# Log-exponential Analogues of Univariate Subdivision Schemes in Lie Groups and their Smoothness Properties

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**Abstract.** The necessity to process data which live in nonlinear geometries (e.g. motion capture data, unit vectors, subspaces, positive definite matrices) has led to some recent developments in nonlinear multiscale representation and subdivision algorithms. The present paper analyzes convergence and $C^1$ and $C^2$ smoothness of subdivision schemes which operate in matrix groups or general Lie groups, and which are defined by the so-called log-exponential analogy. It is shown that a large class of such schemes has essentially the same smoothness as the linear schemes they are derived from. This work extends previous work on Lie group subdivision schemes – we consider alternative definitions of analogous schemes, arbitrary dilation factors, and symmetry of the nonlinear scheme.

## §1. Introduction

### 1.1. Motivation

For many applications it is necessary to handle data which live in certain nonlinear geometries. Examples of such data types are unit vectors, positions of a rigid body, subspaces of $\mathbb{R}^n$, or points on a surface. In these cases, a manifold "$M$" containing the data would be the unit sphere, the Euclidean motion group, a Grassmann manifold, or a surface. Whenever $M$ is globally embeddable in a vector space $\mathbb{R}^m$, a universal coordinate system is available, and numerical representation of data is no problem. Comparing data by computing distances in $\mathbb{R}^m$ may work well (e.g., for two unit vectors $\mathbf{v}$ and $\mathbf{w}$ the distance $\|\mathbf{v} - \mathbf{w}\|$ makes perfect sense), but there are situations where the notions which are derived from the ambient space, like difference vectors and averages, do not have the desired meaning.

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Subdivision in general is a tool to create continuous limit curves and surfaces from discrete data – prominent applications of this concept are in geometric modeling [8] and the definition of wavelet-type transforms [3]. Thus properties of subdivision schemes, also nonlinear ones, have received much attention. Smoothness of the limits produced by a subdivision process is important for various reasons: One is fairness of shapes for geometric modeling, but another one is that for wavelet-type transforms based on interpolatory subdivision schemes, smoothness of data is related to the decay of wavelet coefficients provided the subdivision scheme in use is smooth enough [11].

### 1.2. Previous Work

The papers [13, 12, 9] present a general theory of convergence and smoothness of univariate and multivariate nonlinear subdivision rules which are *analogous* to linear rules. This analogy is also a central notion in the present paper. The method of investigation is via *proximity* of linear and corresponding nonlinear subdivision schemes: If $S$ is a linear scheme of $C^k$ smoothness, and $T$ is a nonlinear scheme sufficiently close to $S$, then also $T$ enjoys $C^k$ smoothness. In this way, $C^1$ smoothness of a large class of univariate and multivariate schemes has been shown, as well as $C^2$ smoothness of a smaller class of univariate 'factorizable' schemes. This general principle of proximity has been used before, e.g. in the analysis of non-stationary linear schemes in [7], and nonlinear schemes in [1, 16]. Recently, proximity methods have been employed in [17] to show $C^k$ smoothness of *interpolatory* schemes which work in the sphere and related manifolds, and which are constructed from a linear scheme by adding a projection step after each round of subdivision. The paper [15] analyzes subdivision schemes operating in Lie groups, namely *log-exponential analogues* of linear subdivision schemes. These have been introduced in [4] and are discussed in [11]. There are several ways to define such analogues – while the definition employed by [15] converts a sequence of group elements into a sequence of vectors and works with them, the present paper as well as [11] uses a different construction which among other things is better suited for generalization to the multivariate setting [9]. Another difference between the present paper and [15] is that here we allow arbitrary dilation factors.

## §2. Log-exponential Analogues of Linear Subdivision Rules

We consider a linear subdivision rule $S$ with dilation factor $N$, which takes as input a sequence $p = (p_i)_{i\in\mathbb{Z}}$ of points and constructs a refined sequence "$Sp$" from it: $Sp_{Ni+k} = \sum_{j\in\mathbb{Z}} a_{Ni+k-Nj} p_j$. We consider only the case that the mask $(a_i)_{i\in\mathbb{Z}}$ has only finitely many nonzero coefficients. It is well known that $S$ can be convergent only if it is affinely invariant, i.e., $\sum_i a_{j-Ni} = 1$ for all $j$. Consequently,

$$Sp_{Ni+k} = p_i + \sum\nolimits_{j\in\mathbb{Z}} a_{Ni+k-Nj}(p_j - p_i), \qquad (p_i \in \mathbb{R}^m) \tag{1}$$

The alternative definition of $S$ by (1) is well suited to define a Lie group analogue with, as shall be demonstrated by the next subsection.

