EQUIFORM KINEMATICS
AND THE GEOMETRY OF LINE ELEMENTS

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Abstract. The present paper studies Plücker coordinates for line elements in Euclidean three-space. The well known relation between line geometry and kinematics is generalized to equiform motions and the geometry of line elements. We consider bundles and linear complexes of line elements and survey their properties.

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1. Introduction and motivation

The geometry of lines is a classical topic (see e.g. [24]) which is of interest not only for its own sake. Given the nature of its object of study – the lines of Euclidean or projective three-space – it is natural that frequently problems on the borderlines between mathematics, computer science, and engineering are solved with line-geometric methods. Especially we would like to mention recent work in Computer Vision [12, 22, 27], on reverse engineering and reconstruction of kinematic surfaces [10, 11, 17, 19, 21], on approximation and interpolation in line space, [1, 5, 13, 16, 18], and in general, on geometric computing with lines [8, 15]. Examples of applications of line geometry are also collected in the monograph [20]. The number of applications where lines and points on them (i.e., *line elements*) appear together raises interest in the geometry of line elements. This paper generalizes the concept of Plücker coordinates to the case of line elements and establishes some basic facts. We emphasize the relation with equiform kinematics, thus generalizing the well known relations between Euclidean kinematics [6] and classical line geometry [20].

Our interest in the geometry of line elements has its origin in our investigation of problems related to the recognition, classification and segmentation of surfaces given by point cloud data, typically obtained by laser scanning. For such data, surface normals can be estimated numerically, or are even delivered by software used for modern 3D photography. The methods used in the Computer Vision community for recognition and reconstruction of special surface types often employ the Hough transform [7], augmented by geometric tools like the Gaussian image, Laguerre geometry [14], and line geometry [2, 20, 17].

These methods have recently been extended to the geometry of line elements [4], which the present paper provides mathematical basis for. The paper [4] contains many examples of

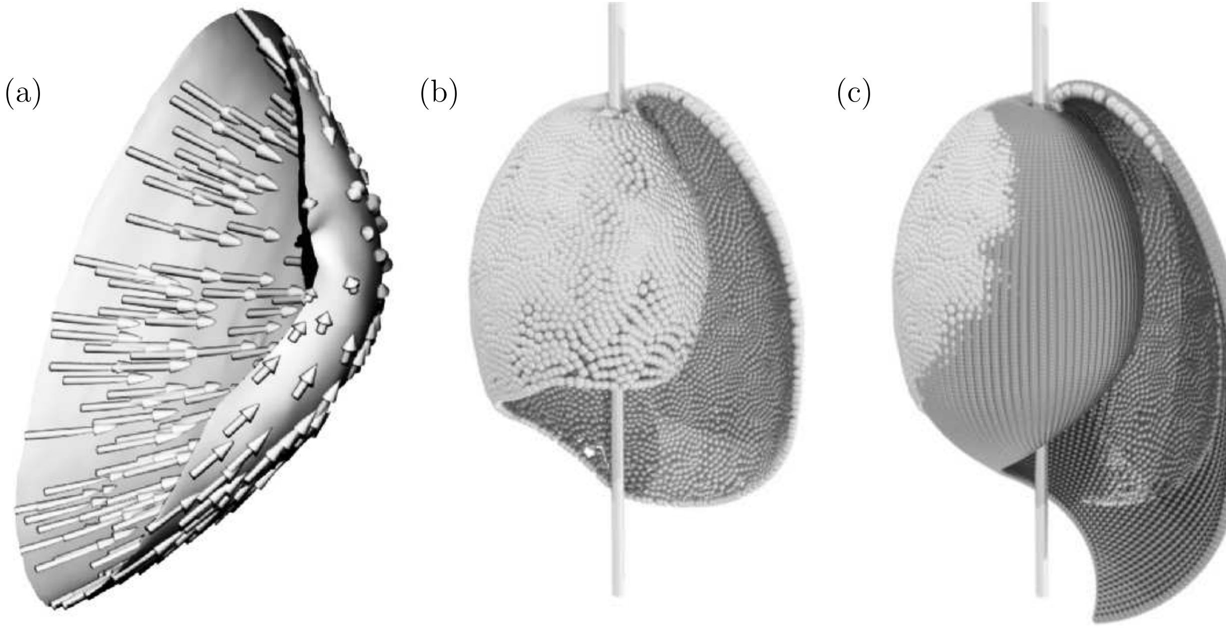

FIGURE 1. (a) Spiral surfaces possess a smooth family $(A(t), a(t), \alpha(t))$ of automorphic equiform motions. The velocity vector field $v(y)$ illustrated in this figure is almost tangent to the shell of a specimen of *saxidomus nutalli*, showing that this shell is almost an exact spiral surface. (b) Surface recognition and classification by means of point cloud data obtained from the marine snail *bulla ampulla*. The spiral axis has been found by numerically estimating surface normals and finding an equiform motion which fits these the surface normal elements. (see [4]). (c) Surface reconstruction. A spiral surface approximating the given point cloud data has been computed (see [4]).

the use of line element geometry for surface recognition, reconstruction, and segmentation. One example is given by Fig. 1.

