On the semidiscrete differential geometry of A-surfaces and K-surfaces

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Abstract. In the category of semidiscrete surfaces with one discrete and one smooth parameter we discuss the asymptotic parametrizations, their Lelieuvre vector fields, and especially the case of constant negative Gaussian curvature. In many aspects these considerations are analogous to the well known purely smooth and purely discrete cases, while in other aspects the semidiscrete case exhibits a different behaviour. One particular example is the derived T-surface, the possibility to define Gaussian curvature via the Lelieuvre normal vector field, and the use of the T-surface's regression curves in the proof that constant Gaussian curvature is characterized by the Chebyshev property. We further identify an integral of curvatures which satisfies a semidiscrete Hirota equation.

Mathematics Subject Classification (2010). Primary 53A05; Secondary 37K.

Keywords. Semidiscrete surface, asymptotic surface, K-surface, pseudosphere.

1. Introduction

An important topic in discrete differential geometry is the study of smooth surface parametrizations $g : \mathbb{R}^d \to \mathbb{R}^n$ which arise as limits of discrete mappings (*nets*) of the form $x : \varepsilon\mathbb{Z}^d \to \mathbb{R}^n$, as $\varepsilon \to 0$. This approach was initiated by R. Sauer, whose work is summarized in his textbook [16]. For instance, the *conjugate* parametrizations g characterized by the condition

$$\{\partial_i g, \partial_j g, \partial_{ij} g\} \text{ linearly dependent} \tag{1.1}$$

for all $i, j = 1, \dots, d$, $i \neq j$ arise as limits of Q-nets which are characterized by planarity of each elementary quadrilateral. This planarity is equivalently

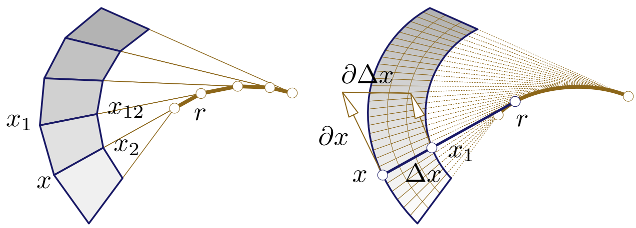

FIGURE 1. *Left:* A Q-net $x : \mathbb{Z}\times\{0,1\} \to \mathbb{R}^3$ with regression points r. *Right:* A semidiscrete Q-surface $x : \{0,1\} \times \mathbb{R}$ with regression curve r is realized as a developable strip whose singularity is the curve r.

expressed in terms of forward differences as

$$\{\Delta_i x, \Delta_j x, \Delta_{ij} x\} \text{ linearly dependent.} \tag{1.2}$$

Another example are asymptotic parametrizations g, characterized by

$$\{\partial_i g, \partial_j g, \partial_{ii} g, \partial_{jj} g\} \text{ co-planar,}$$

and which arise as limits of discrete A-nets which have planar vertex stars. With the notation Δ_i and $\Delta_{\bar{i}}$ for the forward and backward differences, this is expressed as

$$\Delta_i x, \Delta_j x, \Delta_{\bar{i}} x, \Delta_{\bar{j}} x \text{ co-planar.}$$

An important aspect is the study of surface *transformations* in the sense of [8] which means a pair or sequence of surfaces which are in some relation. This can be formalized as the study of mappings $\mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^n$. It turns out the these, like surfaces, can be seen as limits of discrete objects. The possibility of such a semidiscrete limit has been first treated by by [3] and [4]. A systematic textbook treatment of this theory and the modern viewpoint of integrable systems is contained in [7].

Partial limits of d-dimensional nets being established as a device for treating surface transformations, it is natural to ask general questions about the limit of d-dimensional nets which appear when d_1 coordinates remain discrete and $d_2 = d - d_1$ coordinates go to a continuous limit. It is especially the case $d_1 = d_2 = 1$ of mappings $x : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}^n$ that did not receive much attention so far, which is presumably because it does not contribute much to transformation theory. Such mappings can be seen as a sequence of curves, or as a smooth family of polygons. We refer to them as *semidiscrete surfaces*. In particular the semidiscrete analogs of Q-nets have a great potential for applications, as demonstrated by [12, 13]. Their theory, and the theory of their circular and conical reductions, is similar to the respective theory of circular and conical meshes [9, 14, 6].

The present paper investigates the class of *asymptotic* semidiscrete surfaces which arise as partial limits of A-nets, and the interesting class of *surfaces of constant Gaussian curvature*, whose discrete counterpart is the K-nets

[5, 7]. The semidiscrete theory combines, as is to be expected, elements of the smooth and the discrete theories. Part of this paper will come as no surprise if one realizes that asymptotic surfaces and their transforms, and in particular K-surfaces and their Bäcklund transforms, are approximated by their respective discrete analogues [7, Ch. 5]. It is however interesting to observe the interplay of features not present in the discrete case (like a well-defined Gauss curvature) with features not present the continuous case (like T-nets derived from A-nets). It is also interesting to see how the equivalence "Chebyshev A-net $\iff$ constant Gauss curvature" is shown via the special developable surfaces derived from T-nets.

Part of the paper is based on the relation between K-surfaces and the sine-Gordon equation and its relatives. For the discrete case we again refer to the textbook [7], but we also mention that already [10, 11] treated discrete and semidiscrete versions of the sine-Gordon eqation.

