Galilei Laguerre Geometry and Rational Circular Offset Surfaces

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Abstract

In this paper the Laguerre geometry of three-dimensional Galilei space, that is, the geometry of oriented planes and cycles of an affine Cayley-Klein space with absolute line and two conjugate complex absolute points on it, is studied. Several different models are presented: A cyclographic model and an isotropic model together with their duals and an interpretation in terms of geometrical optics of Galilei space. We also have a closer look on Laguerre transformations. Galilei geometry arises naturally in the context of rational surfaces which possess rational circular offset surfaces, and it can also be used for modeling rational circular offset surfaces, e. g., with Galilei cyclides.

1 Introduction

There are two reasons why Laguerre geometry of Galilei space is interesting. First the geometry of planes in Galilei space and the different models of isotropic Möbius geometry connected with it deserve interest.

Second, within the search for rational surfaces with rational offsets, which has its origin in computer aided geometric design applications, we have to treat the case of a circular offset surface, which, for instance, is traced out by a point of the axis of a cylindrical milling tool when shaping a given surface. The geometry of planes of Galilei space here occurs naturally.

On the one hand, we rely on Röschel [13] and Brauner [2]. On the other hand, we follow the work done by Pottmann and Peternell [8, 9, 10, 11].

2 Fundamentals of Galilei Laguerre Geometry

2.1 Galilei Space

We repeat some definitions and elementary properties of Galilei space [13].

Definition Galilei space is three-dimensional real projective space $P_3(\mathbb{R})$ together with the geometry induced by the conformal Galilei group H_8 and its subgroup, the Galilei motion group B_6. H_8 consists of the transformations which, in homogeneous coordinates, are given by

$$\begin{aligned} x_0' &= x_0, \\ x_1' &= rx_0 + \rho\cos(\phi)x_1 + \rho\sin(\phi)x_2 + ux_3, \\ x_2' &= sx_0 - \rho\sin(\phi)x_1 + \rho\cos(\phi)x_2 + vx_3, \\ x_3' &= tx_0 + wx_3, \end{aligned} \tag{1}$$

and B_6 is the subgroup defined by $\rho = w = 1$.

In the terminology of Cayley-Klein geometries [5], Galilei space is the real part of three-dimensional complex projective space $P_3(\mathbb{C})$, equipped with the absolute figure

$$\omega \supset f \supset \{I_1, I_2\}, \tag{2}$$

where ω is the absolute plane $x_0 = 0$, f is the *horizontal absolute line* $x_0 = x_3 = 0$, and I_1, I_2 are the *horizontal absolute points* $(0, 1, \pm i, 0)\mathbb{C}$.

Galilei motions leave the two Galilei distances invariant:

$$d_1^2((x_1, y_1, z_1), (x_2, y_2, z_2)) = (z_2 - z_1)^2, \tag{3}$$

$$d_2^2((x_1, y_1, z), (x_2, y_2, z)) = (x_1 - x_2)^2 + (y_1 - y_2)^2. \tag{4}$$

2.2 Galilei Laguerre Space

Galilei Laguerre geometry is the geometry of oriented non-horizontal planes in Galilei space.

Definition An *oriented hyperplane* ε of $P_n(\mathbb{R})$ is a set

$$\varepsilon = (e_0, \ldots, e_n)\mathbb{R}^+, \quad e_i \in \mathbb{R}, \quad \varepsilon \neq (0, \ldots, 0).$$

The *hyperplane* corresponding to the oriented hyperplane $(e_0, \ldots, e_n)\mathbb{R}^+$ has the equation

$$e_0 x_0 + \ldots e_n x_n = 0. \tag{5}$$

A point $p = (x_0, \ldots, x_n)\mathbb{R} \in P_n(\mathbb{R})$ is *incident* with the oriented hyperplane $(e_0, \ldots, e_n)\mathbb{R}^+$, if it fulfills (5). If $n = 3$, an oriented (hyper-)plane is *horizontal*, if the corresponding plane contains the line $x_0 = x_3 = 0$. Then three-dimensional *Galilei Laguerre space* $\mathcal{G}$ is the set of non-horizontal oriented planes:

$$\mathcal{G} = \{(e_0, e_1, e_2, e_3)\mathbb{R}^+ | (e_1, e_2) \neq (0, 0)\}.$$

Obviously for every $\varepsilon \in \mathcal{G}$ there is exactly one coordinate vector (e_0, e_1, e_2, e_3) which describes ε and which satisfies $e_1^2 + e_2^2 = 1$.

Definition The *cycle* $c = (m_x, m_y, m_z, r)$, $m_x, m_y, m_z, r \in \mathbb{R}$, is the set

$$c = \{(e_0, \ldots, e_3)\mathbb{R}^+ \in \mathcal{G} | e_1^2 + e_2^2 = 1, \; e_0 + e_1 m_x + e_2 m_y + e_3 m_z + r = 0\}. \tag{6}$$

The *carrier* of the cycle is the circle of radius $|r|$, which is contained in the horizontal plane $x_3/x_0 = m_z$ and has center (m_x, m_y, m_z).

If $r \neq 0$, the planes of a cycle c are tangent to the carrier of c. If $r = 0$, all $\varepsilon \in c$ are incident with $(1, m_x, m_y, m_z)\mathbb{R}$.