### 2.1. Group analogues for addition, subtraction, and subdivision

Geometrically, adding a vector $v$ to a point $p$ in $\mathbb{R}^n$ amounts to shooting a ray $c_{p,v}(t) = p + tv$ from $p$ in direction of $v$, and following that ray for the appropriate length, until we have reached the point $c_{p,v}(1) = p + v$. The operation of computing a difference vector of points $p, q$ is to solve the equation $c_{p,v}(1) = q$, which results in $v = q - p$. For a matrix group or general Lie group, a ray system is given by the curves $c_{p,v}(t) = p\exp(tv)$, where $v$ is an element of the Lie algebra $\mathfrak{g}$, i.e., the tangent space of the group unit. This leads to the operations $p \oplus v$ and $p \ominus q$, where $p, q$ are points and $v$ is a vector:

$$p \oplus v := p\exp(v), \quad p \ominus q := \log(q^{-1}q), \quad (p, q \in G, v \in \mathfrak{g}). \tag{2}$$

These operations (but not the notation) are also employed in [15, 11]. We use them to further define the midpoint of points $p, q$, which is denoted by "mean$(p, q)$":

$$v = q \ominus p \implies \mathrm{mean}(p, q) := p \oplus (v/2) \tag{3}$$

The expression "mean$(p, q)$" is symmetric in its two arguments, because $p \ominus q = -q \ominus p$, and $\mathrm{mean}(p, q) = p\exp v \exp(-v/2) = q\exp((p \ominus q)/2) = \mathrm{mean}(q, p)$. Of course, the logarithm in $G$ is defined only in a (not too small) neighbourhood of the unit. For instance the group $\mathrm{GL}_n$ has a globally defined logarithm in the subset of matrices which do not have non-positive real eigenvalues. The definitions of (2) are invariant with respect to left translations in the group: For all $g, p, q \in G, v \in \mathfrak{g}$ we have $gp \ominus gq = p \ominus q$, $gp \oplus v = g(p \oplus v)$, and $g\,\mathrm{mean}(p, q) = \mathrm{mean}(gp, gq)$. A natural and invariant analogue "$T$" of the scheme $S$ which operates on group-valued data consequently is given by

$$Tp_{Ni+k} = p_i \oplus \sum\nolimits_{j\in\mathbb{Z}} a_{Ni+k-Nj}(p_j \ominus p_i), \quad (p_i \in G). \tag{4}$$

**Remark 1.** In a Riemannian manifold it would be natural to use the geodesic line emanating from the point $p$ and having initial tangent vector $v$ as the ray $c_{p,v}(t)$. Such rays obey the homogeneity law $c_{p,tv}(s) = c_{p,v}(ts)$ like the ones defined above. This is common to curves defined as solutions of initial value problems of a certain type of second order differential equation in a manifold known as *spray*.

**Remark 2.** The paper [15] uses a different way of constructing a log-exponential analogue. The scheme $S$ is not expressed in terms of differences $p_j - p_i$ as in (1), but those differences are further expressed in terms of differences $\Delta p_j = p_{j+1} - p_j$. This has the advantage that in both the linear and the nonlinear cases, the sequence $(p_i)_{i\in\mathbb{Z}}$ is uniquely determined by a single point $p_{i_0}$ and the sequence of difference vectors $p_{i+1} - p_i \in \mathbb{R}^m$ or $p_{i+1} \ominus p_i \in \mathfrak{g}$, respectively.