2. The group of equiform transformations

This section describes equiform transformations, which means affine transformations whose linear part is composed from an orthogonal transformation and a homothetical transformation. Such an equiform transformation maps points $x \in \mathbb{R}^3$ according to

$$x \mapsto \alpha A x + a, \quad A \in \mathrm{SO}_3,\ a \in \mathbb{R}^3,\ \alpha \in \mathbb{R}^+. \tag{1}$$

A smooth one-parameter equiform motion moves a point x via $y(t) = \alpha(t)A(t)x + a(t)$. The velocity $\dot{y}(t)$, if expressed in terms of $y(t)$, has the form

$$v(y) = \dot{A}A^T y + \frac{\dot{\alpha}}{\alpha} y - \dot{A}A^T a - \frac{\dot{\alpha}}{\alpha} a + \dot{a}. \tag{2}$$

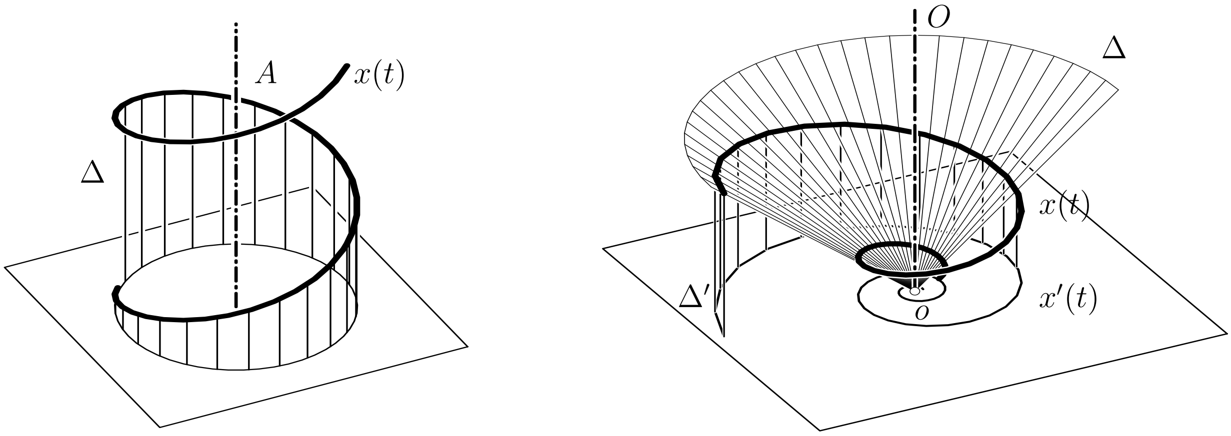

FIGURE 2. Uniform equiform motions with paths $x(t), x'(t)$ and invariant surfaces Δ, Δ'. Left: Helical motion with axis A. Right: Spiral motion with center o and axis O.

Such a velocity vector field is also illustrated in Fig. 1. Since A is orthogonal, the matrix $\dot{A}A^T := C^\times$ is skew-symmetric and the product $C^\times x$ can be written in the form $c \times x$:

$$v(y) = c \times y + \gamma y + \overline{c} \quad (\gamma = \frac{\dot{\alpha}}{\alpha},\ \overline{c} = \dot{A}A^T a - \frac{\dot{\alpha}}{\alpha} a + \dot{a}). \tag{3}$$

This expression for the velocity vector field is similar to the well known Euclidean case (see e.g. [20], §3.4.1). It follows from the general theory of Lie transformation groups [3] that any triple $(c, \overline{c}, \gamma) \in \mathbb{R}^7$ defines a unique *uniform* equiform motion (a one-parameter subgroup of the equiform group) $(A(t), a(t), \alpha(t))$ which has the property that the velocities in (3) do not depend on t, and $A(0) = E_3$, $a(0) = 0$, $\alpha(0) = 1$

2.1. **Uniform equiform motions.** In the following we give a complete list of normal forms of uniform equiform motions, where 'normal form' refers to equiform equivalence. The classification is similar to the well known Euclidean case. An equiform coordinate transformation $y = \tau T z + t$ transforms the velocity vector field (3) into

$$\widetilde{v}(z) = d \times z + \overline{d} + \delta \text{ with } d = T^{-1}c,\ \overline{d} = T^{-1}(c \times t + \overline{c} + \gamma t),\ \delta = \gamma. \tag{4}$$

We are going to choose τ, T, t such that $d, \overline{d}$ have simple coordinates. The corresponding subgroup will be denoted by $(B(t), b(t), \beta(t))$.

Case 1: $\boldsymbol{\gamma = 0,\ c \neq 0}$. This is the Euclidean case. We choose t such that $\overline{d} \parallel d$ and T such that $d = (0, 0, \omega)^T$, $\overline{d} = (0, 0, v)^T$. Then

$$B(t) = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix},\ b(t) = \begin{bmatrix} 0 \\ 0 \\ vt \end{bmatrix},\ \beta(t) = 1 \quad (\omega \neq 0). \tag{5}$$

For $v \neq 0$ this is a helical motion (see Fig. 2, left), otherwise a rotation.

Case 2: $\boldsymbol{\gamma = 0,\ c = 0,\ \bar{c} \neq 0}$. We have $d = (0,0,0)^T$, $\delta = 0$, and it is easy to find T such that $\bar{d} = (0,0,v)^T$. Then $B(t) = E_3$, $\beta(t) = 1$, and $b(t) = (0,0,vt)^T$. This is the case of a uniform translation.

Case 3: $\boldsymbol{\gamma \neq 0,\ c \neq 0}$. The equation $c \times t + \bar{c} + \gamma t = 0$ has the unique solution $t = -(C^\times + \gamma E_3)^{-1}\bar{c}$, as $\det(C^\times + \gamma E_3) = \gamma(\gamma^2 + \langle c, c\rangle) \neq 0$. Thus we can achieve $\bar{d} = 0$, and we choose T such that $d = (0,0,\omega)^T$. It follows that $B(t)$ is the same as in (5), $b(t) = 0$, and $\beta(t) = \exp(\gamma t)$. This is the generic case of a uniform spiral motion, as illustrated in Fig. 2, right.

The orbits of curves under such one-parameter subgroups are spiral surfaces [26], which nature approximates in shells whose growth is governed by scale-invariant processes. This is one of the rare physical manifestations of equiform geometry (see Fig. 1).

Case 4: $\boldsymbol{\gamma \neq 0,\ c = 0}$. It is easy to find t such that $\bar{d} = 0$. Then $B(t) = E_3$, $\beta(t) = \exp(\gamma t)$, and $b(t) = 0$. This is a subgroup of central similarities.

