IVORY'S THEOREM IN HYPERBOLIC SPACES

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Abstract. According to the planar version of Ivory's Theorem the family of confocal conics has the property that in each quadrangle formed by two pairs of conics the diagonals are of equal length. It turned out that this theorem is closely related to self-adjoint affine transformations. This point of view is capable of generalization to hyperbolic and other spaces.

1. Introduction

The planar Euclidean version of Ivory's Theorem states that the two diagonals X_1X_2' and X_2X_1' of any curvilinear quadrangle formed by four confocal conics have the same length (this is illustrated in Fig. 1). Another version of this theorem is the following: Assume that k, k' are confocal ellipses or confocal hyperbolas (two conics *of the same type*). Then there is an affine transformation α with $\alpha(k) = k'$ and such that whenever a conic of the other type, but confocal with k and k' intersects the conic k in the point X_1, it will intersect k' in the point $\alpha(X_1)$, and both intersections are orthogonal. Now Ivory's Theorem states equal distances

$$\overline{\alpha(X_1)\,X_2} = \overline{X_1\,\alpha(X_2)} \quad \text{for all} \quad X_1, X_2 \in k.$$

This statement holds also for singular α when $k' = \alpha(k)$ degenerates into a set of points located on an axis of symmetry of k.

In 1809 J. Ivory proved the three-dimensional version of this theorem by straightforward calculation and by using an appropriate parametrization ([7], see also [5, 11, 4, 1]).

This theorem holds in the n-dimensional Euclidean space ($n > 1$, see e.g. [9]). It has been shown by [8] that it is also true in the pseudo-Euclidean (Minkowski) plane. The aim of this paper is to prove Ivory's Theorem in hyperbolic spaces $\mathbb{H}^n$, and indeed in a more general class of pseudo-Riemannian spaces of constant curvature.

2. Definitions and basic results

2.1. **Scalar products.** We consider the real n-dimensional projective space $\mathbb{P}^n$, which is the set of *points* $X = \mathbf{x}\mathbb{R}$ with $\mathbf{x} \in \mathbb{R}^{n+1} \setminus \{0\}$. We assume that $\mathbb{R}^{n+1}$ is endowed with a *scalar product*, i.e., a nondegenerate symmetric bilinear form

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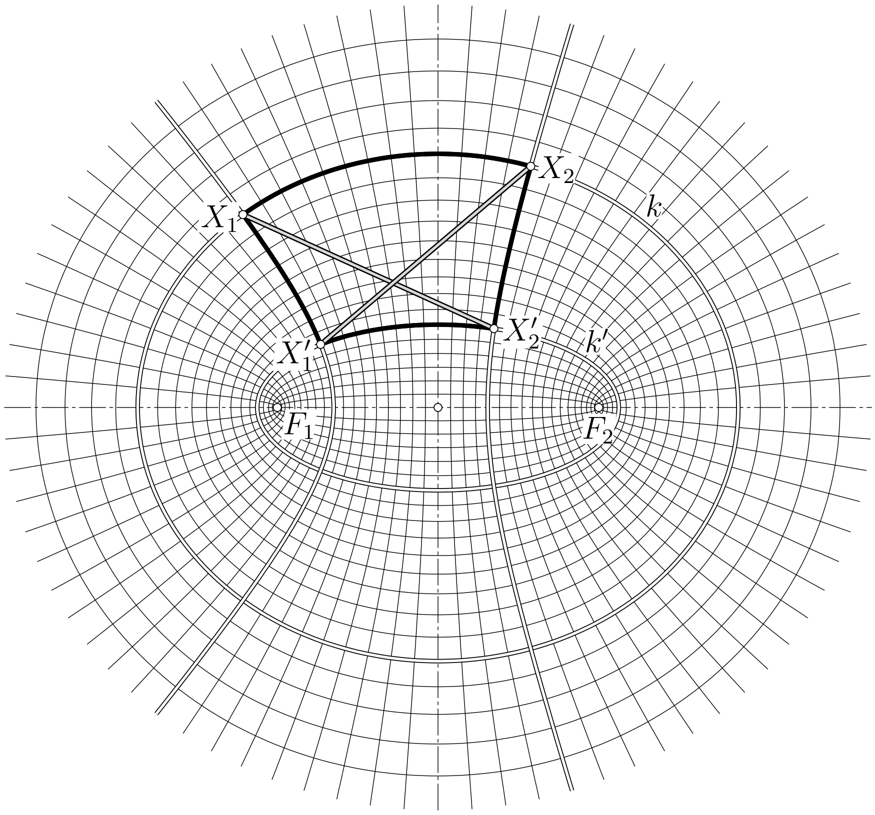