**Remark 3.** The vector space $\mathbb{R}^n$ is a group with addition as basic operation. It fits in the matrix group formalism after identifying a point $p \in \mathbb{R}^n$ with the matrix $\widetilde{p} = \left(\begin{smallmatrix} 1 & 0 \\ p & E \end{smallmatrix}\right) \in \mathbb{R}^{(n+1)\times(n+1)}$, and a vector $v \in \mathbb{R}^n$ with the matrix $\widehat{v} = \left(\begin{smallmatrix} 0 & 0 \\ v & 0 \end{smallmatrix}\right) \in \mathbb{R}^{(n+1)\times(n+1)}$, because then $\widetilde{p+q} = \widetilde{p} \cdot \widetilde{q}$. Further properties are $\widetilde{p}^{-1} = \widetilde{-p}$; $\exp(\widehat{v}) = \sum_{k\geq 0} \widehat{v}^k/k! = \widetilde{v}$ (because $\widehat{v}^2 = 0$); $\widetilde{p+v} = \widetilde{p}\widetilde{v} = \widetilde{p}\exp\widehat{v} = \widetilde{p} \oplus \widehat{v}$; and $\widetilde{p} \ominus \widetilde{q} = \log(\widetilde{p}^{-1}\widetilde{q}) = \log(\widetilde{p-q}) = \widehat{p-q}$. So in this particular matrix group the schemes $T$ analogous to a linear scheme $S$ are actually equal to $S$.

### 2.2. Symmetry of subdivision schemes

The subdivision rule $S$ might have the property that reversing the order in which the input data are indexed has no influence on the result of subdivision (i.e., $q_i = p_{-i} \implies Sq_j = Sp_{j_0-j}$, for some fixed index $j_0$ and all $j \in \mathbb{Z}$). It depends on the number of points of the sequence $p_i$ which contribute the the points $Sp_i$ whether or not also the nonlinear analogue $T$ has this symmetry property. If $T$ is not symmetric, we may consider a modified analogue defined by (6), which is based on the alternative representation of $S$ given by (5):

$$Sp_{Ni+k} = \frac{1}{2}(p_i + p_{i+1}) + \sum\nolimits_{j\in\mathbb{Z}} a_{Ni+k-Nj}(p_j - \frac{1}{2}(p_i + p_{i+1})) \tag{5}$$

$$Tp_{Ni+k} = \mathrm{mean}(p_i, p_{i+1}) \oplus \sum\nolimits_{j\in\mathbb{Z}} a_{Ni+k-Nj}(p_j \ominus \mathrm{mean}(p_i, p_{i+1})). \tag{6}$$

**Example 1.** Consider the interpolatory four-point scheme with weight $w$, which is given by the rules $Sp_{2i} = p_i$ for the even points and $Sp_{2i+1} = \frac{1}{16}(-w(p_{i-1} + p_{i+2}) + (\frac{1}{2} + w)(p_i + p_{i+1}))$ for the odd points (see [6]). Its nonlinear analogues have the general form below. We let the 'base point' $b_i$ equal $p_i$ when using (4), and equal $\mathrm{mean}(p_i, p_{i+1})$, when using (6):

$$\begin{aligned} Tp_{2i} = b_i \oplus (p_i \ominus b_i) = p_i,\ Tp_{2i+1} = b_i \oplus \big(& -w(p_{i-1} \ominus b_i) \\ & - w(p_{i+2} \ominus b_i) + (\frac{1}{2} + w)(p_i \ominus b_i) + (\frac{1}{2} + w)(p_{i+1} \ominus b_i)\big). \end{aligned}$$

Obviously, the analogue constructed with (6) is invariant with respect to inversion of indices.

## §3. Proximity Inequalities

For our smoothness analysis we employ concepts commonly used in the smoothness analysis of a linear curve subdivision scheme $S$, such as the *symbol* $a(z) = \sum a_j z^j$, and the *derived scheme* $S^*$ defined by the commutation relation $N\Delta S = S^*\Delta S$, where $\Delta p$ is defined by $\Delta p_i = p_{i+1} - p_i$. $S^*$ has the symbol $Nz^{N-1}a(z)/(1 + z + \cdots + z^{N-1})$. A further notion is the *norm* of a scheme, which is computable by $\|S\| = \max_j \sum_i |a_{j-Ni}|$. For more information and properties, we refer to the literature [5, 8].