3. Plücker coordinates of line elements

Let L be a line in Euclidean three-space passing through a point x. In order to assign coordinates to the *line element* (L, x), we extend the familiar definition of Plücker coordinates [20, 24]:

Definition 1. *The triple $(l, \bar{l}, \lambda) \in \mathbb{R}^7$ is called the Plücker coordinates of the line element (L, x) in $\mathbb{R}^3$, if $l \neq 0$ is parallel to L, $\bar{l} = x \times l$, and $\lambda = \langle x, l\rangle$.*

Obviously these coordinates are homogeneous. It is elementary to verify that

$$x = p(l,\bar{l}) + \frac{\lambda}{\langle l, l\rangle} l, \quad \text{with} \quad p(l,\bar{l}) = \frac{1}{\langle l, l\rangle} l \times \bar{l}. \tag{6}$$

The point $p(l,\bar{l})$ is the pedal point of the origin on the line L. It is well known that Plücker coordinates satisfy $\langle l, \bar{l}\rangle = 0$, and that all $(l, \bar{l})$ with $\langle l, \bar{l}\rangle = 0$ and $l \neq 0$ occur as coordinates of lines in $\mathbb{R}^3$. Thus, $(l, \bar{l}, \lambda)$ is the Plücker coordinate vector of a line element, if and only if

$$\langle l, \bar{l}\rangle = 0, \quad l \neq 0. \tag{7}$$

Equ. (7) describes part of a quadratic cone in projective space $\mathbb{P}^6$ whose base is the Klein quadric. Note that in this paper we do not consider line elements whose constitutents are "at infinity". In fact it is not so easy to extend Plücker coordinates of lines to Plücker coordinates of line elements – some aspects of this problem are discussed in Section 5 below. We therefore do not follow an approach similar to [25], where Euclidean line geometry is treated from the viewpoint of projective extension.

A line element becomes *oriented*, if the corresponding line has an orientation. In coordinates, this is realized by identifying $(l, \bar{l}, \lambda)$ and $\mu(l, \bar{l}, \lambda)$ if and only if $\mu > 0$, or alternatively by the restriction $\|l\| = 1$.

The equiform transformation (1) transforms the line element $(l, \bar{l}, \lambda)$ into $(l', \bar{l}', \lambda')$ with $x' = \alpha A x + a$, $l' = Al$, $\bar{l}' = x' \times l'$, $\lambda' = \langle x', l' \rangle$. In block matrix form, this transformation reads

$$\begin{bmatrix} l' \\ \bar{l}' \\ \lambda' \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ A^\times A & \alpha A & 0 \\ a^T A & 0^T & \alpha \end{bmatrix} \begin{bmatrix} l \\ \bar{l} \\ \lambda \end{bmatrix} \quad (A^\times x = a \times x). \tag{8}$$

Equ. (8) obviously applies to *oriented* line elements as well, for both ways of coordinatizing them. When considering orientation-reversing equiform mappings as well, one allows that $A \in \mathrm{O}_3$. Still, (8) is valid.

3.1. **The geometric meaning of $\boldsymbol{l}$, $\boldsymbol{\bar{l}}$, and $\boldsymbol{\lambda}$.** By construction, the set of line elements $(L, x) = (l, \bar{l}, \lambda)$ with l, λ fixed is described by $L \parallel l$ and x contained in the plane with equation $\langle x, l \rangle = \lambda$. We recognize $(-\lambda, l)$ as homogeneous coordinates for that plane. If $(l, \bar{l}, \lambda)$ is considered oriented, so is the plane.

Now suppose that $\bar{l} \neq 0$ and λ are given, and we are looking for the set of line elements $(L, x) = (l, \bar{l}, \lambda)$ with given $\bar{l}$ and λ. The lines whose Plücker coordinates $(l, \bar{l})$ have the given $\bar{l}$, are those contained in the plane $\bar{l}^\perp$. The footpoint $p = p(l, \bar{l})$ and the point x of (6) satisfy the relations

$$\|l\| = \frac{\|p\|}{\|\bar{l}\|}, \quad \|x - p\| = \frac{\lambda \|\bar{l}\|}{\|p\|}.$$

We see that the mapping $p \mapsto l$ is an equiform transformation within $\bar{l}^\perp$, but the mapping $p \mapsto x$ is not. It is obvious that the set of line elements $(L, x) = (l, \bar{l}, \lambda)$ with fixed $\bar{l} \neq 0$ and λ is invariant under rotations about the axis L and so is a union of Kasner's turbines [9, 23]. This notion means the set of line elements generated by rotating one line element (L, x) about an axis orthogonal to L. The case $\bar{l} = 0$ leads to all line elements (L, x) with $0 \in L$. If we think of oriented line elements, the results are similar: we get the set of oriented lines contained in the plane $\bar{l}^\perp$ which are oriented such that $\det(p(l, \bar{l}), l, \bar{l}) \geq 0$.

3.2. **Generalized bundles.** It is interesting to study certain linear subspaces of the (quadratic) coordinate space of line elements. The term 'bundle of lines' employed in the definition below means either the set of lines which pass through a point of $\mathbb{R}^3$, or the set of lines parallel to a given line.

Definition 2. *A set of line elements (L, x) is called a generalized bundle, if its lines constitute a bundle and its coordinates are contained in a three-dimensional linear subspace of $\mathbb{R}^7$.*

In view of homogeneity of line element coordinates, a bundle of line elements has dimension two. In case the bundle of Def. 2 has a proper vertex q, we may choose lines parallel to the canonical basis vectors e_1, e_2, e_3 and see that the corresponding coordinate subspace is spanned by the columns of a matrix of the form

$$\left[\begin{array}{ccc} e_1 & e_2 & e_3 \\ m_1 & m_2 & m_3 \\ \hline & p^T & \end{array}\right] = \begin{bmatrix} E_3 \\ C \\ p^T \end{bmatrix} \in \mathbb{R}^{7\times 3}, \quad m_i = q \times e_i. \tag{9}$$

If all lines of the bundle are parallel to $v \in \mathbb{R}^3$, there is an analogous matrix of the form

$$\begin{bmatrix} v & 0 & 0 \\ 0 & m_2 & m_3 \\ \alpha & 0 & 0 \end{bmatrix} \in \mathbb{R}^{7\times 3}, \tag{10}$$

where $m_2, m_3 \in \mathbb{R}^3$ span $v^\perp$.