2.3 The Cyclographic Model

Definition The cyclographic map ζ maps the oriented planes of $\mathcal{G}$ to hyperplanes of $P_4(\mathbb{R})$:

$$\zeta : \mathcal{G} \to P_4(\mathbb{R})^*, \quad (e_0, e_1, e_2, e_3)\mathbb{R}^+ \mapsto (e_0, e_1, e_2, e_3, \sqrt{e_1^2 + e_2^2})\mathbb{R}. \qquad (7)$$

By (6), the following is clear:

Lemma 2.1 *ζ maps the cycle $c = (m_x, m_y, m_z, r)$ to the set $\zeta(c)$ of those hyperplanes of $\zeta(\mathcal{G})$, which pass through the point $(1, m_x, m_y, m_z, r)\mathbb{R}$.*

We define a quadric

$$\Omega : x_0 = x_3 = x_1^2 + x_2^2 - x_4^2 = 0. \qquad (8)$$

Then for all cycles $c = (m_x, m_y, m_z, r)$, we consider the quadratic cone $\Gamma(c)$ with vertex $(1, m_x, m_y, m_z, r)\mathbb{R}$ and base quadric Ω. It can be parametrized by

$$\begin{aligned} \Gamma(c) = \quad & \{\lambda(1, m_x + r\cos t, m_y + r\sin t, m_z, 0) + \mu(1, m_x, m_y, m_z, r), \\ & t, \lambda, \mu \in \mathbb{R}, \quad \lambda\mu \neq 0\}. \end{aligned} \qquad (9)$$

We embed $\mathbb{R}^3$ in $P_4(\mathbb{R})$ by letting $(x_1, x_2, x_3) \mapsto (1, x_1, x_2, x_3, 0)\mathbb{R}$. Then, by closer inspection of the definitions, the following holds:

Lemma 2.2 *$\zeta(\mathcal{G})$ consists of those hyperplanes $H \in P_4(\mathbb{R})^*$, which fulfil $|H \cap \Omega| = 1$. For all cycles c, the carrier of c coincides with $\Gamma(c) \cap \mathbb{R}^3$.*

2.4 The Blaschke Cylinder

Definition Let δ be the duality defined by

$$\delta : \begin{cases} P_4(\mathbb{R})^* \to P_4(\mathbb{R}), \\ (e_0, e_1, e_2, e_3, e_4)\mathbb{R} \mapsto (e_4, e_1, e_2, e_3, e_0)\mathbb{R}. \end{cases} \qquad (10)$$

The set $\delta \circ \zeta(\mathcal{G})$ is called *Blaschke cylinder*, which is justified by the following

Lemma 2.3

1. *$\delta \circ \zeta(\mathcal{G})$ is contained in the affine part $\mathbb{R}^4$ of $P_4(\mathbb{R})$ and coincides with the cylinder $\Delta = S^1 \times \mathbb{R}^2$, whose projective extension has the equation*

$$\Delta : x_1^2 + x_2^2 = x_0^2. \qquad (11)$$

2. *We define $\mathcal{H}$ to be the set of hyperplanes of $P_4(\mathbb{R})$ which do not contain the point $(0, 0, 0, 0, 1)\mathbb{R}$. Then for all cycles c there is an $H(c) \in \mathcal{H}$ such that $\zeta(c) = \Delta \cap H(c)$ and vice versa.*

Proof: The first part is clear by (7). For the second part, we note that by Lemma 2.1 and (10), the points of $\delta \circ \zeta(c)$ are contained in the hyperplane $(r, m_x, m_y, m_z, 1)\mathbb{R}$. □

2.5 The Isotropic Model

It is well known that a quadric can be represented as conformal closure (in some well-defined sense dependent on the quadric) of an affine space. In order to perform this task for the Blaschke cylinder Δ, we apply a stereographic projection to it.

Consider the point $w = (0, 1, 0, 0) \in \Delta$ and its generator subspace

$$W = T_w(\Delta) \cap \Delta : e_1 = 0,\ e_2 = 1. \tag{12}$$

Definition We embed $\mathbb{R}^3$ in $P_4(\mathbb{R})$ by letting $(x, y, z) \mapsto (1, x, 0, y, z)\mathbb{R}$. Then σ is the projection of $\Delta \setminus W$ with center w to $\mathbb{R}^3$.

The stereographic projection σ is in coordinates described as follows:

$$\sigma : \begin{cases} \Delta \setminus W \to \mathbb{R}^3, \\ (1, e_1, e_2, e_3, e_0)\mathbb{R} \mapsto \dfrac{1}{1 - e_2}(e_1, e_3, e_0). \end{cases} \tag{13}$$

If c is the cycle (m_1, m_2, m_3, r), then $\sigma \circ \delta \circ \zeta(c)$ is the set Ψ of points, which, in coordinates, is given by

$$\Psi : z = f_c(x, y) = -\frac{r + m_2}{2}x^2 - m_1 x - m_3 y - \frac{r - m_2}{2}. \tag{14}$$

If $r \neq -m_2$, then Ψ is a parabolic cylinder, whose projective closure has the vertex $(0, 0, -m_3, 1)\mathbb{R}$. If $r + m_2 = 0$, then Ψ is a plane. Equation 14 shows that the $\sigma \circ \delta \circ \zeta$-images of cycles are precisely the Möbius spheres of twofold isotropic space in the sense of [2].

Oriented planes $(e_0, 0, 1, e_3)\mathbb{R}^+$ do not have an image under $\sigma \circ \delta \circ \zeta$. This leads to the following definition:

Definition The disjoint union $I_3 = \mathbb{R}^3 \cup W$ is called *isotropic model* of Galilei Laguerre geometry.