The proximity methods according to [13, 14, 12] require to establish certain inequalities involving the distance of schemes $S$, and $T$, in order to conclude smoothness of $T$ from smoothness of $S$. With the notation $\|p\| = \sup_i \|p_i\|$ and $d(p) = \sup_i \|p_{i+1} - p_i\|$, these inequalities are

$$\|Sp - Tp\| \leq Cd(p)^2, \tag{7}$$

$$\|\Delta Sp - \Delta Tp\| \leq C(d(p)d(\Delta p) + d(p)^3), \quad (C > 0). \tag{8}$$

They are required to be true whenever $d(p)$ is small enough. Usually (7), which is needed for showing convergence and $C^1$ smoothness of $T$, is easy to show, but (8), required for $C^2$ smoothness of $T$, is harder. The reader will note that (7) and (8) indiscriminately compare points $Sp_i$ of $\mathbb{R}^n$ and points $Tp_i$ which are supposed to be elements of the Lie group $G$. This way of putting the proximity inequalities therefore makes sense only if $G$ is a matrix group, i.e., a subgroup of $\mathrm{GL}_n$: Then every $p \in G$ at the same time is an element of $\mathbb{R}^{n\times n} = \mathbb{R}^m$ with $m = n^2$. The canonical Euclidean norm in $\mathbb{R}^n$ corresponds to the Frobenius norm $\|g\|^2 = \mathrm{tr}(gg^T)$ in the matrix group $G$. However, it turns out that, without loss of generality, analysis can be restricted to the case of matrix groups (cf. the proof of [15, Theorem 5]).

### 3.1. The exponential function in matrix groups

We here derive several inequalities concerning the matrix exponential function and the matrix logarithm which are needed later.

**Lemma 1.** *Assume that $U$ is a bounded neighbourhood of the unit in the matrix group $G$ where the matrix logarithm is a diffeomorphism, and that $p_i$ is a $G$-valued sequence. Then there are constants $C, C', C'' > 0$ such that*

$$\|p_j \ominus b_i\| \leq Cd(p), \qquad (b_i = p_i \text{ or } b_i = \mathrm{mean}(p_i, p_{i+1})) \tag{9}$$

$$\|(p_j \ominus b_i) - 2(p_{j+1} \ominus b_i) + (p_{j+2} \ominus b_i)\| \leq C'd(\Delta p) + C''d(p)^2, \tag{10}$$

*if all differences $p \ominus q$ are such that $p^{-1}q \in U$, and $|i - j|$ is bounded.*

**Proof:** The inequality (9) follows immediately from the relations $x^{-1} = e - x + o(\|x\|^2)$, $\log(e+x) = x + o(\|x\|^2)$, and $\exp(x) = e + x + o(\|x\|^2)$. For (10), we use the abbreviation $v_i^j = p_j \ominus b_i$. The exponential series implies that there is $\gamma_1 > 0$ such that $\|(v_i^j - 2v_i^{j+1} + v_i^{j+2}) - (\exp v_i^j - 2\exp v_i^{j+1} + \exp v_i^{j+2})\| \leq \gamma_1 \sup_j \|v_i^j\|^2$, which in turn is bounded by $\gamma_2 d(p)^2$ ($\gamma_2 = C^2\gamma_1$). We conclude that $\|v_i^j - 2v_i^{j+1} + v_i^{j+2}\| \leq \|\exp v_i^j - 2\exp v_i^{j+1} + \exp v_i^{j+2}\| + \gamma_2 d(p)^2 = \|b_i^{-1}(\Delta^2 p_j)\| + \gamma_2 d(p)^2 \leq \|b_i^{-1}\|\|\Delta^2 p\| + \gamma_2 d(p)^2$. If we let $C' := \sup \|b_i^{-1}\|$, this is what we wanted to show. □

### 3.2. Some facts concerning Laurent polynomials

The proof of the inequality (8) which is the key for $C^2$ smoothness will need some properties of Laurent polynomials. In the following, $N$ is the dilation factor of the subdivision scheme $S$.