FIGURE 1. IVORY's Theorem in the Euclidean plane $\mathbb{E}^2$

$\langle\ ,\ \rangle$. If the scalar product has signature $(- + \cdots +)$, then the n-dimensional *hyperbolic space* is given by the subset

$$\mathbb{H}^n = \{\mathbf{x}\mathbb{R} \mid \langle \mathbf{x}, \mathbf{x} \rangle < 0\}$$

of $\mathbb{P}^n$. The hyperbolic distance $d_h(\mathbf{x}, \mathbf{y})$ of points of $\mathbb{H}^n$ is given by

$$\cosh d_h(\mathbf{x}, \mathbf{y}) = \Big| \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}} \Big|. \tag{1}$$

The case of signature $(+ \cdots +)$ leads to the *elliptic metric* d_e in $\mathbb{P}^n$ defined by

$$\cos d_e(\mathbf{x}, \mathbf{y}) = \Big| \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}} \Big|. \tag{2}$$

Remark. Standard spherical geometry is a twofold covering of the elliptic n-space. We may identify $\mathbb{H}^n$ with one of the two components of the set $\{\mathbf{x} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1\}$. This embedding induces a Riemannian metric in $\mathbb{H}^n$, where distances of points are given by (1). For general information on hyperbolic spaces see [2].

The signatures $(+-\cdots-)$ and $(-\cdots-)$ likewise generate hyperbolic and elliptic geometries, with the obivous modifications in the definition of the metric. Signatures of the form $(++--\pm\cdots\pm)$ do not lead to Riemannian manifolds, but we may still consider expressions of the form

$$\delta(X,Y)=\delta(\mathbf{x},\mathbf{y})=\frac{\langle\mathbf{x},\mathbf{y}\rangle}{\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle\langle\mathbf{y},\mathbf{y}\rangle|}} \tag{3}$$

which serve as a substitute for the metric, and which are meaningful for all points $X=\mathbf{x}\mathbb{R}, Y=\mathbf{y}\mathbb{R}\in\mathbb{P}^n$ with $\langle\mathbf{x},\mathbf{x}\rangle,\langle\mathbf{y},\mathbf{y}\rangle\neq 0$. Points $\mathbf{x}\mathbb{R}$ with $\langle\mathbf{x},\mathbf{x}\rangle=0$ are called *absolute.*

The *hyperplanes* in $\mathbb{P}^n$ are the zero sets of linear forms $\mathbf{a}^*\in\mathbb{R}^{(n+1)*}$. We may represent $\mathbf{a}^*$ by its *gradient vector* $\mathbf{v}$ such that $\mathbf{a}^*(\mathbf{x})=\langle\mathbf{v},\mathbf{x}\rangle$. The two hyperplanes with gradient vectors $\mathbf{v}$, $\mathbf{w}$, resp., are called *orthogonal* if $\langle\mathbf{v},\mathbf{w}\rangle=0$. Self-orthogonal hyperplanes are called *absolute.*

Recall that *endomorphisms* l and l^* are called *adjoint*, if $\langle\mathbf{x},l(\mathbf{y})\rangle=\langle l^*(\mathbf{x}),\mathbf{y}\rangle$ for all $\mathbf{x},\mathbf{y}$. When we write bilinear forms as $\sigma(\mathbf{x},\mathbf{y})=\langle\mathbf{x},l(\mathbf{y})\rangle$ with a linear endomorphism l, then σ is symmetric if and only if l is selfadjoint.

Definition 1. *A (nondegenerate) quadric Φ is the zero set of a (nondegenerate) symmetric bilinear form $\sigma(\mathbf{x},\mathbf{y})=\langle\mathbf{x},l(\mathbf{y})\rangle$, with a selfadjoint (nonsingular) linear endomorphism l.*

$l=\mathrm{id}$ corresponds to the *absolute quadric* Ω, the set of absolute points.

2.2. **Selfadjoint endomorphisms.** We are going to describe *simultaneous normal forms* of the symmetric bilinear form $\langle\ ,\ \rangle$ and the selfadjoint linear mappings l — this means the choice of a coordinate system such that both have simple coordinate matrices. We use the notation I_k for the $k\times k$ unit matrix, and

$$J_k(t,s)=\begin{bmatrix} t & s & & \\ & \ddots & \ddots & \\ & & t & s \\ & & & t\end{bmatrix},\quad S_k=\begin{bmatrix} & & 1\\ & \iddots & \\ 1 & & \end{bmatrix},\quad R_2(a,b)=\begin{bmatrix} a & b\\ -b & a\end{bmatrix},$$

$$R_{2k}(a,b,s)=\begin{bmatrix} R_2(a,b) & sI_2 & & \\ & \ddots & \ddots & \\ & & R_2(a,b) & sI_2 \\ & & & R_2(a,b)\end{bmatrix}.$$

The lower right index indicates the size of the matrix.