Then I_3 is the conformal closure of affine $\mathbb{R}^3$ in the sense of twofold isotropic Möbius geometry [2]. We extend σ by:

$$\overline{\sigma}(x) := \begin{cases} \sigma(x), \text{ if } x \notin W, \\ x, \text{ if } x \in W, \end{cases} \tag{15}$$

and define

$$\Lambda : \mathcal{G} \to I_3, \quad e \mapsto \overline{\sigma} \circ \delta \circ \zeta(e). \tag{16}$$

Then for all cycles c, the set $\Lambda(c) \subset I_3$ consists of the graph surface $\{x, y, f_c(x, y)\}$ together with $\delta \circ \zeta(c) \cap W$.

The inverse map Λ^{-1} maps points of I_3 to oriented planes of $\mathcal{G}$, and is, in the affine part $\mathbb{R}^3$ of I_3 described by:

$$\Lambda^{-1} : (x, y, z) \mapsto \frac{1}{1 + x^2}(2z, 2x, x^2 - 1, 2y)\mathbb{R}^+. \tag{17}$$

2.6 The Dual Isotropic Model and Geometrical Optics

A plane $h = (h_0, h_1, h_2, h_3)\mathbb{R}^+ \in \mathcal{G}$, $h_1^2 + h_2^2 = 1$, intersects the horizontal plane $x_3/x_0 = c$ in the oriented line $(h_0 + ch_3, h_1, h_2)\mathbb{R}^+$. We use this for the following

Definition Let the oriented lines $a = (a_0, a_1, a_2)\mathbb{R}^+$, $b = (b_0, b_1, b_2)\mathbb{R}^+$ of $\mathbb{R}^2$ and the line $c = (c_0, c_1, c_2)\mathbb{R}$ be given. Assume that $a_1^2 + a_2^2 = b_1^2 + b_2^2 = 1$, and denote the non-oriented lines corresponding to a, b by a', b'. Then c is called *bisector* of a and b, if c is a euclidean bisector of a' and b', and there is a $\lambda \in \mathbb{R}$ such that $(c_1, c_2) = \lambda(a_1 + b_1, a_2 + b_2)$.

The non-horizontal plane $\varepsilon_1 \in P_3(\mathbb{R})^*$ is the *bisector* of $\varepsilon_2, \varepsilon_3 \in \mathcal{G}$ if and only if for all horizontal planes $\eta \subset \mathbb{R}^3$, the intersection line $\eta \cap \varepsilon_1$ is the bisector of the oriented intersection lines $\eta \cap \varepsilon_2$ and $\eta \cap \varepsilon_3$.

The connection between this definition and the next one will become apparent later. The dual isotropic model of Galilei Laguerre space is the conformal closure of a subset of dual three-dimensional projective space. The oriented planes of $\mathcal{G}$ whose $\delta \circ \zeta$-image is not in W (cf. (12)), are again represented by planes.

We define a mapping β by

$$\begin{cases} \beta : \mathcal{G} \setminus (\delta \circ \zeta)^{-1}(W) \to P_3(\mathbb{R})^*, \\ (e_0, e_1, e_2, e_3)\mathbb{R}^+ \mapsto (e_0, e_1, e_2 - \sqrt{e_1^2 + e_2^2}, e_3)\mathbb{R}. \end{cases} \tag{18}$$

and define

Definition The *dual isotropic model* I_3^* of Galilei Laguerre geometry is the disjoint union $I_3^* = \beta(\mathcal{G} \setminus (\delta \circ \zeta)^{-1}(W)) \cup W$. The mapping Λ^* is defined by

$$\Lambda^* : \mathcal{G} \to I_3^* : \quad \begin{cases} g \mapsto \beta(g) \text{ if } \delta \circ \zeta(g) \notin W \\ \delta \circ \zeta(g) \quad \text{if } \delta \circ \zeta(g) \in W \end{cases} \tag{19}$$

The name *dual isotropic model* is explained by the following

Lemma 2.4 *Let π be the duality given by*

$$\pi : P_3(\mathbb{R}) \to P_3(\mathbb{R})^*, \quad (x_0, x_1, x_2, x_3)\mathbb{R} \mapsto (x_3, x_1, -x_0, x_2)\mathbb{R}. \tag{20}$$

Then for all $g \in \mathcal{G}$ such that $\Lambda(g) \in \mathbb{R}^3 \subset I_3$, $\pi \circ \Lambda(g) = \Lambda^*(g)$.

Proof: Clear from (7), (13) and (20). □

Lemma 2.5 *The mapping Λ^* is one-to-one. Let $g_0 = (0, 1, 0, 0)\mathbb{R}^+$. For all $g \in \mathcal{G}$ with $\delta \circ \zeta(g) \notin W$, the plane $\Lambda^*(g)$ is the bisector of g and g_0.*

Proof: Because of Lemma 2.4, Λ^* is one-to-one. The statement about the bisector property is verified by an elementary calculation. □

We want to show a connection between Λ^* and geometrical optics. For this purpose, we consider an immersed surface $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, such that (i) for the normal vector field $n_f = f_x \times f_y = (n_1, n_2, n_3)$ always $n_1 \neq 0$ holds, and (ii) the (euclidean) Gaussian curvature never vanishes. Then we have the following

Lemma 2.6 *If f has the above properties, the family of Λ^*-images of the oriented tangent planes*

$$(x, y) \mapsto f^*(x, y) = \Lambda^*((-\langle n_f, f\rangle, n_1, n_2, n_3)\mathbb{R}^+) \tag{21}$$

envelopes a surface which is then called the dual isotropic image of f.