**Lemma 2.** *Let $a(z) = \sum a_j z^j$ be the symbol of the subdivision scheme $S$, and assume that $a(\zeta^i) = a'(\zeta^i) = a''(\zeta^i) = 0$ for all $i = 1, \ldots, N-1$, where $\zeta^N = 1$ and $\zeta^k \neq 1$ for $k = 1, \ldots, N-1$ (i.e., $\zeta$ is a primitive $N$th root of the unit). Then for all $k \in \mathbb{Z}$,*

$$(\sum_i i a_{k+1-Ni})^2 - (\sum_i i a_{k-Ni})^2 = \sum i^2 (a_{k+1-Ni} - a_{k-Ni}). \qquad (11)$$

**Proof:** We define $S_{1,k} = \sum_i i a_{k-Ni}$, $S_{2,k} = \sum_i i^2 a_{k-Ni}$, and observe that (11) can be expressed as $S_{1,k+1}^2 - S_{1,k}^2 = S_{2,k} - S_{2,k+1}$. This relation is verified as follows: We define the Laurent polynomials $a_k(z) = \sum_i a_{k-Ni} z^{k-Ni}$, and compute $a_k'(1) = \sum_i (k-Ni) a_{k-Ni} = k - N \sum i a_{k-Ni}$ (the last equality was by affine invariance), whence $S_{1,k} = \frac{1}{N}(k - a_k'(1))$. We further observe that $a_k(z)$ can be written as an average: $a_k(z) = (a(z) + \zeta^k \sum_{j=1}^{N-1} a(\zeta^j z))/N$. Differentiation yields $a_k'(z) = \big(a'(z) + \zeta^k \cdot \sum_{j=1}^{N-1} \zeta^j a'(\zeta^j z)\big)/N$, which implies $a_k'(1) = a'(1)/N$ because of our assumption that $a'(\zeta^j) = 0$ for $j = 1, \ldots, N-1$. Thus we have shown $S_{1,k} = \frac{1}{N}(k - \frac{a'(1)}{N})$. A similar computation shows that $S_{2,k} = \frac{1}{N^2}(\frac{a''(1)}{N^2} - k(k-1) + (2k-1)(k - \frac{a'(1)}{N}))$. Now (11) can be verified directly. □

**Lemma 3.** *The bivariate Laurent polynomial $A(x,y)$ can be written in the form $A(x,y) = (1-x)^2 a(x,y) + (1-y)^2 b(x,y)$ with Laurent polynomials $a(x,y)$, $b(x,y)$ if and only if $A(1,1) = A_x(1,1) = A_y(1,1) = A_{xy}(1,1) = 0$. Here the index indicates differentiation.*

**Proof:** The 'if' part is trivial. For the 'only if' part we first observe that multiplication of $A$ with powers of $x$ and $y$ does not affect the result. Thus we can without loss of generality assume that $A$ is a polynomial, and the desired expression follows from the Taylor expansion of $A$ at $(x,y) = (1,1)$ (every monomial of which is a multiple of either $(1-x)^2$ or $(1-y)^2$). □

### 3.3. The proof of proximity inequalities

Having established Lemmas 1–3, we are now ready to show under which circumstances inequalities of the form (7) and (8) hold.

**Lemma 4.** *Consider a matrix group $G$, a linear subdivision rule $S$, and the nonlinear subdivision rule $T$ which is defined either by* (4) *or* (6). *If $S$ has derived schemes up to order three, then $S$ and $T$ satisfy the proximity conditions* (7) *and* (8) *for all bounded input data $(p_i)_{i\in\mathbb{Z}}$ where successive points $p_i, p_{i+1}$ are close enough.*

**Proof:** The inequality (7) has been shown in [9], and is anyway much easier than (8). In order to show (8), we have to give an upper bound for $F_{l+1} - F_l$, where $F_l = Tp_l - Sp_l$. We do this by considering that expression as a function of the vectors $v_i^l = p_l \ominus b_i$, where the base point either equals $p_i$ or $\mathrm{mean}(p_i, p_{i+1})$, depending on whether we use (4) or (6) as definition of $T$. Recall that

$$Tp_{Ni+k} - Sp_{Ni+k} = b_i \exp(\sum a_{Ni+k-Nj} v_i^j) - b_i - \sum a_{Ni+k-Nj}(b_i e^{v_i^j} - b_i).$$

It is easy to see that when expanding the exponential series here, that the first order terms vanish. In our analysis of second order terms in the expansion of $F_{l+1} - F_l$, we consider two cases.

