^{(r-1)k}}\sum_{n=0}^{k-1}N^{(r-1)n}\sum\nolimits_{(\alpha_1,\dots,\alpha_r)\in\Omega_r}\|\Delta T^n p\|^{\alpha_1}\cdots\|\Delta^r T^n p\|^{\alpha_r}.\end{aligned}$$

This time, in contrast to the estimate for $\widetilde{D}_k$ above, we are able to estimate $\|\Delta^r T^n p\|$ by (22) and do not have to resort to the crude estimate $\|\Delta^r T^n p\| \le 2\|\Delta^{r-1}T^n p\|$. Again we employ the inequality $\sum_{j=3}^r\alpha_j \le \eta_r$. Thus we obtain

$$\begin{aligned}\widetilde{F}_k &\le \frac{C'}{N^{(r-1)k}}\sum_{n=0}^{k-1}N^{(r-1)n}\sum_{(\alpha_1,\dots,\alpha_r)\in\Omega_r}\left(\frac{\delta'_{1,0}h}{N^n}\right)^{\alpha_1}\left(\frac{\delta'_{2,0}h^2}{N^{2n}}\right)^{\alpha_2}\left(\frac{\delta'_{3,\epsilon}h^3}{N^{(3-\epsilon)n}}\right)^{\alpha_3}\cdots\left(\frac{\delta'_{r,\epsilon}h^r}{N^{(r-\epsilon)n}}\right)^{\alpha_r}\\ &\le \frac{C''}{N^{(r-1)k}}\sum_{n=0}^{k-1}N^{(r-1)n}\sum_{(\alpha_1,\dots,\alpha_r)\in\Omega_r}\frac{h^{\alpha_1+2\alpha_2+\cdots+r\alpha_r}}{(N^{\alpha_1+2\alpha_2+\cdots+r\alpha_r-\sum_{j=3}^r\alpha_j\epsilon})^n}\end{aligned}$$

$$\leq \frac{C''}{N^{(r-1)k}} \sum_{n=0}^{k-1} N^{(r-1)n} \sum_{(\alpha_1,\ldots,\alpha_r)\in\Omega_r} \frac{h^{r+1}}{(N^{r+1-\eta_r\epsilon})^n}.$$

We now choose $\epsilon$ such that $\eta_r\epsilon < 1$, therefore $N^{r+1-\eta_r\epsilon} \geq N^r$, and we get

$$\widetilde{F}_k \leq \frac{C'''h^{r+1}}{N^{(r-1)k}} \sum_{n=0}^{k-1} N^{-n} \leq \text{const} \cdot \frac{h^{r+1}}{N^{(r-1)k}}.$$

This implies (23):

$$\left\| \frac{N^{(r-1)k}}{h^{r-1}} \Delta^{r-1}(S^k - T^k)p \right\| = \frac{N^{(r-1)k}}{h^{r-1}} \widetilde{F}_k \leq \text{const} \cdot h^2,$$

and the proof is complete. □

COROLLARY 8. *Under the same conditions as in Lemma 6, $(S^\infty f - T^\infty f)^{(r-1)}$ exists and is bounded by a constant times $h^2$.*

## 3. Approximation order

For interpolatory subdivision schemes $S, T$ which fulfill a proximity inequality of order $r$, we show that if $S$ is a linear $C^r$ scheme, then the nonlinear scheme $T$ has approximation order $r+1$.

THEOREM 9. *Let $S, T, h_0$ be as in Lemma 6, with $S, T$ interpolatory schemes. Assume that a vector-valued function $f : \mathbb{R} \to V$ with bounded derivatives up to order $r$ is sampled at density $h$, where $h < h_0$. Then the limit function $T^\infty f$ fulfills*

$$\|f - T^\infty f\| \leq \text{const} \cdot h^{r+1}. \tag{24}$$

PROOF. In view of Lemma 1, we can without loss of generality assume that the considered subdivision scheme obeys (14). According to Corollary 8, $\|(S^\infty f - T^\infty f)^{(r-1)}\| \leq \text{const} \cdot h^2$. Because both $S$ and $T$ are interpolatory, $f(ih) = S^\infty f(ih) = T^\infty f(ih)$ for all integers $i$. We consider the function $\phi = T^\infty f - S^\infty f$ and apply Taylor's formula to the equation $0 = \phi((i+j)h)$ for all $i, j \in \mathbb{Z}$:

$$0 = jh\phi'(ih) + \cdots + \frac{(jh)^{r-2}}{(r-2)!}\phi^{(r-2)}(ih) + \frac{(jh)^{r-1}}{(r-1)!}\phi^{(r-1)}(\theta_{ij}),$$

where $\theta_{ij} \in (ih, (i+j)h)$. With $j = 1, 2, \ldots, r-2$ we get the following system of equations. It is to be understood component-wise in the coefficients of the vector functions $\phi', \ldots, \phi^{(r-1)}$:

$$\begin{bmatrix} 1 & \cdots & 1^{r-2} \\ \vdots & \ddots & \vdots \\ r-2 & \cdots & (r-2)^{r-2} \end{bmatrix} \begin{bmatrix} h\phi'(ih) \\ \vdots \\ \frac{h^{r-2}\phi^{(r-2)}(ih)}{(r-2)!} \end{bmatrix} = \frac{-h^{r-1}}{(r-1)!} \begin{bmatrix} 1^{r-1}\phi^{(r-1)}(\theta_{i,1}) \\ \vdots \\ (r-2)^{r-2}\phi^{(r-2)}(\theta_{i,r-2}) \end{bmatrix},$$

The matrix of this system, denoted by $V_{r-2}$, is a Vandermonde matrix and therefore regular. The symbol $\|V_{r-2}^{-1}\|$ denotes the norm of its inverse with respect to the maximum norm. In view of Corollary 8 we obtain

$$\max_{1\leq k\leq r-2}\left\|\frac{h^k\phi^{(k)}(ih)}{k!}\right\| \leq \|V_{r-2}^{-1}\|\cdot \text{const}\cdot h^{r+1} \implies \|\phi^{(k)}(ih)\| \leq \text{const}\cdot h^{r+1-k}$$

for $k = 1,\dots,r-2$. Taylor's formula now implies that

$$\begin{aligned} |\phi(ih+\tau)| &\leq \text{const}\cdot(\tau h^r + \frac{\tau^2}{2!}h^{r-1} + \cdots + \frac{\tau^{r-2}}{(r-2)!}h^3 + \frac{\tau^{r-1}}{(r-1)!}h^2) \\ &\leq \text{const}\cdot h^{r+1} \quad \text{for } \tau\in[0,h], i\in\mathbb{Z}. \end{aligned}$$

Thus we have shown the desired approximation order. □

Example 2. The linear $(2k+2)$-point Dubuc-Deslauriers schemes have approximation order $2k+2$, as they reproduce polynomials up to degree $2k+1$ [**1**]. They enjoy $C^r$ smoothness, where $r$ grows with $k$, but is much lower than $2k+2$. The projection analogues of these schemes according to [**4**] fulfill the conditions of Theorem 9, so they have approximation order $r+1$. The same applies to their log-exponential analogues in Lie groups, where the proximity conditions (13) were established by [**12**].

*Remark on the multivariate case:* The methods presented in this paper can in principle be adapted to show analogous results for multivariate interpolatory subdivision schemes in the regular grid case. Without going into details, we mention that the method of [**5, 3**] to prove multivariate statements along the lines of univariate ones works also for the present paper. When subdividing on $s$-dimensional grids, we essentially have to replace the forward difference operator $\Delta$ by a vector operator $\mathbf{\Delta}$, whose components are forward differences in the $s$ directions, and to deal with corresponding derived schemes with matrix masks. We should add that the proximity conditions employed in [**3**] to show $C^r$ smoothness of interpolatory schemes in Lie groups are different from the ones required in this paper, but a conversion is possible. For the Taylor expansion argument in the proof of Theorem 9 one has to choose a set of indices $j\in\mathbb{Z}^s$ suitable for polynomial interpolation.

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