Proof: From $n_1 \neq 0$ it follows that the $\delta \circ \zeta$-image of the oriented tangent plane is not in W, so its Λ^*-image is in the projective part of I_3^*. The Gaussian curvature of f is not zero, so the differential of the (euclidean) spherical mapping $(x, y) \mapsto n_f(x, y)/\|n_f(x, y)\|$ is nonsingular. This implies that also the differential of the mapping, which maps (x, y) to the unit normal vector of $f^*(x, y)$, is nonsingular, and the envelope surface exists. □

We consider a bundle of light rays, which is (Galilei) reflected at the surface f. A surface which intersects the reflected rays (Galilei) orthogonally, is called *anticaustic* surface with respect to the given light rays. It remains to define 'reflection' and 'orthogonal' in Galilei geometry. This can simply be done by stating that reflection at a vertical plane is defined by euclidean reflection, and all other reflections are images of this one under Galilei motions. Orthogonality between a horizontal line and a plane is defined in exactly the same way. Then we have the following:

Theorem 2.7 *Let f be an immersed surface $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ satisfying the conditions of Lemma 2.6. Then f is an anticaustic surface of the dual isotropic image of f with respect to the bundle of parallel light rays emanating from $(0, 1, 0, 0)\mathbb{R}$.*

Proof: The light rays are horizontal, and therefore stay horizontal after the reflection. An elementary calculation shows that the Λ^*-images of the oriented tangent planes along a horizontal level curve of the surface f touch the dual isotropic image of f along a horizontal curve. Therefore, and because of the definition of the bisector plane together with Lemma 2.5, it is sufficient to restrict attention to the situation in a horizontal plane, which is well known (see e.g. [10]). □

3 Galilei Laguerre Transformations

Definition A *Galilei Laguerre transformation* is a bijection $\mathcal{G} \to \mathcal{G}$ which induces a bijection in the set of cycles. The group of Galilei Laguerre transformations will be denoted by $\mathcal{L}$.

Theorem 3.1 *For all Galilei Laguerre transformations $f \in \mathcal{L}$ there is a projective automorphism κ of P_4 with $\kappa(\Delta) = \Delta$, such that the $\delta \circ \zeta$-image of f coincides with $\kappa|\Delta$.*

Proof: The $\delta \circ \zeta$-image g of an $f \in \mathcal{L}$ is a bijection $\Delta \to \Delta$. By Lemma 2.3, g induces a bijection in $\mathcal{H}$, also denoted by g. We are going to show that g is an automorphism of a circle geometry in the sense of [3], p. 992.

In the terminology of [3], a 2-dimensional plane $\varepsilon \not\subset \Delta$ intersects Δ in a proper circle $k = \varepsilon \cap \Delta$, if the affine span $[k]$ equals ε. We denote the absolute plane $x_0 = 0$ by ω, the line $x_0 = x_1 = x_2 = 0$ by f, and the point $(0,0,0,0,1)\mathbb{R}$ by u. The proper circles of Δ are therefore those conics ($\varepsilon \cap f = \emptyset$) and pairs of parallel lines ($\varepsilon \cap f = \{p\}$), which are not entirely contained in a generator subspace. Two proper circles k_1, k_2 *touch* at a if either $a \in k_1 = k_2$ or $[k_1] \cap [k_2]$ has dimension 1 and is tangent to Δ in a. A conic k_1 and a pair k_2 of parallel lines never touch, because $[k_1] \cap [k_2]$ then would contain a point of $f \in [k_1]$.

If three points $p_1, p_2, p_3 \in \Delta$ span a 2-dimensional generator subspace U, there is no $H \in \mathcal{H}$ such that $p_i \in \mathcal{H}$. Therefore there is no $H' \in \mathcal{H}$ such that $g(p_i) \in H'$ and g thus maps U into another generator subspace. g^{-1} enjoys the same properties as g, thus the set $X \subset \Delta$ is contained in a generator subspace if and only if $g(X)$ is.

For every irreducible conic $k \subset \Delta$ there exist $H_1, H_2 \in \mathcal{H}$, such that $H_1 \cap H_2 \cap \Delta = k$, and k is not contained in a generator subspace. Therefore neither is $g(k) = g(H_1) \cap g(H_2) \cap \Delta$, and $g(k)$ is a proper circle.

If k consists of two parallel lines l_1, l_2, such that $[k] \cap f \neq u$, there exist $H_1, H_2 \in \mathcal{H}$ such that $k = H_1 \cap H_2 \cap \Delta$. There is no generator subspace which contains k, and there are generator subspaces $U_i \supset l_i$, $i = 1, 2$. Thus $g(k)$ is not contained in a generator subspace, but $g(l_i)$ is. Because of $g(l_i) \subset g(H_1) \cap g(H_2)$, we have $g(l_1) \| g(l_2)$ and $g(k)$ is a proper circle with $u \notin [g(k)]$.

For all lines $l \ni u$, also $g(l) \ni u$, because for any two points $p, q \in l$ there is no $H \in \mathcal{H}$ such that $p, q \in H$, and therefore there is no $H \in \mathcal{H}$ such that $g(p), g(q) \in H$, and vice versa. Thus also those proper circles k which pass through u are mapped to proper circles $g(k) \ni u$.

We are going to show that g is compatible with the touching relation: Assume that the proper circles $k_1 \neq k_2$ are conics. Let $V = [k_1] \cap [k_2]$ and let U_a be the generator subspace of $a \in \Delta$. Then k_1 and k_2 touch at a, if and only if $\dim V = 1$ and $V \cap \Delta \subset U_a$. The intersection $V \cap f$ is empty because neither $[k_1]$ nor $[k_2]$ intersect f. This implies that (i) there is exactly one $H \in \mathcal{H}$ with $k_i \subset H$ and (ii) within this H, whose intersection $H \cap \Delta$ is a non-degenerate elliptic cylinder $Z \supset k_i$, the set $k_1 \cap k_2$ has no points outside the Z-generator line $H \cap U_a$.