**Case 1:** $l = Ni + k$ *with* $k \in \{0, \ldots, N-2\}$ *and* $i \in \mathbb{Z}$. Here both terms $Tp_{l+1}$ and $Tp_l$ are defined via the *same* base point $b_i$, and one easily finds, by considering the first terms of the exponential series, that the second order Taylor polynomial of $F_{l+1} - F_l$ has the form

$$\frac{b_i}{2}\big[(\sum\nolimits_j a_{Ni+k+1-Nj} v_i^j)^2 - (\sum\nolimits_j a_{Ni+k-Nj} v_i^j)^2 \\ - \sum\nolimits_j (a_{Ni+k+1-Nj} - a_{Ni+k-Nj})(v_i^j)^2\big].$$

**Case 2:** $l = Ni + N - 1$ *for some integer* $i$. In this case, the base points which occur in the definition of $Tp_{l+1}$ and $Tp_l$ are different, and there occur vectors of type $v_i^j$ as well as vectors of type $v_{i+1}^j$. By definition,

$$p_l = b_{i+1} \exp v_{i+1}^l = b_i \exp v_i^l = b_{i+1} \exp v_{i+1}^i \exp v_i^l. \qquad (12)$$

The Campbell-Hausdorff formula (cf. [10]), which in general reads $\exp(v) \cdot \exp(w) = \exp(v + w + \frac{1}{2}[v, w] + \cdots)$, implies that $v_{i+1}^j = v_{i+1}^i + v_i^j + \frac{1}{2}[v_{i+1}^i, v_i^j] + \cdots$, where the dots indicate higher order terms. In order to relate the different base points, we use $p_{i+1} = p_i \exp v_i^{i+1}$ and $\mathrm{mean}(p_{i+1}, p_{i+2}) = \mathrm{mean}(p_i, p_{i+1}) \exp(\frac{1}{2} v_i^{i+1}) \exp(\frac{1}{2} v_{i+1}^{i+2})$. When we use these formulas to replace all references to $b_{i+1}$ and $v_{i+1}^l$ in $F_{l+1}$ by expressions whose dominant term is $b_i$ or the $v_i^l$'s, we see that vectors of type

$v_{i+1}^l$ occur only in higher order terms, and that the second order Taylor expansion of $F_{l+1} - F_l$ is the same as in case 1.

We continue to discuss the second order Taylor polynomial of $F_{l+1} - F_l$, which has the form $\frac{1}{2} b_i \sum A_{rs} v_i^r v_i^s$, with

$$A_{rs} = a_{Ni+k+1-Ns} a_{Ni+k+1-Nr} - a_{Ni+k-Ns} a_{Ni+k-Nr} \quad (s \neq r)$$
$$A_{rs} = a_{Ni+k+1-Ns}^2 - a_{Ni+k-Ns}^2 + (a_{Ni+k+1-Ns} - a_{Ni+k-Ns}) \quad (s = r).$$

As $\|b_i\|$ is bounded, it does not affect the desired upper bound, so we discard this factor. We want to rewrite $\sum_{r,s} A_{rs} v_i^r v_i^s$ in the form

$$\sum\nolimits_{r,s} \big(a_{rs} v_i^r (v_i^s - 2v_i^{s+1} + v_i^{s+2}) + b_{rs} (v_i^r - 2v_i^{r+1} + v_i^{r+2}) v_i^s\big) \qquad (13)$$

with as yet unknown coefficients $a_{rs}$ and $b_{rs}$. This is converted into a statement on the generating functions $A(x, y) = A_{rs} x^r y^s$, $a(x, y) = a_{rs} x^r y^s$, and $b(x, y) = b_{rs} x^r y^s$: We want to write $A(x, y)$ in the form $(1 - y)^2 a(x, y) + (1 - x)^2 b(x, y)$. By Lemma 3, this is possible if and only if $A(1, 1) = A_x(1, 1) = A_y(1, 1) = A_{xy}(1, 1) = 0$. It is now an elementary if somewhat tedious task to evaluate these values and derivatives: The second order Taylor polynomial of $F_l$ alone is easily seen to have the generating function $\widetilde{A}(x, y) = (\sum a_{l-Nr} x^r)(\sum a_{l-Ns} y^s) - \sum a_{l-Nr} x^r y^r$, and it is clear that $\widetilde{A}(1, 1) = \widetilde{A}_x(1, 1) = \widetilde{A}_y(1, 1) = 0$, when we observe that the affine invariance of the scheme $S$ implies $\sum_i a_{i-Nj} = 1$ for all $j$. Therefore also $A$ has this property.