2. Semidiscrete asymptotic surfaces.

2.1. Basic definitions.

For a semidiscrete surface $x(k,t)$ where k is an integer parameter and t runs in the reals, we use the notation

$$(\partial x)(k,t) = \frac{d}{dt} x(k,t),$$
$$x_1(k,t) = x(k+1,t), \quad x_{\bar{1}}(k,t) = x(k-1,t), \quad \Delta x = x_1 - x.$$

For visualization purposes, we associate with x a piecewise ruled surface, consisting of the union of segments $x(k,t)x(k+1,t)$ This *strip model* associated with x is interesting also for other reasons. In case of semidiscrete Q-surfaces, which are defined by

$$\{\Delta x, \partial x, \partial \Delta x\} \quad \text{linearly dependent,} \tag{2.1}$$

the associated ruled surface strips are *developable* (see Figure 1). The locus of their singularities is given by

$$r = x + u^* \Delta x, \quad \text{where} \quad u^* = -\frac{\partial x \times \Delta x}{\Delta \partial x \times \Delta x} = -\frac{\operatorname{area}(\partial x, \Delta x)}{\operatorname{area}(\partial x_1, \Delta x)}. \tag{2.2}$$

The analogy of (2.1) with Equations (1.1) and (1.2) is obvious (the application potential of such *developable strip models* for geometry processing has been shown by [12, 13]). We proceed with the definition of semidiscrete A-surfaces, which is the main topic of the present paper.

Definition 2.1.1. *A semidiscrete surface is asymptotic, i.e., an A-surface, if and only if vectors*

$$\partial x, \ \partial^2 x, \ \Delta x = x_1 - x, \ \Delta x_{\bar{1}} = x - x_{\bar{1}}$$

are co-planar.

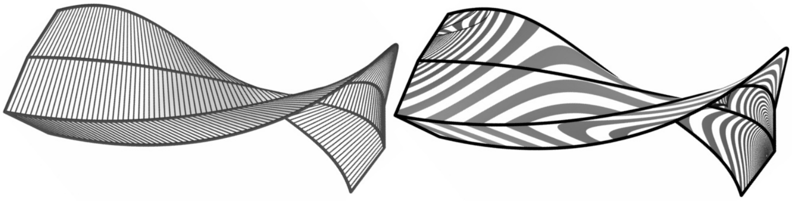

FIGURE 2. A semidiscrete A-net. *Left:* Rulings connecting points $x(k,t)$ and $x(k+1,t)$. *Right:* Reflection of a striped pattern shows that this surface is globally C^1, since reflection lines are continuous.

Note that an equivalent definition is co-planarity of $\{\partial x, \partial^2 x, \Delta x, \Delta^2 x_{\overline{1}}\}$. Like for Q-surfaces, there is a direct analogy to both the smooth and discrete cases. In contrast to the developable strip models which are only piecewise smooth, the defining property of A-surfaces implies that each point, even on the boundary of strips, has a unique tangent plane. Thus the strip model is a C^1 surface, provided each single strip is a regular ruled surface parametrization, and successive strips lie to either side of their common boundary curve. This is the case in Figure 2, but for instance the discrete pseudospheres of Figure 5 do not everywhere obey this condition.

2.2. Lelieuvre vector fields

Lelieuvre vector fields (L-fields) are a device useful for the study of A-nets. They can already be defined for a curve $c(t)$, as a normal vector field $u(t)$ with

$$\dot{c} = \pm u \times \dot{u}.$$

Such an L-vector field is almost determined by its direction. If n is some unit normal vector field n, then $u = \lambda n$ is found by $u \times \dot{u} = \lambda n \times (\dot{\lambda} n + \lambda \dot{n}) = \lambda^2 n \times \dot{n}$ $\implies \pm\langle \dot{c}, \dot{c}\rangle = \det(n, \dot{n}, \dot{c})\lambda^2$. It follows that

$$\dot{c} = \varepsilon u \times \dot{u} \implies u = \pm\Big(\varepsilon \frac{\langle \dot{c}, \dot{c}\rangle}{\det(n, \dot{n}, \dot{c})}\Big)^{1/2} \cdot n \quad (\varepsilon = \pm 1). \tag{2.3}$$

Apparently the choice of ε is determined by the sign of $\det(n, \dot{n}, \dot{c})$. It is well known that smooth asymptotic parametrizations possess normal vector fields which are multiple L-fields. The following is taken from [2]:

Proposition 2.2.1. *Assume that $x(t_1, t_2)$ is a smooth asymptotic parametrization such that $\partial_1 x$, $\partial_2 x$, $\partial_{12} x$ constitute a positive basis. Then the vector field $u = \frac{\partial_1 x \times \partial_2 x}{F}$, where $F = \det(\partial_1 x, \partial_2 x, \partial_{12} x)^{1/2}$, has the Lelieuvre property with respect to both parameter lines: $u \times \partial_1 u = \partial_1 x$, $u \times \partial_2 u = -\partial_2 x$. Either condition determines u up to sign. The Gaussian curvature K of x obeys $K = -\|u\|^{-4}$.*

The orientation of the above mentioned basis can be reversed by applying a mirror reflection to the surface. Discrete surfaces possess Lelieuvre vector fields similar to smooth ones. We state the result found e.g. in [7]:

Proposition 2.2.2. *Suppose $x : \mathbb{Z}^2 \to \mathbb{R}^3$ is a discrete asymptotic parametrization. Then there exists a Lelieuvre vector field $u : \mathbb{Z}^2 \to \mathbb{R}^3$ such that $\Delta_1 x = u \times \Delta_1 u$, $\Delta_2 x = -u \times \Delta_2 u$. It is unique up to rescaling with a factor $\alpha(i,j) = const \cdot (-1)^{i+j}$.*

We show below that for semidiscrete surfaces a Lelieuvre vector field u can be found, provided $\det(\partial\Delta x, \partial x, \Delta x) > 0$. If this is not the case, apply a mirror reflection to x to make this construction work.

Proposition 2.2.3. *For a semidiscrete asymptotic surface $x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$, define the Lelieuvre vector field $u : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$ by*

$$u = \frac{\partial x \times \Delta x}{F} \quad \textit{where} \tag{2.4}$$

$$F = \det(\partial x_1, \partial x, \Delta x)^{1/2} = \det(\partial\Delta x, \partial x, \Delta x)^{1/2}. \tag{2.5}$$

It has the properties

$$\partial x = u \times \partial u, \quad \Delta x = -u \times \Delta u, \tag{2.6}$$

and is determined uniquely up to sign by (2.6). *The expression*

$$u = \frac{1}{F_{\bar{1}}}(\partial x \times \Delta x_{\bar{1}}). \tag{2.7}$$

is an alternative description of the same surface.