The converse is also true: if $k_1 \neq k_2$ with $a \in k_1 \cap k_2$, and there is an $H \in \mathcal{H}$ containing the k_i, then $\dim V = 1$; and if $k_1 \cap k_2 \subset H \cap U_a$, then V is tangent to Δ, which means that the k_i touch in a.

Conditions (i) and (ii) are invariant with respect to g, and therefore so is the touching relation, when restricted to conics. Pairs of lines, however, touch if and only if one of their lines coincide, so g leaves the touching relation invariant. Thus g is an automorphism of circle geometries in the sense of [3], and we can use Schröder's theorem ([3], p. 992 or [14]) to deduce the existence of a projective automorphism κ of P_4 with $\kappa(\Delta) = \Delta$ and $\kappa|\Delta = g$. □

In [2], H. Brauner considered Möbius geometry of twofold isotropic space. It turns out that isotropic Möbius geometry and Galilei Laguerre geometry are isomorphic. The result in [2], which describes all isotropic Möbius transformations as collineations automorphic for some quadric, which is of the same projective type as the Blaschke cylinder, has been obtained by assuming continuity. This is the reason why we re-proved it on basis of [14].

We are able to write down the $\delta \circ \zeta$-image of Galilei Laguerre transformations:

Theorem 3.2 *In homogeneous coordinates, the projective automorphisms κ of the Blaschke cylinder are given by a matrix*

$$A = \left(\begin{array}{c|c} A_0 & 0 \\ \hline A_1 & A_2 \end{array} \right), \tag{22}$$

where A_0 is the matrix of a projective automorphism of the unit circle $x_0^2 = x_1^2 + x_2^2$, and $A_2 \in \mathrm{GL}(2, \mathbb{R})$. Thus $\mathcal{L}$ is a 13 dimensional transformation group.

Proof: For all x_3, x_4, $\kappa((0,0,0,x_3,x_4)\mathbb{R}) = (0,0,0,x_3',x_4')\mathbb{R}$ holds, therefore the upper right corner in (22) is zero.

Embed $P_2(\mathbb{R})$ into $P_4(\mathbb{R})$ by letting $(x_0,x_1,x_2)\mathbb{R} \mapsto (x_0,x_1,x_2,0,0)\mathbb{R}$ and let $\pi : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by κ followed by the central projection with center $x_0 = x_1 = x_2 = 0$ onto $P_2(\mathbb{R})$. Then $\pi|S^1$ is a projective automorphism of the unit circle. A is invertible if and only both A_0 and A_2 are.

Obviously these conditions are also sufficient for κ being an automorphism of Δ and therefore $\dim \mathcal{L} = 13$. □

4 Rational Circular Offset Surfaces

In analogy to the Pythagorean-Normal surfaces of [8, 11] we now will consider rational surfaces with rational *circular offsets* [9]. A circular offset surface is formed by the offset curves at distance d to the level curves of the original surface in horizontal planes. If we define the Galilei normals to a surface as the horizontal lines perpendicular to level curves, the definition of circular offset surfaces is in accordance with the usual definition of an offset surface as a surface whose distance to the given surface, measured along the common surface normals, is constant.

Definition A two-dimensional *rational parametrization* p is a tuple $(p_0(x,y), \ldots, p_n(x,y))$ of rational functions such that not all of them are zero. $p(x,y)$ then is defined for all (x,y) except for those which are contained in an algebraic subset of dimension at most one. We say that p is defined for *almost all* (x,y) and the mentioned algebraic set is *negligible*. The (possibly degenerate) *rational surface* $\Phi(p)$ of $P_n(\mathbb{R})$ defined by p is the smallest projective algebraic variety which contains all points $(p_0(x,y), \ldots, p_n(x,y))\mathbb{R}$. If $\Phi(p)$ is not entirely contained in the absolute plane $x_0 = 0$, the set $\mathbb{R}^n \cap \Phi(p)$ will also be called rational surface (of $\mathbb{R}^n$).

If p parametrizes a rational surface of $\mathbb{R}^n$, for almost all parameter values (x,y), the point $p(x,y)\mathbb{R}$ is defined and actually contained in $\mathbb{R}^n \subset P_n(\mathbb{R})$.

Definition If $p(x,y) \neq 0$, the hyperplane $h = (h_0, \ldots, h_3)\mathbb{R} \in P_3(\mathbb{R})^*$ is *tangent* to $p = (p_0, \ldots, p_3)$ at $p(x,y)\mathbb{R}$, if $\langle h, p(x,y)\rangle = \langle h, \frac{\partial}{\partial x} p(x,y)\rangle = \langle h, \frac{\partial}{\partial y} p(x,y)\rangle = 0$. If $p \notin \omega$, the vector $\lambda(h_1, h_2, 0)$, $\lambda \in \mathbb{R}$, is then called *Galilei normal vector* of p in $p(x,y)\mathbb{R}$.

Definition A rational surface $\Phi(p)$ of $\mathbb{R}^3$ is called a *rational circular offset surface*, if there are rational functions $n_1(x,y)$, $n_2(x,y)$ and $r(x,y)$, r nonzero, such that

1. for all (x,y) such that $n_i(x,y)$ is defined and $p(x,y) \neq 0$, the vector $(n_1(x,y), n_2(x,y), 0)$ is Galilei normal to p in $p(x,y)\mathbb{R}$ and
2. the equation $n_1^2 + n_2^2 = r^2$ holds in $\mathbb{R}(x,y)$.

