Computing quadrilateral and conical meshes

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It is well known that the network of principal curvature lines can be discretized by circular meshes, i.e., quadrilateral meshes with planar faces, where the vertices of each face are co-circular [1]. A new discretization is *conical meshes*, where we require faces adjacent to a vertex to be co-conical; to be precise: the oriented planes which carry those faces are tangent to an oriented cone of revolution [2]. In the smooth case, infinitesimally neighbouring surface normals along a principal curvature line are co-planar (this is a characterization) – in the discrete case, neighbouring axes of circles/cones of a circular/conical mesh are co-planar.

In the S^3 model of Möbius geometry, co-circular vertices lie in $S^3 \cap U$, with $\dim U = 2$. Analogously, in the Blaschke cylinder model $S^2 \times \mathbb{R}$ of Laguerre geometry, co-conical faces appear as points which lie in $(S^3 \times \mathbb{R}) \cap U$, with $\dim U = 2$. In this way, both the circular and conical meshes appear as quadrilateral meshes in the appropriate geometric model. Möbius/Laguerre transformations transform circular/conical meshes into meshes of the same property, an important example of a Laguerre transformation being the offsetting operation. The latter leads to applications of conical meshes in architectural design.

For a conical mesh, the unit normal vectors of the faces constitute a circular mesh in S^2, which implies that 3D consistency of the conical condition follows directly from Miquel's theorem. It is interesting to note that the rhombic networks of [3] which are models of surfaces of constant curvature have diagonals which constitute a mesh which is both circular and conical.

We express planarity/circularity/conicality of a mesh in terms of the angles $\phi_{e,f}$ enclosed by edges e, f of the mesh ($0 \leq \phi_{e,f} \leq \pi$): A face with boundary edges $e_1, \dots, e_n$ is planar and convex $\iff \sum \phi_{e_i, e_{i+1}} = (n-2)\pi$ (Fenchel's theorem). In the case $n = 4$, it is in addition circular $\iff$ the sums of opposite angles equal π. If $e_1, \dots, e_4$ are the edges emanating successively from a vertex, then this vertex is conical $\iff \phi_{e_1,e_2} + \phi_{e_3,e_4} = \phi_{e_2,e_3} + \phi_{e_4,e_1}$ (Lexell's theorem).

By summing up the squares of these conditions we arrive at a nonnegative geometry functional $F_G(v_1, \dots)$ on the vertices where $F_G = 0$ characterizes meshes of the required properties. In order to perturb a given mesh such that it becomes planar/circular/conical, we numerically optimize in the space of vertices such that $F_G \to 0$ and in addition $F_P, F_F \to \min$, where F_P and F_F are nonnegative functionals expressing distance from a target surface and mesh fairness, resp.

References

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