A short computation shows that the condition $A_{x,y}(1, 1) = 0$ is equivalent to (11). As $S$ was assumed to have derived schemes up to oder three, the symbol $a(z)$ has a factor of the form $(1 + z + \cdots z^{N-1})$, i.e., $a(\zeta^k) = a'(\zeta^k) = a''(\zeta^k) = 0$ for a primitive $N$th root $\zeta$ and $k = 1, \ldots, N - 1$. We therefore can use Lemma 2 to conclude that the conditions of Lemma 3 are fulfilled, and we can indeed write the second order Taylor polynomial of $F_l$ in the form (13). The inequality (10) of Lemma 1 now implies that $\sup \|F_l\| = \|\Delta Sp - \Delta Tp\| \leq C(d(p)d(\Delta p) + d(p)^3)$ up to higher order terms (which again are of magnitude $d(p)^3$). So we have shown the desired proximity condition. □

## §4. Results

In the previous section we have collected all inequalities which are necessary to show a theorem on smoothness of nonlinear subdivision schemes which in its scope and generality is analogous to Theorems 5 and 6 of [15].

**Theorem.** *Let $G$ be a matrix group, and $S$ a convergent linear curve subdivision scheme with finite mask and dilation factor $N \geq 2$. Define a nonlinear log-exponential analogue $T$ of $S$ either by* (4)*, or by* (6)*. Then,*

1. *For any sequence $(p_i)_{i\in\mathbb{Z}}$, where successive points are close enough, the scheme $T$ produces continuous limit curves.*
2. *If the second derived scheme $S^{**}$ exists, and for some nonzero integer $k$, $\|S^{*k}\| < N^{k/2}$, $\|S^{**k}\| < N^k$, then all continuous limit curves produced by the scheme $T$ are continuously differentiable.*
3. *If the third iterated derived scheme $S^{***}$ exists and for some nonzero integer $k$, $\|S^{*k}\| < N^{k/3}$, $\|S^{*k}\|\,\|S^{**k}\| < N^k$, $\|S^{***k}\| < N^k$, then all continuous limit curves produced by $T$ enjoy $C^2$ smoothness.*

*The statement of the theorem remains true if 'matrix group' is replaced by 'finite-dimensional Lie group'.*

**Proof:** The proof of this theorem is completely analogous to the proof of Theorems 5 and 6 in [15]. In that paper, we establish that proximity inequalities hold in exactly the same way as in our Lemma 4. Theorem 5 of [15] shows the result for matrix Lie groups and therefore also for embedded local matrix groups. In order to show the theorem for a general Lie group $G$, we use the following argument. The result is local, and the construction of the log-exponential analogue is invariant by left multiplication. Thus it suffices to show the theorem for a small neighbourhood of the identity. By Ado's theorem, the Lie algebra $\mathfrak{g}$ of $G$ is isomorphic to a Lie algebra $\mathfrak{g}'$ of $n \times n$ matrices and $G$ is locally around the identity isomorphic to the local matrix group "$\exp(\mathfrak{g}')$". We see that the theorem applies to $G$ also. Theorem 6 of [15] uses a different argument.

Note that the sequence $(p_i)_{i\in\mathbb{Z}}$ of input data no longer has to be bounded (in contrast to previous lemmas). This is because only finitely many data points can influence a certain compact interval of the limit curve, so we may w.l.o.g. assume boundedness of input data. □

**Example 2.** The Deslauriers-Dubuc scheme "$S$" with parameter $D = 2$ [2] has the symbol $a(z) = (3z^4 - 18z^3 + 38z^2 - 18z + 3)(1 + z)^6/(256z^5)$ and dilation factor $N = 2$. Its log-exp analogue $T$ is used in [11] for the construction of nonlinear wavelet transforms. $T$'s limit curves are $C^2$, because our theorem applies to $S$ for $k = 2$: $\|S^{*2}\| \approx 1.487 < 2^{k/3} \approx 1.587$, $\|S^{*2}\|\,\|S^{**2}\| \approx 3.917 < 2^k = 4$, and $\|S^{***2}\| \approx 2.844 < 2^k = 4$.

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