For a given generalized bundle of line elements (L, x) it is interesting to observe the location of all points x:

Lemma 3.1. *A generalized bundle of line elements (L, x) either consists of the lines incident with a point $q \in \mathbb{R}^3$ such that x is contained in a sphere with center $(p+q)/2$ and radius $\|p-q\|/2$, with p, q from (9); or of the lines parallel to a vector $v \in \mathbb{R}^3$ such that x is contained in the plane with equation $\langle x, v\rangle = \alpha$, with v, α from (10).*

Proof. We first consider the case (9). With $p = (p_1, p_2, p_3)^T$, the general line element (L, x) in the bundle has coordinates $(l, q \times l, \sum l_i p_i)$. By construction, $\sum l_i x_i = \sum l_i p_i$, which implies that $x - p \perp l$. It follows that x is contained in a Thales sphere with diameter pq. In the case (10), the general line element (L, x) has coordinates $(l, \bar{l}, \lambda)$ with $l = \gamma v$, $\lambda = \gamma\alpha$. Obviously, $\langle x, v\rangle = \frac{1}{\gamma}\langle x, l\rangle = \frac{1}{\gamma}\lambda = \alpha$. □

The result of Lemma 3.1 is illustrated in Fig. 4, left.

3.3. **Linear mappings of Plücker coordinates.** A linear automorphism of $\mathbb{R}^7$ which transforms the set

$$\{(l, \bar{l}, \lambda) \in \mathbb{R}^7 \mid \langle l, \bar{l}\rangle = 0\} \tag{11}$$

into itself, is called a *linear mapping* of line elements. Similar to the phenomenon that restricting an automorphism of a projective space P to an affine space $A \subset P$ does not map A into A, also a linear mapping of line elements will in general map some coordinate vectors of line elements to coordinate vectors of the type $(0, \bar{l}, \lambda)$ which no longer represent line elements. Note that a linear mapping of line elements is not not, in general, induced by a point-to-point mapping of affine of projective three-space.

An example of a linear mapping of line elements which *is* induced by a point-to-point mapping (by an equiform transformation, to be precise) is given by (8). It turns out that a general affine transformation does not give rise to a linear mapping of line elements in the same way:

Lemma 3.2. *An affine mapping $x \mapsto Ax + a$ with $A \in \mathbb{R}^{3\times 3}$ induces a linear mapping of line elements if and only if it is a similarity transformation.*

Proof. Assume that line element coordinates are mapped according to $(l, \bar{l}, \lambda) \mapsto (k, \bar{k}, \varkappa)$. Then $k = Al$, $\bar{k} = K'l + K''\bar{l}$, but by (6),

$$\varkappa = \langle A \frac{l \times \bar{l} + \lambda l}{\langle l, l\rangle} + a, Al\rangle = \frac{1}{\langle l, l\rangle}(\det(l, \bar{l}, A^T Al) + \lambda\langle l, A^T Al\rangle) + a^T Al.$$

This dependence is linear if and only if $A^T Al$ is a multiple of l, for all l, i.e., if and only if $A^T A$ is a multiple of E_3. □

Lemma 3.3. *A linear automorphism φ of $\mathbb{R}^7$ with block matrix representation*

$$\begin{bmatrix} l' \\ \bar{l}' \\ \lambda' \end{bmatrix} = \begin{bmatrix} K & L & a \\ P & Q & b \\ u^T & v^T & \omega \end{bmatrix} \begin{bmatrix} l \\ \bar{l} \\ \lambda \end{bmatrix}, \tag{12}$$

is a linear mapping of line elements, if and only if $a = b = 0$, both $K^T P$ and $L^T Q$ are skew-symmetric, and $K^T Q + P^T L = \varkappa E_3$ with $\varkappa \neq 0$.

Proof. The mapping φ is a linear mapping of line elements, if and only if it leaves the relation $\langle l, \bar{l}\rangle = 0$ invariant. It is straightforward to describe the group of linear automorphisms of a quadratic surface. For the convenience of the reader, we describe the argument here:

With the canonical projection $\pi : \mathbb{R}^7 \to \mathbb{R}^6$, $\widetilde{\varphi} : (l, \bar{l}) \mapsto \pi \circ \varphi(l, \bar{l}, 0)$ is a linear automorphism of the Klein quadric, whence the conditions on K, L, P, and Q. Especially the upper left 2×2 block $\left[\begin{smallmatrix} K & L \\ P & Q \end{smallmatrix}\right]$ is regular.

The expression $\langle l', \bar{l}'\rangle$ expands to $\varkappa\langle l, \bar{l}\rangle + \lambda[a^T\ b^T] \cdot \left[\begin{smallmatrix} P & Q \\ K & L \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} l \\ \bar{l} \end{smallmatrix}\right] + \lambda^2\langle a, b\rangle$. Now $\langle l', \bar{l}'\rangle = 0 \iff \langle l, \bar{l}\rangle = 0$ for all l, $\bar{l}$, λ, if and only if $a = b = 0$. □

Corollary 1. *A linear mapping φ of line elements with the block matrix representation* (12) *determines a unique automorphism $\widetilde{\varphi}$ of the Grassmann manifold of lines in projective space $\mathbb{P}^3$, which in Plücker coordinates reads*

$$\begin{bmatrix} l \\ \bar{l} \end{bmatrix} \mapsto \begin{bmatrix} K & L \\ P & Q \end{bmatrix} \begin{bmatrix} l \\ \bar{l} \end{bmatrix}.$$

The mapping $\widetilde{\varphi}$ in turn is induced by either a projective automorphism $\varkappa$ of $\mathbb{P}^3$ or a correlation $\varkappa^\star$ of $\mathbb{P}^3$ onto its dual. Conversely, for all such $\widetilde{\varphi}$, there is a six-dimensional affine space of φ's.