The name *rational circular offset surface* is justified by the following

Lemma 4.1 *If p is a parametrization of a rational circular offset surface, then indeed for all $d \in \mathbb{R}$ there exist rational parametrizations p_d of surfaces, which have the property that the distance to $\Phi(p)$, measured along the common Galilei normal, equals d.*

Proof: Without loss of generality we can achieve $n_1^2 + n_2^2 = 1$ by changing n_1 and n_2 to n_1/r and n_2/r, respectively. Then an easy calculation shows that the surface given by

$$p_d = (1, \frac{p_1}{p_0} + dn_1, \frac{p_2}{p_0} + dn_2, \frac{p_3}{p_0}),$$

has the following properties: (i) for almost all (x,y), the point $p_d(x,y)$ is defined. (ii) if both are defined, the Galilei distance between $p_d(x,y)\mathbb{R}$ and $p(x,y)\mathbb{R}$ equals d. □

We are now going to define the surface $\Phi^*(p)$ *dual* to a rational surface $\Phi(p)$. Because of possible degeneracies, we have to distinguish several cases. We define

$$p_x(x,y) = \frac{\partial}{\partial x}\left(\frac{p_1}{p_0}, \frac{p_2}{p_0}, \frac{p_3}{p_0}\right) \text{ and } p_y(x,y) = \frac{\partial}{\partial y}\left(\frac{p_1}{p_0}, \frac{p_2}{p_0}, \frac{p_3}{p_0}\right). \qquad (23)$$

1. If $\dim \Phi(p) = 2$, for almost all (x,y) the vectors $p_x(x,y)$ and $p_y(x,y)$ are linearly independent and $n_p(x,y) = p_x(x,y) \times p_y(x,y)$ is nonzero.
2. If $\dim \Phi(p) = 1$, at least one of p_x and p_y, say p_x, is nonzero. There exists a vector $n_0 \in \mathbb{R}^3$ which is independent of $p_x(x,y)$ for almost all (x,y). Thus for almost all (x,y) the vector $n_p(x,y) = p_x(x,y) \times n_0$ is nonzero.
3. If $\dim \Phi(p) = 0$, let $n_p(x,y) = (x,y,1)$.

Definition If p and n_p are as defined above, the rational surface $\Phi(p)^*$ of $P_3(\mathbb{R})^*$ parametrized by

$$p^* = (-n_1p_1 - n_2p_2 - n_3p_3, n_1p_0, n_2p_0, n_3p_0) \qquad (24)$$

is called the surface *dual* to $\Phi(p)$.

We leave the verification that p^* is actually well-defined to the reader. We have the following

Lemma 4.2 *When defined as in (24), $p^*(x, y)$ is tangent to p in $p(x, y)$. Both the iterated dual p^{**} and p define the same surface. If p defines a rational circular offset surface with Galilei normal vector field m_1, m_2, there is a rational function m_3 such that p^* and $(-m_1p_1 - m_2p_2 - m_3p_3, m_1p_0, m_2p_0, m_3p_0)$ define the same surface $\Phi(p^*)$.*

Proof: In regular points $p(x, y)$ we can use differential geometry to verify the lemma, and the set of parameter values corresponding to singular points is negligible.

Not both n_1, n_2 are zero, so assume that n_1 is nonzero. Both $(n_1, n_2, 0)$ and $(m_1, m_2, 0)$ are Galilei normal to p and therefore $(m_1, m_2, 0) = (m_1/n_1)(n_1, n_2, 0)$ holds in $\mathbb{R}(x, y)$. Then define $m_3 = m_1n_3/n_1$. □

We are going to show that Galilei Laguerre transformations do not destroy the property of being a rational circular offset surface. But first we have to define what dual surface means in terms of Galilei Laguerre space.

Definition Let $p^* = (h_0, h_1, h_2, h_3)$ be a rational parametrization of a surface in $P_3(\mathbb{R})^*$ which does not entirely consist of horizontal planes. Then the set of those $p^*(x, y)\mathbb{R}^+$, which are not horizontal, is called *dual surface* to p considered as a subset of $\mathcal{G}$.

Then the following theorems hold:

Lemma 4.3 *The transfer $\delta \circ \zeta$ from Galilei Laguerre space to the Blaschke cylinder Δ maps the duals of rational circular offset surfaces to rational surfaces which are contained in Δ, and vice versa.*

Proof: Without loss of generality we can assume a parametrization h of the dual surface which satisfies $h_1^2 + h_2^2 = 1$. Then $\delta \circ \zeta(h(x, y)\mathbb{R}^+) \in \Delta$. Conversely, for a rational parametrization $q = (q_0, q_1, q_2, q_3, q_4)$ of a surface contained in Δ, q_0 is nonzero, so without loss of generality $q_0 = 1$. This implies $q_1^2 + q_2^2 = 1$ and $p^* = (\delta \circ \zeta)^{-1}(q)$ defines a surface parametrized by p^{**}, which is a rational circular offset surface. □

Theorem 4.4 *Galilei Laguerre transformations map the duals of rational circular offset surfaces to the duals of rational circular offset surfaces.*

Proof: The $\delta \circ \zeta$-image of a Galilei Laguerre transformation is a projective automorphism of Δ and therefore preserves rationality. The theorem follows immediately from Lemma 4.3. □

The stereographic projection σ defines a bijective map between rational surfaces in $\Delta \setminus W$ and rational surfaces in the isotropic model $I_3 \setminus W$, which gives us a simple construction of rational circular offset surfaces.