Proof. The desired relation $\partial x = u \times \partial u$ implies $u = \partial x \times \Delta x / F$, for some yet unknown factor F. The computation

$$\begin{aligned} u \times \partial u &= u \times \Big(\frac{1}{F}(\partial^2 x \times \Delta x + \partial x \times \Delta \partial x) \Big) \\ &= \frac{1}{F^2} \Big(\Delta x \det(\partial x, \partial^2 x, \Delta x) - \partial x \det(\Delta x, \partial x, \Delta \partial x) \Big), \end{aligned}$$

observing $\det(\partial x, \partial^2 x, \Delta x) = 0$, implies (2.4), (2.5), which serve as the definition of u. Alternatively we let $u = \partial x \times \Delta x_{\bar{1}} / \widetilde{F}$ and verify the alternate expression by an analogous computation. The right hand equality in (2.6) is verified by

$$u \times u_1 = \frac{(\partial x \times \Delta x) \times (\partial x_1 \times \Delta x)}{\det(\partial x_1, \partial x, \Delta x)} = \frac{\Delta x \det(\partial x, \partial x_1, \Delta x)}{\det(\partial x_1, \partial x, \Delta x)} = -\Delta x.$$

The vector field $u(k,t)$ for each k is determined up to sign, and by (2.6) this sign cannot depend on k. ☐

Proposition 2.2.1 states a relation between the Lelieuvre vector field and Gaussian curvature, which is not present in the discrete case because of the ambiguity in the vector field's length. However, due to Prop. 2.2.3, we may define Gaussian curvature as follows:

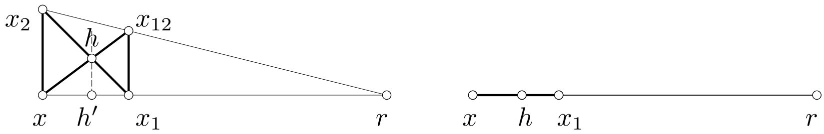

FIGURE 3. As a quadrilateral degenerates, harmonic position of x, x_1, h', r remains an invariant. In the special case that r is at infinity, h' is the midpoint of x, x_1.

Definition 2.2.4. *The Gaussian curvature K of a semidiscrete asymptotic surface x with Lelieuvre field u is given by*

$$K = -\|u\|^{-4}.$$

We see in Section 3 that this definition is "right" in the sense that $K = \text{const.}$ is equivalent to the semidiscrete analogue of the Chebyshev property which is well known in both the smooth and discrete cases.

2.3. Moutard surfaces and trapezoidal surfaces

The defining condition of M-surfaces or *Moutard* surfaces is $\partial_{12}u = q \cdot u$ for some function q in the smooth case, and $\Delta_1\Delta_2 u = \alpha(u_1 + u_2)$, for some function α, in the discrete case. Here the subscript j indicates an index shift by one of the j-th parameter. Semidiscrete M-surfaces are defined as follows:

Definition 2.3.1. *A semidiscrete surface x with the property $\Delta\partial x = \beta(x + x_1)$ for some $\beta : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is called an M-surface.*

Lemma 2.3.2. *The Lelieuvre vector field of an A-surface is an M-surface. Conversely, every M-surface u is the Lelieuvre vector field of an A-surface.*

Proof. By differentiating (2.6) we obtain $\Delta\partial x = u_1 \times \partial u_1 - u \times \partial u = -\partial u \times \Delta u - u \times \Delta\partial u \implies (u_1 + u) \times \Delta\partial u = 0$. Conversely, if u is given, then x can be defined by (2.6). This definition is consistent because of the Moutard condition. The asymptotic condition is satisfied because ∂x, Δx, $\Delta x_{\bar{1}}$ are orthogonal to u by construction, and so is $\partial^2 x = u \times \partial^2 u$. □

In the discrete case it is known that multiplying every other element of a Lelieuvre vector field by -1 creates a *trapezoidal* net, or T-net [7]. It has the defining property that the diagonals of each elementary quadrilateral are parallel. Such a construction is also possible in the semidiscrete case. In order to find the right semidiscrete version of the T-net condition, we look at Figure 3 and specialize to the case of parallel diagonals ($r = \infty$). This leads to

Definition 2.3.3. *A semidiscrete surface s is a T-surface, if each associated ruled strip is developable, with line of regression $(x + x_1)/2$.*

In the notation of (2.2), the T-condition reads $u^* = \frac{1}{2}$ and simplifies to

$$(\partial x_1 + \partial x) \times \Delta x = 0 \tag{2.8}$$

Apparently this also implies the developability condition

$$\det(\partial x, \Delta x, \partial x_1) = 0.$$

Proposition 2.3.4. *For a semidiscrete A-surface x and the corresponding Lelieuvre vector field u, the vector field $v(k,t) = (-1)^k u(k,t)$ is a trapezoidal surface, with curve of regression given by $r = (-1)^{k+1}\Delta u/2$.*

Proof. The Moutard equation directly translates to $(u_1 + u) \times (\partial u_1 - \partial u) = 0$, i.e., $\Delta v \times (\partial v_1 + \partial v) = 0$. □

Remark. A smooth A-surface x has an associated affine normal vector field $y = \partial_{12}x/F$ which obeys $\langle u, y\rangle = 1$ [2]. In the semidiscrete case an analogous construction is possible: we let $y = \Delta\partial x/F$ and have $\langle u, y\rangle = 1$.

Remark. Quantities associated with discrete surfaces may be attached to vertices, edges or faces. This is especially relevant for non-regular combinatorics, but also in the regular case we can detect the type of such a quantity by its symmetries. Semidiscrete surfaces have only vertices and edges. It turns out that the Lelieuvre vector field is associated to *edges*, since any reflection $(k,t) \mapsto (2k_0 + 1 - k, t_0 - t)$ in the parameter domain leaves F as defined by (2.5) invariant, and this reflection transforms the two equivalent definitions (2.4), (2.7) of u into each other.