Proof. This is obvious from the coordinate representation given in the previous lemma and from the well known coordinate representations of automorphisms of line space: the conditions on the matrices K, L, P, Q are the same in both cases. For given $\widetilde{\varphi}$ we may choose $u, v \in \mathbb{R}^3$, $\omega \neq 0$ arbitrarily. □

Lemma 3.4. *The linear mapping φ of Lemma 3.3 maps generalized bundles with proper vertices to generalized bundles with proper vertices, if and only if the matrices K, L, P, Q coincide with those of a similarity transformation, as given by* (8).

Proof. First it is obvious that such mappings have the required properties. In order to show the reverse implication, we consider the mapping $\widetilde{\varphi}$ of Cor. 1. It is induced by an affine mapping, as φ maps bundles with proper vertices to bundles with proper vertices. It follows that in the block matrix (12), $L = 0$ and K is regular. The image of a subspace of type (9) is given by

$$\begin{bmatrix} K \\ P + QC \\ \star \end{bmatrix} \sim \begin{bmatrix} E_3 \\ K^{-1}(P + QC) \\ \star \end{bmatrix},$$

where the symbol '$\sim$' means that the linear span of the columns of the matrix does not change. This is a subspace of type (9), if and only if the second block is skew-symmetric. This means that $K(P + QC)^T = -(P + QC)K^T$ for all choices of skew-symmetric matrices C, i.e., $KP^T + KC^TQ^T + PK^T + QCK^T = 0$. As KP^T is skew-symmetric anyway, this condition reduces to the skew symmetry of QCK^T for all skew-symmetric C. By Lemma 3.3, $K^TQ = \varkappa E_3$, and consequently we have $K^T = Q^{-1}/\varkappa$. The above condition now reads: QCQ^{-1} skew-symmetric, i.e., $C(Q^TQ) = (Q^TQ)C$. It is easy to verify that a matrix which commutes with all skew-symmetric ones, is a multiple of E_3. Thus we have shown $Q^TQ = \mu E_3$, and the result follows. □

4. Linear complexes of line elements

The set of lines whose Plücker coordinates $(l, \bar{l})$ satisfy a homogeneous linear equation $\langle l, \bar{c}\rangle + \langle \bar{l}, c\rangle = 0$ is called the linear line complex with coordinates $(c, \bar{c})$ [20, 24]. We generalize this and define:

Definition 3. *The set of line elements $(l, \bar{l}, \lambda)$ which satisfy*

$$\langle c, \bar{l}\rangle + \langle \bar{c}, l\rangle + \gamma\lambda = 0 \tag{13}$$

is called the linear complex of line elements with coordinates $(c, \bar{c}, \gamma)$.

If a complex C with equation (13) is given, and $\gamma \neq 0$, then for every line $L = (l, \bar{l})$ in Euclidean space there is a point $x \in L$ such that $(L, x) \in C$. In case $\gamma \neq 0$, the condition that $(L, x) \in C$ refers to the line L alone, and $(L, x) \in C$ if and only if L is contained in the complex of lines whose equation is (13). Thus the set of *lines* associated to the line elements of a complex in the sense of Def. 3 can have dimensions 3 or 4, depending on γ.

In Euclidean kinematics, the path normals of a smooth motion at a fixed instant comprise a linear line complex. This connection between Euclidean motions and line complexes generalizes to equiform motions and line elements: We call (L, y) a path normal element at y, if L is orthogonal the velocity vector $v(y)$ (cf. (3)).

Theorem 1. *At any regular instant of a smooth one-parameter equiform motion with velocity vector field $v(y)$ from* (3)*, the set of path normal elements of points equals the linear complex of line elements with coordinates $(c, \bar{c}, \gamma)$.*

Proof. The condition that the line element $(l, \bar{l}, \lambda)$ is orthogonal to $v(y)$, reads $0 = \langle v(y), l\rangle = \langle c \times y + \bar{c} + \gamma y, l\rangle = \det(c, y, l) + \langle \bar{c}, l\rangle + \gamma\langle y, l\rangle = \langle c, \bar{l}\rangle + \langle \bar{c}, l\rangle + \gamma\lambda$. □

Obviously, all linear complexes of line elements occur in this way. The group of equiform transformations $x \mapsto \tau T x + t$ acts on the set of linear complexes of line elements in a natural way. In view of Th. 1, this action is given by Equ. (4), and the classification of complexes is reduced to that of velocity vector fields:

Theorem 2. *Up to equiform equivalence, there are the following homogeneous coordinates of linear complexes of line elements:*

$$(c, \bar{c}, \gamma) = (0, 0, 1; 0, 0, p; 0) \quad (p \in \mathbb{R}), \tag{14}$$

$$(c, \bar{c}, \gamma) = (0, 0, 0; 0, 0, 1; 0), \tag{15}$$

$$(c, \bar{c}, \gamma) = (0, 0, 1; 0, 0, 0; p) \quad (p \neq 0), \tag{16}$$

$$(c, \bar{c}, \gamma) = (0, 0, 0; 0, 0, 0; 1). \tag{17}$$

Proof. The list of normal forms of velocity vector fields given earlier in this paper corresponds to the four cases above. Two different cases cannot be equivalent, because neither the action of the equiform group nor multiplication with a factor changes the vanishing of $\|c\|$ or γ. Likewise p is an invariant in both (14) and (16). □

A linear complex $(c, \bar{c}, \gamma)$ of line elements corresponds to a spiral motion if $c \neq 0$ and $\gamma \neq 0$, as demonstrated in Sec. 2.1: The spiral center, which after the coordinate transformation to normal form has coordinates $(0, 0, 0)^T$, obviously is given by $o = -(C^\times + \gamma E_3)^{-1}\bar{c}$. It is elementary to verify that this expression is the same as

$$o = \frac{1}{\gamma\mu}(\gamma c \times \bar{c} - \gamma^2 \bar{c} - \langle c, \bar{c}\rangle c), \quad \text{with } \mu = \gamma^2 + \langle c, c\rangle. \tag{18}$$

The spiral axis is parallel to c, and so we get the following line element coordinates for the *axis element* consisting of axis and center:

$$(c, o \times c, \langle o, c\rangle) = \Big(c, \frac{1}{\mu}(\langle c, c\rangle \bar{c} - \langle c, \bar{c}\rangle c + \gamma\, c \times \bar{c}), -\frac{1}{\gamma}\langle c, \bar{c}\rangle\Big). \tag{19}$$

In the case $\gamma = 0$, (19) is replaced by the well known expression $(c, \frac{1}{\mu}(\langle c, c\rangle \bar{c} - \langle c, \bar{c}\rangle c)) = (c, \bar{c} - \frac{\langle c, \bar{c}\rangle}{\langle c, c\rangle} c)$ for the Plücker coordinates of the axis of a helical motion (see Fig. 2).