Theorem 4.5 *The geometric transformation Λ^{-1}, which describes the change from the isotropic model of Galilei Laguerre space to the standard model, maps rational surfaces to the duals of rational circular offset surfaces, with the single exception of one degenerate rational surface consisting of a horizontal point at infinity.*

Proof: The stereographic projection preserves rationality and the only rational surface which could lie entirely in W is one which corresponds to a bundle with vertex situated on the absolute horizontal line. The converse is obvious. □

5 Special Rational Circular Offset Surfaces

5.1 Normal Forms of Quadratic Graphs

We consider the graph $\Psi = \{(x, y, f(x, y)\}$ of the quadratic function

$$f : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto \sum_{i+j \leq 2} a_{ij} x^i y^j \tag{25}$$

as a subset of $\mathbb{R}^3 \subset I_3$ and ask for its Λ-preimage in $\mathcal{G}$. It will turn out that $\Lambda^{-1}(\Psi)$ consists of the set of tangent planes of a cubic surface with special properties (see next section). In order to simplify the discussion, we are looking for geometric transformations τ which do not destroy the Galilei geometric properties of $\Lambda^{-1}\Psi$, and transform Ψ with the Λ-image of τ. The result is

Lemma 5.1 *We are able to transform the graph surface to the graph of one the following normal forms:*

$$1. \qquad z = ax^2 + by^2, \quad a, b \in \mathbb{R},\ b \neq 0 \tag{26}$$

$$2. \qquad z = cxy, \quad c \in \mathbb{R} \setminus 0. \tag{27}$$

$$3. \qquad z = d, \quad d \in \mathbb{R}. \tag{28}$$

Proof: The transformation is achieved by by letting $\phi = v = 0$ in (1) and adjusting the other parameters appropriately. □

5.2 Parabolic Galilei Cyclides

We consider the isotropic model I_3 of Galilei Laguerre Geometry and the normal form (26) of quadratic graphs. Λ^{-1} transforms this graph into a rational parametrization of a surface of class 4, whose corresponding point surface Φ has the homogeneous equation

$$\Phi : \ x_3^2 x_2 + 2a x_3^2 x_0 + 2b(x_1^2 + x_2^2) x_0 + 4ab x_2 x_0^2 = 0. \tag{29}$$

If necessary, we also consider the complex extension $\Phi_{\mathbb{C}} \subset P_3(\mathbb{C})$, which is defined by the same equation. These surfaces are called *parabolic Galilei cyclides.* For cyclides in Galilei space, see also [7].

Equation 29 obviously defines an algebraic surface of degree three. Its intersection with the plane $\omega : x_0 = 0$ at infinity is

$$\Phi \cap \omega : x_3^2 x_2 = x_0 = 0, \tag{30}$$

so Φ touches ω in all points of the horizontal line $x_3 = x_0 = 0$. The line $l : x = 0, y = -2a$ is easily seen to be contained in Φ.

Definition An algebraic surface is called a *Blutel's surface* [4], if almost all of it is covered by a one parameter family of conic sections, such that for all of these conics the tangent planes in the points of the conic are tangent to a quadratic cone.

Lemma 5.2 *The surface Φ carries a one-parameter family of horizontal circles and a one-parameter family of parabolas in vertical planes all of which contain the line l. Φ is a Blutel's surface with respect to both families of conics.*

Proof: Horizontal planes intersect the affine part of Φ in circles or points. The intersections of Φ with planes $\varepsilon \supset l$ consist, by the surface's degree, of l and a conic. These conics touch ω in their horizontal points at infinity. They are, therefore, parabolas.

We want to show that the planes tangent to Φ in the points of a horizontal circle form a (Galilei) cone of revolution. For this purpose we notice that for all $v \in \mathbb{R}$, along the plane $y = \text{const} = v$, the quadratic graph (26) touches a parabolic cylinder with ideal vertex situated on the line $x_0 = x_1 = 0$, the Λ^{-1}-image of which is a cycle. The carrier of this cycle therefore is contained in Φ. The tangent planes along this circle are, by Λ^{-1}, those with coordinates $(2(au^2 + bv^2), 2u, u^2 - 1, 2v)\mathbb{R}$ $(v \in \mathbb{R})$. The line $x = 0, y = -2a$ intersects them in a point p independent of u. This shows that along its horizontal circles, Φ is touched by cones whose vertices p are situated on l.

Analogously it is easily verified that Φ is touched by horizontal parabolic cylinders along the parabolas mentioned above. □

Lemma 5.3 *The complex extension $\Phi_{\mathbb{C}}$ either possesses two conic singularities or one biplanar singularity situated on l. Additionally, $\Phi_{\mathbb{C}}$ has conic singularities in the absolute points $J_{1,2} = (0, i, \pm 1, 0)\mathbb{R}$.*

Proof: The first part is proved by looking at the intersection points of l and one (and then, all) of the parabolas mentioned above. If one (and then, all) of the parabolas touch l in a point p, $\{p\}$ is the intersection of Φ with the horizontal plane $\varepsilon \ni p$. The intersection $\Phi_{\mathbb{C}} \cap \varepsilon$, however, consists of $[pJ_1] \cup [pJ_2]$, which defines the two tangent planes of p.

The second part is clear from the fact that the midpoints of the horizontal circles contained in Φ are situated on a parabola, which then serves as a base curve for the tangent cones of both J_1 and J_2. □

Figure 1 shows an example of a parabolic Galilei cyclide. Those special Galilei transformations, which have been used to transform a general quadratic graph

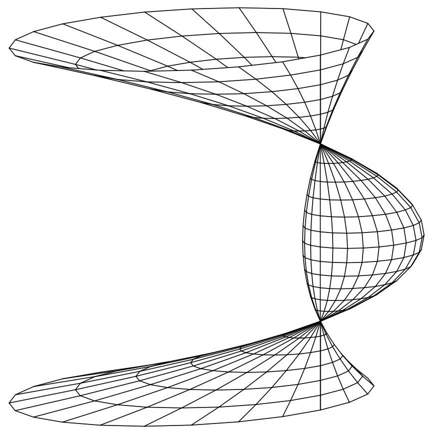

Figure 1: Parabolic Galilei Cyclide

to its normal form, are affine transformations, whose restrictions to the horizontal planes are translations. Thus all Λ^{-1}-images of quadratic graphs are parabolic Galilei cyclides carrying circles in horizontal planes, whose centers are situated on a parabola with horizontal axis in the plane $x = 0$.