3. Semidiscrete surfaces of constant Gaussian curvature

Among the smooth A-surfaces, those of constant Gaussian curvature — the K-surfaces — are characterized by an L-field u of constant norm, or equivalently, by the Chebyshev property

$$\partial_2 \|\partial_1 x\| = \partial_1 \|\partial_2 x\| = 0.$$

The proof of this characterization is based on the integrability conditions of Mainardi and Codazzi [1]. After suitable re-parametrization and scaling one obtains the sine-Gordon equation

$$\partial_{12}\varphi = \sin\varphi, \quad \text{where } \varphi = \sphericalangle(\partial_1 x, \partial_2 x). \tag{3.1}$$

Also in the discrete case, where u is not unique, we can define a K-net by the existence of a Lelieuvre vector field of constant norm, or equivalently as an A-net where $\Delta_1 \|\Delta_2 x\| = \Delta_2 \|\Delta_1 x\| = 0$ (i.e., edge lengths depend on one variable only). Further, the angle $\varphi = \sphericalangle(\Delta_1 x, \Delta_2 x)$ fulfills a discrete sine-Gordon equation, which contains the angles $\alpha_1 = \sphericalangle(u, u_1)$, $\alpha_2 = \sphericalangle(u, u_2)$. These K-nets have been the object of study for a long time (see [7, 5], and [15, 17] for earlier work). A particular reason for the interest in K-nets is their relation to integrable systems and their Bäcklund transforms.

For semidiscrete surfaces, we have already given a definition of Gaussian curvature by Def. 2.2.4, so semidiscrete K-surfaces are defined. We are going to show that they have properties similar to the smooth and discrete cases. We start this task by collecting some relations between the Lelieuvre vector field and curvatures.

3.1. The Chebyshev property of semidiscrete K-surfaces.

We introduce the moving frame $\Phi : \mathbb{Z} \times \mathbb{R} \to SO_3$ associated with the curves $x(k, \cdot)$: The first rows e^1, e^2 are found by applying Gram-Schmidt orthonormalization to ∂x, Δx, and we let $e^3 = e^1 \times e^2$. This leads either to the ordinary Frenet frame of the curves $x(k, \cdot)$, or to a modified frame which is rotated by 180 degrees about the tangent, depending on whether Δx and the principal normal point to the same side of the tangent or not. The frame Φ moves according to

$$\partial\Phi = v \begin{pmatrix} 0 & -\kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & -\tau & 0 \end{pmatrix} \Phi, \quad v = \|\partial x\|. \tag{3.2}$$

Here τ is the torsion of the curves $x(k, \cdot)$, their curvature being given by $\pm\kappa$.

Lemma 3.1.1. *The Gaussian curvature K of a semidiscrete A-surface x is related to the torsion τ of the curves $t \mapsto x(k,t)$ via $K = -\tau^2$.*

Proof. The L-field u by construction is orthogonal to both ∂x and Δx, so $u = \lambda e^3$. The computation $\partial x = u \times \partial u = (\lambda e^3) \times ((\partial\lambda)e^3 + \lambda\partial e^3) = \lambda^2 v\tau e^1$ shows that $e^1 = \lambda^2 \tau e^1$, which implies $\tau = \lambda^{-2}$ and

$$K = -\|u\|^{-4} = -\lambda^{-4} = -\tau^2. \qquad \square$$

The following theorem states that for semidiscrete surfaces the condition $K = \text{const}$ implies the Chebyshev property, which is well known from the smooth and discrete cases.

Theorem 3.1.2. *If the Gaussian curvature of the semidiscrete A-surface x is constant. then x has the Chebyshev property, namely*

$$\Delta \|\partial x\| = \partial \|\Delta x\| = 0. \tag{3.3}$$

Both the Lelieuvre vector field u and the associated T-surface $v(k,t) = (-1)^k u(k,t)$ have the Chebyshev property:

$$\Delta \|\partial u\| = 0, \qquad \partial \|u_1 + u\| = \partial \|u_1 - u\| = 0. \tag{3.4}$$

The regression curves of the T-surface are spherical curves.

Proof. It is sufficient to consider $K = -1$. Then $\|u\| = 1$, and

$$0 = \|u_1\|^2 - \|u\|^2 = \langle u_1 + u, u_1 - u\rangle.$$

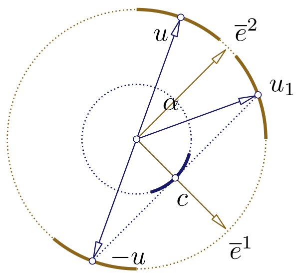

FIGURE 4. Illustration of Equation (3.5).

By Prop. 2.3.4, the ruled surface defined by generator curves u_1 and $-u$ is developable with regression curve $c = \frac{1}{2}\Delta u$. This means that vectors ∂c and $u + u_1$ are parallel:

$$\langle \partial c, \Delta u\rangle = 0 \implies \langle \partial \Delta u, \Delta u\rangle = 0 \implies \partial \left\|\Delta u\right\| = 0.$$

Consequently the following derived quantities, expressed in terms of the angle α between successive Lelieuvre field curves u, u_1 do not depend on the continuous parameter (see Fig. 4 for an illustration):

$$\sin\frac{\alpha}{2} = \frac{1}{2}\left\|u_1 - u\right\| = \left\|c\right\|, \quad \cos\frac{\alpha}{2} = \frac{1}{2}\left\|u_1 + u\right\|. \tag{3.5}$$

We let

$$\overline{e}^1 = \frac{1}{2\sin(\alpha/2)}(u_1 - u), \qquad \overline{e}^2 = \frac{1}{2\cos(\alpha/2)}(u_1 + u), \qquad \overline{e}^3 = \overline{e}^1 \times \overline{e}^2.$$