4.1. **Concurrent and co-planar line elements in a complex.** The intersection of a linear complex of *lines* with a planar field of lines is a pencil, i.e., the set of path normals of a Euclidean motion within that plane. It turns out that the latter formulation generalizes to line elements:

Theorem 3. *The line elements of a linear complex contained in a plane are the path normal elements of a planar spiral motion.*

Proof. Without loss of generality we consider only the plane $x_3 = 0$. The Plücker coordinates of line elements (L, x) in that plane have the form $(l_1, l_2, 0; 0, 0, \bar{l}_3; \lambda)$ with $\bar{l}_3 = x_1 l_2 - x_2 l_1$. The line elements belonging to a linear complex satisfy

$$\bar{c}_1 l_1 + \bar{c}_2 l_2 + c_3 \bar{l}_3 + \gamma\lambda = 0. \tag{20}$$

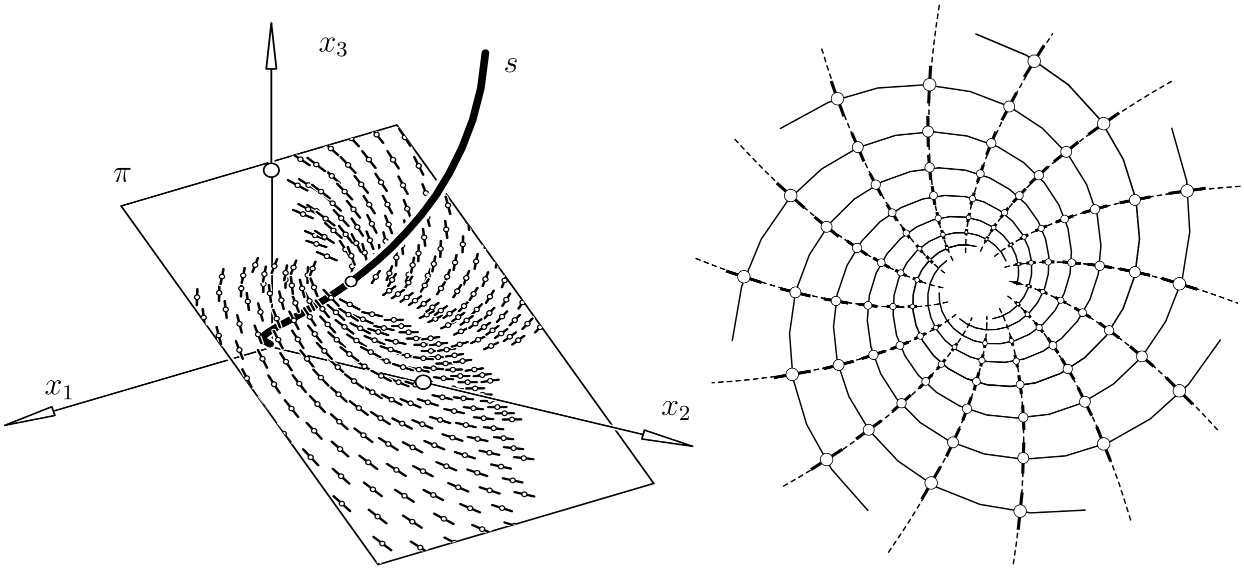

FIGURE 3. Path normal elements of a planar spiral motion. Left: planar section of a linear complex of line elements. Right: point paths.

The velocity vector $v(x)$ of a point x under a general planar spiral motion reads

$$v(x) = (\bar{c}_1, \bar{c}_2, 0)^T + c_3(-x_2, x_1, 0)^T + \gamma(x_1, x_2, 0)^T. \tag{21}$$

The condition that the line element (L, x) above is orthogonal to $v(x)$ is expressed by

$$\langle v(x), l\rangle = \bar{c}_1 l_1 + \bar{c}_2 l_2 + c_3(x_1 l_2 - l_1 x_2) + \gamma\langle l, x\rangle = 0, \tag{22}$$

which is the same as (21). □

Lemma 4.1. *If $C = (c, \bar{c}, \gamma)$ is a linear complex of line elements with $\gamma \neq 0$, then for all lines L there is a unique point x such that $(L, x) \in C$.*

If $\gamma = 0$, we consider the linear complex $C' = (c, \bar{c})$ of lines. If $L \in C'$, for all $x \in L$ we have $(L, x) \in C$, otherwise there is no x with $(L, x) \in C$.