The general parabolic cyclide then is the image of Φ under a Galilei transformation. Not all of them are Λ^{-1}-images of quadratic graphs. For all of them, however, there is a special choice of the center of the stereographic projection σ, such that they become Λ^{-1}-images.

The Λ-preimages of the quadratic graphs of the remaining normal forms of (26) are described by the following

Lemma 5.4 *The normal form (27) is the Λ-image of the surface dual to the cubic conoidal ruled surface with equation*

$$(x^2 - c^2)y - 2xz = 0. \tag{31}$$

The normal form (28) corresponds to the cycle $(0, d, 0, -d)$.

Proof: The result is verified by elementary calculations. The second part also follows directly from (14). □

5.3 Galilei Cyclides

The most general Galilei cyclides are, in their affine part, generated by circles in horizontal planes $z = c$, whose centers are situated on a parabola with horizontal axis and contained in a non-horizontal plane, and whose radius is a quadratic polynomial function of c. They either are Galilei cylinders/cones of revolution or are the images of the parabolic cyclide Φ under Galilei transformations and the quadratic transformation

$$\psi : (x, y, z) \mapsto (x, y + \beta z + \gamma z^2, z).$$

Those Galilei cyclides will be called *proper* ones. Let $p = (m_x + r\cos\phi, m_y + r\sin\phi, m_z)$ and $v = (n_x - r\cos\phi, n_y - r\sin\phi, n_z)$. Then the tangent vector $(p; v)$ is mapped by $d\psi$ to $(\psi(p); w)$ with $w = v + (0, n_z(\beta + 2m_z\gamma), 0)$. This tells us that a surface touching a Galilei cone of revolution along a horizontal circle does not lose this property when being transformed by ψ.

Furthermore, the second component of w depends linearly on m_z. This tells us that if the vertices of all those cones are situated on a line l, they also do so after the ψ-transform. This implies:

Theorem 5.5 *A proper Galilei cyclide is a twofold Blutel's surface carrying a horizontal family of circles. The vertices of the cones tangent to the surface along these circles are situated on a straight line.*

6 Modeling Rational Circular Offset Surfaces

6.1 Galilei Canal Surfaces

In analogy to [10], where a canal surface has been discretised and approximated by a sequence of Dupin cyclides, we can ask for a discretisation and approximation of Galilei canal surfaces by Galilei cyclides:

Definition Surfaces of the form

$$f : I \times \mathbb{R} \to \mathbb{R}^3, \quad (u, \phi) \mapsto (m_x(u) + r(u)\cos(\phi), m_y(u) + r(u)\sin(\phi), u).$$

are called *Galilei canal surfaces.*

A Galilei cyclide touches such a surface along a horizontal circle $u = \text{const.}$, if and only if both surfaces have both the circle and the tangent cone along this circle in common.

Definition If f is defined as above, its *medial axis* is the curve

$$c_f : I \to \mathbb{R}^3, \quad u \mapsto (m_x(u), m_y(u), r(u)).$$

Clearly, two Galilei canal surfaces f and g touch each other along the circle $z = u$, if and only if their medial axes touch each other in the point $c_f(u) = c_g(u)$.

If the curve of centers $(m_x(u), m_z(u), u)$ is of the form $a + ub + u^2c$ with $a, b, c \in \mathbb{R}^3$ such that c is horizontal and b is not, c_f is a parabola or a straight line, and f becomes a parametrization of a Galilei cyclide. Conversely, every parabola of $\mathbb{R}^3$ corresponds to a proper Galilei cyclide, and every straight line to a Galilei cylinder/cone of revolution. Thus the problem of approximating a Galilei canal surface is equivalent to the problem of approximating a curve of $\mathbb{R}^3$ by a quadratic B-spline, which has been studied extensively. A picture is shown in figure 2.

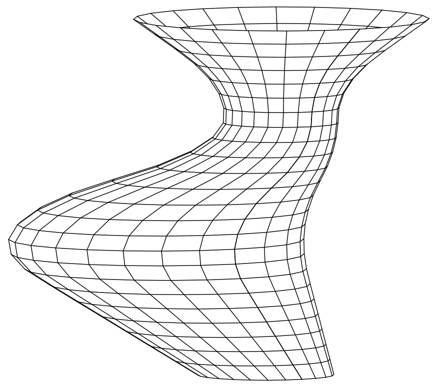

Figure 2: Piecewise Galilei Cyclide Surface

6.2 A Powell-Sabin Method

Let $\mathcal{T}$ be a triangulation of a polygonal subset $U \subseteq \mathbb{R}^2$ and let for each vertex v_i a value z_i and a linear functional $d_i \in \mathbb{R}^{2*}$ be given. Then the algorithm of Powell and Sabin [6] gives a piecewise quadratic $C^1(U)$ function f with the property $f(v_i) = z_i$ and $df(v_i) = d_i$ for all i. Moreover, the points where f is not C^2 are those of the edges of a six-fold refinement of $\mathcal{T}$. This can be used for interpolating given values and derivatives by a rational circular offset surface, simply by interpolating the appropriate values and derivatives in the isotropic model. For classical Laguerre geometry, this has been done in [8].

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