This implies $c = \sin\frac{\alpha}{2}\,\overline{e}^1$. The frame $\overline{\Phi} = (\overline{e}^1, \overline{e}^2, \overline{e}^3)^T$ moves according to

$$\partial\overline{\Phi} = \overline{v}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & \gamma \\ 0 & -\gamma & 0 \end{pmatrix}\overline{\Phi}, \qquad \overline{v} = \left\|\partial \overline{e}^1\right\|. \tag{3.6}$$

Consequently,

$$u = \cos\frac{\alpha}{2}\,\overline{e}^2 - \sin\frac{\alpha}{2}\,\overline{e}^1 \implies \partial u = \overline{v}(\cos\frac{\alpha}{2}(-\overline{e}^1 + \gamma\overline{e}^3) - \sin\frac{\alpha}{2}\overline{e}^2),$$
$$u_1 = \cos\frac{\alpha}{2}\,\overline{e}^2 + \sin\frac{\alpha}{2}\,\overline{e}^1 \implies \partial u_1 = \overline{v}(\cos\frac{\alpha}{2}(-\overline{e}^1 + \gamma\overline{e}^3) + \sin\frac{\alpha}{2}\,\overline{e}^2).$$

We conclude $\left\|\partial u\right\| = \left\|\partial u_1\right\|$. Together with (3.5) this implies the Chebyshev property for u and the derived T-surface. As to x itself, recall from the proof of Lemma 3.1.1 that $u(k, \cdot)$ is the binormal vector of the curve $x(k, \cdot)$ whose torsion equals 1. The Frenet equations then directly imply

$$\left\|\partial u\right\| = \left\|\partial x\right\|, \tag{3.7}$$

and so $\Delta\left\|\partial x\right\| = 0$. As to $\Delta x = u_1 \times u$, we compute

$$\Delta x = 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\,\overline{e}^1 \times \overline{e}^2 = \sin\alpha\,\overline{e}^3. \tag{3.8}$$

This shows $\partial\left\|\Delta x\right\| = 0$, which completes the proof. □

Corollary 3.1.3. *For a semidiscrete A-surface of constant Gaussian curvature K, the length of ruling segments is bounded by*

$$\|\Delta x\| < |K|^{-1/2}.$$

Proof. Scaling with $C > 0$ causes $\|\Delta x\|$, K to scale with factors C, $1/C^2$, resp. It is therefore sufficient to show the case $K = -1$, which follows from (3.8). □

The converse of Theorem 3.1.2 requires an auxiliary lemma:

Lemma 3.1.4. *Consider curves $t \mapsto e^1(t) \in S^2$, $c(t) = \sigma(t)e^1(t)$, and $u = c - \rho\partial c$, $u_1 = c + \rho\partial c$, with positive functions σ, ρ (the meaning of this is the the ruled strip defined by u, u_1 is a T-surface with c as the curve of regression). Then $\|u \times \partial u\| = \|u_1 \times \partial u_1\|$ and $\partial \|u \times u_1\| = 0 \implies \sigma, \rho = \text{const}$.*

Proof. A parameter transform does not influence the assumptions, so we assume $\|\partial e^1\| = 1$. Extend e^1 to a moving frame $(e^1, e^2, e^3)^T$ which obeys (3.6), i.e., $\partial e^1 = e^2$ and $\partial e^2 = -e^1 + \gamma e^3$. In the following we describe vectors by coordinate columns with respect to this frame. We get

$$c = \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix}, \ \partial c = \begin{bmatrix} \partial\sigma \\ \sigma \\ 0 \end{bmatrix}, \ u \times u_1 = \begin{bmatrix} \sigma - \rho\partial\sigma \\ -\rho\sigma \\ 0 \end{bmatrix} \times \begin{bmatrix} \sigma + \rho\partial\sigma \\ \rho\sigma \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2\sigma^2\rho \end{bmatrix}.$$

It follows that $\sigma^2\rho$ is constant, so $2\sigma\rho\partial\sigma + \sigma^2\partial\rho = 0$, i.e., $2\rho\partial\sigma + \sigma\partial\rho = 0$. We employ this equation when simplifying the second coordinates of $\partial u, \partial u_1$, and get the following expression for $u \times \partial u$ (in case $\varepsilon = -1$) and $u_1 \times \partial u_1$ (in case $\varepsilon = +1$):

$$\begin{bmatrix} \sigma + \varepsilon\rho\partial\sigma \\ \varepsilon\rho\sigma \\ 0 \end{bmatrix} \times \begin{bmatrix} \partial\sigma + \varepsilon\partial(\rho\partial\sigma) - \varepsilon\sigma\rho \\ \sigma \\ \varepsilon\gamma\sigma\rho \end{bmatrix} = \begin{bmatrix} \gamma\rho^2\sigma^2 \\ -\varepsilon(\sigma + \varepsilon\rho\partial\sigma)\gamma\rho\sigma \\ \sigma^2 - \rho\sigma\partial(\rho\partial\sigma) + \rho^2\sigma^2 \end{bmatrix}.$$

The requirement $\|u \times \partial u\| = \|u_1 \times \partial u_1\|$ now reduces to $(\sigma + \rho\partial\sigma)^2 = (-\sigma + \rho\partial\sigma)^2 \iff \sigma\rho\partial\sigma = 0 \iff \partial\sigma = 0$. Therefore σ is constant, and so is ρ. □

Theorem 3.1.5. *If a semidiscrete A-surface x is a Chebyshev net, i.e.,*

$$\Delta \|\partial x\| = \partial \|\Delta x\| = 0,$$

then its Gaussian curvature K is constant, and its Lelieuvre vector field u has constant length and enjoys the Chebyshev property:

$$\|u\| = \text{const}, \quad \Delta \|\partial u\| = \partial \|\Delta u\| = 0.$$

Proof. By Prop. 2.3.4, the ruled surface strip defined by u, u_1 is developable with the curve of regression $c = \frac{1}{2}(u_1 - u)$. The required Chebyshev property means that we can apply Lemma 3.1.4. With constants σ, ρ from there, we get $\|u\|^2 = \sigma^2 + \rho^2 = \|u_1\|^2$, so the Gaussian curvature $K = -\|u\|^{-4}$ is constant. The remaining statements follow either directly or from Theorem 3.1.2. □