Proof. We have to solve the equation $\langle \bar{c}, l\rangle + \langle \bar{l}, c\rangle + \gamma\lambda = 0$. With $L \in C \iff \langle \bar{c}, l\rangle + \langle \bar{l}, c\rangle = 0$ the result follows. □

The point x referred to in Lemma 4.1 is easily computed with (6):

$$x = \frac{1}{\langle l, l\rangle}(l \times \bar{l} - \frac{1}{\gamma}(\langle c, \bar{l}\rangle + \langle l, \bar{c}\rangle)). \tag{23}$$

Lemma 4.2. *The set of points x such that (L, x) is contained in the complex $(c, \bar{c}, \gamma)$ and L is parallel to a fixed vector $l \neq 0$, is a plane, except in the case that $\gamma = 0$ and $l \parallel c$.*

Proof. For any point x, the line element (L, x) parallel to l has coordinates $(l, x \times l, \langle x, l\rangle)$. It is contained in the complex if and only if $0 = \langle \bar{c}, l\rangle + \langle c, x \times l\rangle + \gamma\langle x, l\rangle = \langle \bar{c}, l\rangle + \langle x, l \times c + \gamma l\rangle$. This is a nontrivial linear equation, if $\gamma \neq 0$ or $l \times c \neq 0$. □

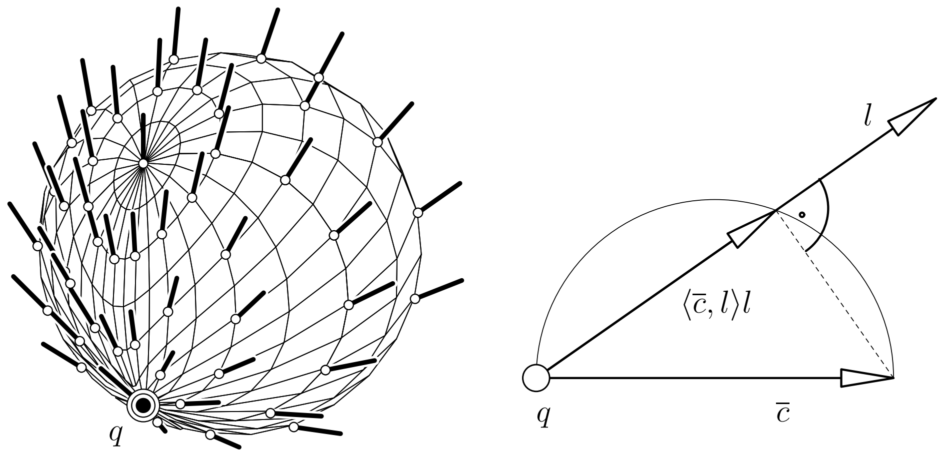

FIGURE 4. Left: Line elements in a generalized bundle. Right: See proof of Th. 4.

The condition that a line $(l, \bar{l})$ is incident with a point q is linear of rank 2. Thus by counting linear equations we see that the line elements (L, x) of a given complex with $q \in L$ in general comprise a generalized bundle in the sense of Def. 2. In analogy to Lemma 3.1 we show:

Theorem 4. *Assume that $q \in \mathbb{R}^3$ and $C = (c, \bar{c}, \gamma)$ $(\gamma \neq 0)$ is a linear complex of line elements. Then the set of x such that there is $(L, x) \in C$ with $q \in L$ is the sphere with diameter qq', where $q' = q - \frac{1}{\gamma}v(q)$ is expressed in terms of the velocity vector field* (3).

Proof. Without loss of generality we let $q = 0$, so $v(q) = \bar{c}$. The conditions imposed on the line element $(L, x) = (l, \bar{l}, \lambda)$ are $\bar{l} = 0$ and

$$\langle \bar{c}, l\rangle + \gamma\lambda = 0. \tag{24}$$

Then (6) implies

$$x = -\frac{1}{\gamma}\frac{\langle \bar{c}, l\rangle}{\langle l, l\rangle} l. \tag{25}$$

Obviously, x is the pedal point of the point $-\frac{1}{\gamma}\bar{c} = q - \frac{1}{\gamma}v(q)$ on the line L. It follows that the set of points x is the Thales sphere with diameter qq' (see Fig. 4, right). □

Corollary 2. *With the complex C from Th. 4, the set of points x such that there is $(L, x) \in C$ with L contained in a given pencil, is a circle.*

Proof. The circle in question is found by intersecting the sphere of Th. 4 with the carrier plane of the pencil. □

4.2. **Intersection of a complex with a line congruence.** Recall that a hyperbolic linear congruence of lines with skew axes A_1, A_2 consists of the lines which intersect both A_1 and A_2 [20, 24].

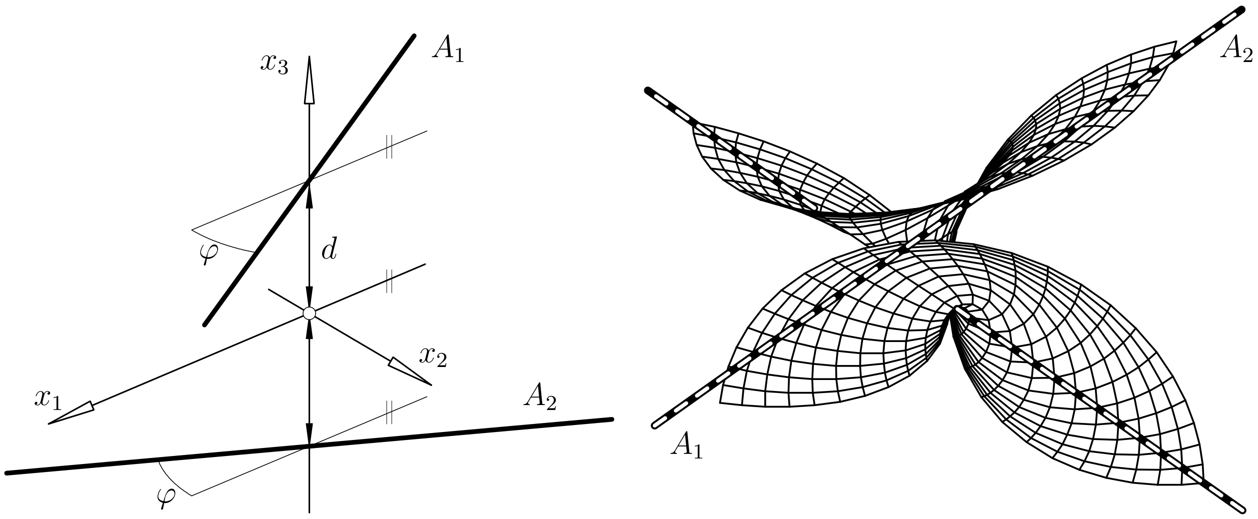

FIGURE 5. Left: Axes of a hyperbolic linear line congruence and the coordinate system used in the proof of 5. Right: The surface Φ of Th. 5 with two one-parameter families of circles.