3.2. Construction of K-surfaces from initial values

Finding a one-strip K-surface $x : \{0, 1\} \times \mathbb{R} \to \mathbb{R}^3$ is easy, since we only have to choose the spherical regression curve c of the derived T-surface and compute u, u_1 by intersecting c's tangent surface with a co-centric sphere. Extension of a K-surface $x(k, t)$ which is defined for values $k \leq k_0$ to $k > k_0$ can be done by means of solving the Moutard equation for its Lelieuvre field, and we can prescribe the values $\alpha(k)$ arbitrarily:

Assume that $\|u(k, t)\| = 1$ for $k = k_0$ and that $u_1(k_0, t) = u(k_0 + 1, t)$ is a solution of the ordinary (Moutard) differential equation

$$\partial u_1 = \partial u - \frac{\langle \partial u, u_1 \rangle}{1 + \cos \alpha}(u + u_1), \tag{3.9}$$

where initial values are chosen such that $\|u_1\| = 1$, $\langle u, u_1 \rangle = \cos \alpha$ for some value $(k, t) = (k_0, t_0)$.

It turns out that these two equalities then hold true for all t: In order to show that a solution curve u_1 of (3.9) does not leave the surface $S = \{(t, x) \in \mathbb{R} \times \mathbb{R}^3 \mid \|x\|^2 = 1, \langle u(t), x \rangle = \cos \alpha\}$, it is sufficient to show that ∂u_1, seen as a function of t and u_1, is everywhere in S tangential to S. For that, we compute

$$\partial \langle u_1, u_1 \rangle = 2\langle \partial u_1, u_1 \rangle = 2\langle \partial u, u_1 \rangle - 2\frac{\langle \partial u, u_1 \rangle}{1 + \cos \alpha}\langle u + u_1, u_1 \rangle = 0,$$

$$\partial \langle u, u_1 \rangle = \langle \partial u, u_1 \rangle + \langle u, \partial u_1 \rangle = \langle \partial u, u_1 \rangle - \frac{\langle \partial u, u_1 \rangle}{1 + \cos \alpha}\langle u, u + u_1 \rangle = 0,$$

since $\langle u, u \rangle = 1$ and $\langle u, \partial u \rangle = 0$. We conclude that u_1 maps into the unit sphere.

Having constructed $u(k, t)$ as a Moutard surface, we can derive an A-surface $x(k, t)$. The Gaussian curvature of x equals $-\|u\|^{-4} = -1$.

3.3. The angle between parameter lines

This section deals with an equation of sine-Gordon type for the angles $\varphi = \sphericalangle(\partial x, \Delta x)$ enclosed by the smooth and the discrete parameter lines. We assume that the union of ruled surface strips associated with a semidiscrete K-surface is a C^1 surface, i.e., $x_{\bar{1}} - x$, $x_1 - x$ lie in two halves of the tangent plane which are defined by the vector ∂x. We can therefore give an orientation to this surface and assume a positive parametrization. The Frenet-type frame $\Phi = (e^1, e^2, e^3)^T$ introduced earlier then has the property that e^1, e^2 is a positive basis of the tangent plane. The Chebyshev property implies that we can re-parametrize such that $\|\partial x\| = 1$. We now consider the angle

$$\varphi = \sphericalangle(\partial x, \Delta x). \tag{3.10}$$

Another immediate consequence of the Chebyshev property is $\langle \partial x_1, \Delta x \rangle = \langle \partial x, \Delta x \rangle$, so

$$\varphi = \sphericalangle(\partial x_1, \Delta x), \tag{3.11}$$

In the following we derive relations between the functions κ and φ, which measure the change in angle of derivatives along parameter lines, and the constant $\tau = \sqrt{-K}$ and the function α, both of which measure the change in angle of normal vectors along parameter lines. Observe that $\partial\alpha = 0$.

Theorem 3.3.1. *Assume that the smooth parameter lines of a semidiscrete K-surface are parametrized by arc length, i.e., $\|\partial x\| = 1$. Then the quantities κ, φ, $\tau = \sqrt{-K}$, and α obey the relation*

$$\kappa_1 + \kappa = -2\partial\varphi, \qquad \kappa_1 - \kappa = -2\tau \sin\varphi \tan\frac{\alpha}{2}. \tag{3.12}$$

Further, the angle φ fulfills

$$\partial\Delta\varphi = \tau \sin\varphi_1 \tan\frac{\alpha_1}{2} + \tau \sin\varphi \tan\frac{\alpha}{2} \tag{3.13}$$

which is a semidiscrete sine-Gordon equation.

Proof. The equality of angles expressed by Equations (3.10), (3.11) means that rotating the frame Φ about the axis

$$\frac{\Delta x}{\|\Delta x\|} = \cos\varphi\, e^1 + \sin\varphi\, e^2 = \cos\varphi\, {e^1}_1 + \sin\varphi\, {e^2}_1 \tag{3.14}$$

yields the frame Φ_1 associated with the curve x_1. The plane orthogonal to the rotation axis has the orthonormal basis $\{r, e^3\}$, where $r = -\sin\varphi\, e^1 + \cos\varphi\, e^2$ and the Lelieuvre vector field u is a multiple of e^3. This rotation maps r, e^3 to $r_1, {e^3}_1$, and it follows that

$$\langle e^3, {e^3}_1\rangle = \cos\alpha, \qquad \langle r_1, e^3\rangle = -\sin\alpha, \qquad \langle r, {e^3}_1\rangle = \sin\alpha.$$

Differentiating (3.14) and using (3.2) with $v = \|\partial x\| = 1$ yields

$$\partial\frac{\Delta x}{\|\Delta x\|} = (\kappa + \partial\varphi) r + \tau \sin\varphi\, e^3 = (\kappa_1 + \partial\varphi) r_1 + \tau\sin\varphi\, {e^3}_1.$$

By taking scalar products with e^3 and ${e^3}_1$ and reordering terms we obtain

$$\kappa_1 + \partial\varphi = -\tau\sin\varphi\tan\frac{\alpha}{2}, \qquad \kappa + \partial\varphi = \tau\sin\varphi\tan\frac{\alpha}{2}.$$

This immediately yields (3.12). The second equation above also implies $\kappa_1 + \partial\varphi_1 = \tau\sin\varphi_1\tan\frac{\alpha_1}{2}$ from which we can eliminate κ_1 and achieve (3.13). $\square$

Remark. If g, g^+ is a Bäcklund pair of smooth K-surfaces, then corresponding asymptotic parameter lines constitute a semidiscrete K-surface $x : \{0,1\} \times \mathbb{R} \to \mathbb{R}^3$, by the well known properties of the Bäcklund transform. The previous theorem can be directly applied to the geodesic curvatures κ, κ^+ of these asymptotic lines and the angle φ enclosed by the parameter lines the vectors connecting corresponding points.

Remark. If a continuous K-surface $x(t_1, t_2)$ occurs as the limit of a semidiscrete one, then $\partial\varphi \to \partial_1\varphi$, $\frac{\Delta\varphi}{\|\Delta x\|} \to \partial_2\varphi$, and also $\frac{\alpha}{\|\Delta x\|} \to \sqrt{-K}$, because the torsion of parameter lines equals $\sqrt{-K}$. Therefore the semidiscrete sine-Gordon equation (3.13) tends to the continuous sine-Gordon equation (3.1).

The semidiscrete sine-Gordon equation (3.13) can of course also be derived from the discrete sine-Gordon equation (see [5, 7]).

Remark. The 'Frenet' frame Φ associated with a K-surface x which is defined in the previous section always contains the surface normal in its third row. Equation (3.2) and the first paragraph of the proof of Theorem 3.3.1 express the motion of this frame. In order to establish the connections with the notation of [5, 7], we employ the SU_2 representation of rotations and identify vectors $a = (a_1, a_2, a_3)$ with matrices $\left(\begin{smallmatrix} -ia_3 & -a_2-ia_1 \\ a_2-ia_1 & ia_3 \end{smallmatrix}\right) \in \mathfrak{su}_2$. Then the rotation $R_{a,\theta}$ about the axis a with $\|a\| = 1$ and the angle θ is expressed as $R_{a,\theta} = \cos\frac{\theta}{2}E_2 + \sin\frac{\theta}{2}\, a \in \mathrm{SU}_2$. In this notation, the motion of Φ reads

$$\partial\Phi = X\Phi, \quad \Phi_1 = Y\Phi, \quad (X \in \mathfrak{su}_2,\ Y \in \mathrm{SU}_2)$$

where the infinitesimal rotation X and the rotation Y have the form

$$X = -\frac{i}{2}\begin{pmatrix} \kappa & \tau \\ \tau & -\kappa \end{pmatrix},$$

$$Y = \cos\frac{\alpha}{2}E_2 + \sin\frac{\alpha}{2}\begin{pmatrix} 0 & -\sin\varphi - i\cos\varphi \\ \sin\varphi - i\cos\varphi & 0 \end{pmatrix}.$$

The transition matrices X, Y are analogous to the ones encountered in the discrete case (cf. Equations (4.33) and (4.34) of [7], bearing in mind that the infinitesimal angle between normal vectors in points $x(k,t)$ and $x(k,t+dt)$ equals $\tau\, dt$).

3.4. Hirota equation and Bäcklund transform.

In order to extend the analogy between the discrete and semidiscrete cases to the Hirota equation which prominently features in the discrete case, one seeks a function $\psi : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ with

$$-\kappa = \partial\psi, \qquad \varphi = \frac{1}{2}(\psi_1 + \psi).$$

The consistency of this equation follows directly from (3.12), and therefore ψ is found by integrating the curvature κ.

Theorem 3.4.1. *Consider a semidiscrete K-surface whose smooth parameter lines are parametrized by arc length, and the associated quantities $\kappa, \varphi, \tau = \sqrt{-K}, \alpha$. Then there is a function $\psi : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ which obeys*

$$\partial\psi = -\kappa, \quad \psi_1 + \psi = 2\varphi.$$

It satisfies

$$\frac{1}{4}\partial\Delta\psi = \frac{\tau}{2}\tan\frac{\alpha}{2}\sin\frac{\psi_1 + \psi}{2}, \tag{3.15}$$

which is a semidiscrete Hirota equation.

Proof. Expanding shows that (3.15) is equivalent to the known equation $-\Delta\kappa = 2\tau\tan\frac{\alpha}{2}\sin\varphi$. ∎

Remark. Equation (3.15) was called a semidiscrete sine-Gordon equation in [10, 11].

We do not discuss the fact that any solution of the Hirota equation leads to a K-surface. It is not difficult to show but anyway follows from the analogous fact known in the discrete theory and a limit argument. Further, it is obvious that any triple consisting of a constant τ, a sequence $\alpha(k)$, and a solution ψ of (3.15) does in fact describe an entire family of solutions with ψ unchanged, but the torsion τ and the angle function α modified by

$$\tau^{(\lambda)} := \lambda\tau, \qquad \tan\frac{\alpha^{(\lambda)}}{2} := \frac{1}{\lambda}\tan\frac{\alpha}{2}.$$

This leads to an associated family of semidiscrete K-surfaces. Another result we state without proof is the semidiscrete version of the Bäcklund transform: Any solution ψ of the Hirota equations gives rise to another one, $\widetilde{\psi}$, related by

$$\frac{\partial(\widetilde{\psi}+\psi)}{2} = \frac{\tau}{\tan\gamma/2}\sin\frac{\widetilde{\psi}-\psi}{2},$$
$$\sin\frac{\Delta(\widetilde{\psi}-\psi)}{4} = \tan\frac{\alpha}{2}\tan\frac{\gamma}{2}\sin\frac{\widetilde{\psi}_1+\widetilde{\psi}+\psi_1+\psi}{4}.$$