Theorem 5. *Let $C = (c, \overline{c}, \gamma)$ be a linear complex of line elements with$\gamma \neq 0$. Consider the set Φ of points $x \in \mathbb{R}^3$ such that there is $(L, x) \in C$ with L contained in a given hyperbolic line congruence. Φ is a cubic surface which carries two one-parameter families of circles.*

Proof. Without loss of generality we assume that the axes A_1 and A_2 of the hyperbolic linear line congruence are parametrized linearly by

$$A_1(u) = (2ku, 2u, d),\ A_2(v) = (2kv, -2v, -d)\ (k, d \neq 0). \tag{26}$$

The congruence is in Plücker coordinates parametrized by $L(u, v) = (l(u, v), \overline{l}(u, v))$ with $l(u, v) = \frac{1}{2}(A_2(v) - A_1(u)) = (k(u - v), u + v, d)$, $\overline{l}(u, v) = \frac{1}{2}(A_2(v) + A_1(u))) \times l(u, v) = (d(u - v), -dk(u + v), 4kuv)$. The point $x(u, v)$ such that the line element $(L(u, v), x(u, v))$ is contained in C is computed with (23). By Lemma 4.1, it is unique. Implicitization of this surface yields the equation $k\gamma(x_1^2 + x_2^2 + x_3^2)x_3 + \dots$, where the dots indicate lower order terms. Since $k, \gamma \neq 0$, Φ is of degree three. For any plane $\varepsilon \supset A_i$, the intersection $\Phi \cap \varepsilon$ consists of A_i plus a degree two curve R_ε. By Cor. 2, those lines $L(u, v)$ which lie in ε lead to a circle of points $x(u, v)$, which is now identified with R_ε. It follows that the parametrization $x(u, v)$ covers Φ entirely. □

From the equation of Φ and also from the fact that Φ carries circles it is obvious that the projective and complex extension of Φ contains the absolute conic.

4.3. **Intersection of complexes.** In this short paragraph we consider the intersection of two complexes of line elements. It has already become apparent above that a complex $(c, \overline{c}, \gamma)$ with $\gamma = 0$ has special properties, which is also the case here. Assume that $C_i = (c_i, \overline{c}_i, \gamma_i)$ $(i = 1, 2)$ are linearly independent coordinate vectors of then different complexes with $(\gamma_1, \gamma_2) \neq (0, 0)$. The linear combination $C := (c, \bar{c}, \gamma) = \gamma_2 C_1 - \gamma_1 C_2$ describes the complex with equation

$$\gamma_2(\langle l, \bar{c}_1\rangle + \langle \bar{l}, c_1\rangle) = \gamma_1(\langle l, \bar{c}_2\rangle + \langle \bar{l}, c_2\rangle), \tag{27}$$

and obviously has $\gamma = 0$. It is actually the equation of a linear complex D of *lines*. If $L = (l, \bar{l})$ is a line in D, then there is λ such that $(l, \bar{l}, \lambda) \in C_1 \cap C_2$: We have $\lambda = -(\langle l, \bar{c}_i\rangle + \langle \bar{l}, \bar{c}_i\rangle)/\gamma_i$ whenever $\gamma_i \neq 0$. It follows that the intersection $C_1 \cap C_2$ consists of a set of line elements whose corresponding set of lines is a linear complex.

5. Projective closure

In line geometry it is well known that the simple definition of Plücker coordinates of lines in Euclidean space via moment vectors is elegantly extended to lines at infinity. The Plücker coordinates $(l, \bar{l})$ of lines are precisely those pairs $(l, \bar{l}) \in \mathbb{R}^6$ with $l \neq 0$ and $\langle l, \bar{l}\rangle = 0$. It turns out that the lines at infinity can be added without difficulty – they get coordinates with $l = 0$.

In the case of line elements, this extension is not as simple. All coordinate 7-tuples $(l, \bar{l}, \lambda) \in \mathbb{R}^7$ with $l \neq 0$ and $\langle l, \bar{l}\rangle = 0$ describe a line element (L, x) with $x \in \mathbb{R}^3$ and L not at infinity. Projective extension adds, among others, the line elements (L, x) with L proper and x at infinity. The limit $x \to \infty$ along the line L leads to coordinates "$(l, \bar{l}, \infty)$", or, when employing homogeneity, the coordinate vector $(0, 0, 1) \in \mathbb{R}^7$, regardless of l and $\bar{l}$. This alone shows that the quadratic surface $\langle l, \bar{l}\rangle = 0$ in six-dimensional projective space $\mathbb{P}^6$ is not an appropriate model.

The point in $\mathbb{P}^3$ with homogeneous coordinates $(x_0, \dots, x_3) = (x_0, x) \in \mathbb{R}^4$ is contained in the line with Plücker coordinates $(l, \bar{l})$ if and only if $\langle x, \bar{l}\rangle = 0$ and $x \times l = x_0 \bar{l}$. These two equations together with $\langle l, \bar{l}\rangle = 0$ define a point model of the set of line elements within $\mathbb{P}^3 \times \mathbb{P}^5$. We leave the investigation of lower-dimensional point models which perhaps have a simpler definition as a topic for future research.

Conclusion

We have introduced Plücker coordinates for line elements and considered certain sets of line elements which are given by linear equations of Plücker coordinates: Linear complexes of line elements, and generalized bundles. Further, we discussed linear mappings of line elements. The relation between Euclidean kinematics and complexes of lines has been generalized to equiform kinematics and complexes of line elements, which also leads to a classification of the linear complexes with respect to the equiform group. In order to better understand the geometry of line elements, we studied the intersection of linear complexes with bundles, fields, and linear congruences, in one case also giving a kinematic interpretation.

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