3.5. Pseudospheres

It is not difficult to verify that the solution $\psi = \text{const.} = 0$ of the Hirota equation has the following Bäcklund transform $\widetilde{\psi}$:

$$\tan\frac{\widetilde{\psi}(k,t)}{4} = \tan\frac{\widetilde{\psi}(0,0)}{4}\exp\Big(\frac{\tau}{\tan(\gamma/2)}t\Big)\prod_{j=0}^{k}\frac{1+\tan\gamma\tan\frac{\alpha_j}{2}}{1-\tan\gamma\tan\frac{\alpha_j}{2}} \qquad (k \geq 0),$$

the case $k < 0$ being similar. It is shown by [5] how to construct a K-surface from $\widetilde{\psi}$ via the *Sym formula.* It is clear from a limit argument that this construction, which is the same for both the smooth and the discrete cases, works also in the semidiscrete case. The result for the special case $\alpha = \text{const}$ is

$$x(k,t) = \frac{r}{1+r^2}\begin{bmatrix}\sin\omega(k,t)\sin(-2k\arctan v - 2t)\\ \sin\omega(k,t)\cos(-2k\arctan v - 2t)\\ \cos\omega(k,t)\end{bmatrix} + \Big(t - \frac{kv}{1+v^2}\Big)\begin{bmatrix}0\\0\\1\end{bmatrix},$$
$$\omega(k,t) = 2\arctan\Big(\exp\big(\frac{2t}{r}\big)\Big(\frac{1-vr}{1+vr}\Big)^k\Big).$$

Regardless of how we arrived at this equation one can verify the K-surface property. The letters r, v for the free parameters are chosen to show the analogy to the example in [5]. Converting the power in the last formula into an exponential, we see that the dependence of $x(k,t)$ on k and t is governed by three linear forms on $\mathbb{Z} \times \mathbb{R}$:

$$a_1^*(k,t) = k \arctan v + t,$$
$$a_2^*(k,t) = -kv/(1+v^2) + t,$$
$$a_3^*(k,t) = k\big(\ln(1-vr) - \ln(1+vr)\big) + (2/r)t.$$

We want to make $x(t,k)$ periodic in some direction. For this purpose we first look for some point $(k_0, t_0) \in \mathbb{Z} \times \mathbb{R}$ such that

$$a_1^*(k_0,t_0) = \pi, \qquad a_2^*(k_0,t_0) = 0.$$

(we could also require $a_1^*(k_0,t_0) = n\pi$, for some integer n). If k_0 is given, this yields an implicit equation for v and an explicit formula for t_0, namely

$$\arctan v + v/(v^2+1) = \pi/k_0, \qquad t_0 = \pi - k_0 \arctan v.$$

Next, we determine r such that a_2^*, a_3^* are linearly dependent. This is an implicit equation for r. We have then successfully converted the K-surface $x(k,t)$ into the form

$$x(k,t) = F(a_1^*(k,t), a_2^*(k,t))$$

with the property

$$x(k+k_0, t+t_0) = F(a_1^*(k,t)+\pi, a_2^*(k,t)) = x(k,t)$$

because $a_1^*(k,t)$ occurs only in the form $\sin(-2a_1^*)$ and $\cos(-2a_1^*)$. By choosing $k_0 = 3, 4, \ldots$ we achieve a sequence of K-surfaces which we call semidiscrete pseudospheres. With the factors

$$\rho = \frac{a_1^*(1,0)}{a_1^*(0,1)}, \qquad \sigma = \frac{a_2^*(1,0)}{a_2^*(0,1)}$$

we also see that

$$\begin{aligned} x(k+1,t) &= F(a_1^*(k,t) + a_1^*(1,0), a_2^*(k,t) + a_2^*(1,0)) \\ &= F(a_1^*(k, t+\rho) a_2^*(k, t+\sigma)) \\ &= F(a_1^*(k, t+\sigma+(\rho-\sigma)) a_2^*(k, t+\sigma)), \end{aligned}$$

so all curves $x(k,\cdot)$ are in fact the same, up to rotation by the angle $2(\rho-\sigma)$ and the parameter change $t \mapsto t+\sigma$. The periodicity of the surfaces implies that the rotation angle between neighbouring curves $x(k,\cdot)$ has the value $2(\rho-\sigma) = \frac{2\pi}{k_0}$.

Remark. We observe one further symmetry: The substitution $t \mapsto -t$ causes the curve $x(0,t)$ to be rotated by 180 degrees about the x_2 axis. It follows that all curves $x(k,\cdot)$ have an analogous symmetry.

Acknowledgments

The author gratefully acknowledges the support of the Austrian Science Fund (FWF) under grant S92-09, which is part of the National Research Network 'Industrial Geometry'.

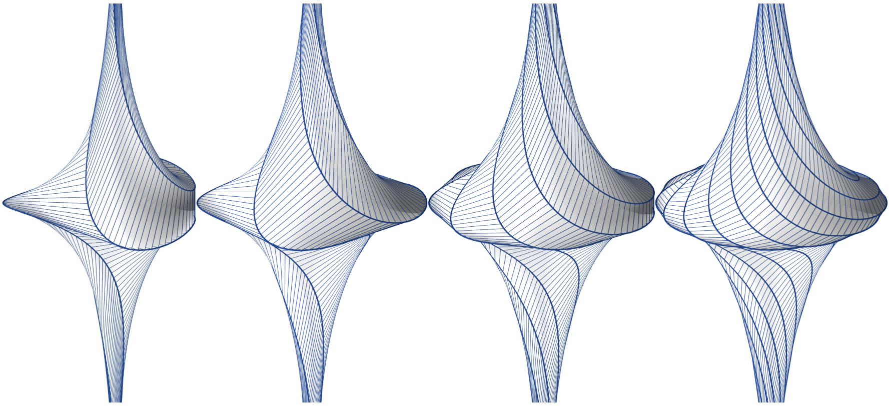

FIGURE 5. Semidiscrete pseudospheres, which are smooth even if their rulings are not. From left to right, $k_0 = 3, 4, 7, 12$.

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