Theorem 1. *For any selfadjoint l, there are coordinates in $\mathbb{R}^{n+1}$ such that $\langle\mathbf{x},\mathbf{y}\rangle=\mathbf{x}^TH\mathbf{y}$, $l(\mathbf{x})=A\cdot\mathbf{x}$, and the matrices A, H have the form*

$$A=\mathrm{diag}(J_{r_0}(t_0,1),\ldots,J_{r_{k-1}}(t_{k-1},1),R_{2r_k}(a_k,b_k,1),\ldots,R_{2r_s}(a_s,b_s,1)), \tag{4}$$

$$H=\mathrm{diag}(\epsilon_0S_{r_0},\ldots,\epsilon_{k-1}S_{r_{k-1}},\epsilon_kS_{2r_k},\ldots,\epsilon_sS_{2r_s}), \tag{5}$$

with $\epsilon_i=\pm1$ and $t_i,a_i,b_i\in\mathbb{R}$, $b_i\neq 0$.

Proof: See [6], Th. 5.3. □

Corollary 1. *In case of a definite scalar product* $\langle\ ,\ \rangle$*, the normal form of Th. 1 is the following:*

$$A = \operatorname{diag}(t_0, \dots, t_n), \quad H = \pm I_{n+1}.$$

In case of a hyperbolic scalar product with signature $\epsilon(-+\cdots+)$ *the normal form of Th. 1 is one of the following:*

$$\begin{array}{llll} (i) & A = \operatorname{diag}(t_0, \dots, t_n) & H = \epsilon\operatorname{diag}(-1, I_n), & \\ (ii) & A = \operatorname{diag}(J_2(t_0, 1), t_2, \dots, t_n) & H = \epsilon\operatorname{diag}(\epsilon_0 S_2, I_{n-1}) & \textit{with } \epsilon_0 = \pm 1, \\ (iii) & A = \operatorname{diag}(R_2(a, b), t_2, \dots, t_n) & H = \epsilon\operatorname{diag}(\epsilon_0 S_2, I_{n-1}) & \textit{with } \epsilon_0 = \pm 1, \\ (iv) & A = \operatorname{diag}(J_3(t_0, 1), t_3, \dots, t_n) & H = \epsilon\operatorname{diag}(S_3, I_{n-2}). & \end{array}$$

Proof: We consider the hyperbolic case first. The signature of each matrix ϵS_r is given by $\epsilon(+-+-\cdots)$, and by summing up we easily compute the signature of the block matrix (5). This signature must equal the signature of $\langle\ ,\ \rangle$. Thus only one of the S_r's can have size greater than one, and this size must be less than four. The result follows immediately from Th. 1.

The elliptic case is similar — besides, this is the spectral theorem for selfadjoint endomorphisms in the presence of a definite scalar product. □

2.3. **Square roots.** Later we will need that certain endomorphisms have square roots of a certain form, so we collect some lemmas here. We will extend the definition of $J_k(t, s)$ to matrices with entries in a ring R in the obvious way. First we define a relation between formal power series with coefficients in a commutative ring R and matrices in $R^{k\times k}$ $(k > 0)$ as follows:

$$(6) \qquad a(x) \sim A \quad \Longleftrightarrow \quad a(x) = \sum a_i x^i,\ A = \begin{bmatrix} a_0 & a_1 & \dots & a_{k-1} \\ & a_0 & \dots & a_{k-2} \\ & & \ddots & \vdots \\ & & & a_0 \end{bmatrix}.$$

It is obvious that

$$(7) \qquad A, B \in R^{k\times k},\ a(x) \sim A,\ b(x) \sim B \implies a(x)b(x) \sim AB.$$

Lemma 1. *Assume that* R *is a* $\mathbb{Q}$*-algebra (i.e., a commutative ring which contains* $\mathbb{Q}$ *as a subring). Assume further that* $t \in R$ *has a square root* $\sqrt{t}$ *as well as an inverse* t^{-1}*. Consider the upper triangular matrix* $A \in R^{k\times k}$ *which according to* (6) *is related to the power series*

$$(8) \qquad a(x) = (t + sx)^{1/2} = \sum\nolimits_{i\ge 0} \binom{1/2}{i} (st^{-1})^i \sqrt{t} x^i \in R[[x]].$$

Then A^2 *equals the Jordan block* $J_k(t, s)$*.*

Proof: Clearly, the matrix $J_k(t, s)$ is related to the power series $t + sx$, and by (7) a square root is found as the matrix related to $(t+sx)^{1/2} = \sqrt{t}(1+st^{-1})^{1/2} = \sqrt{t}\sum_{i\ge 0}\binom{1/2}{i}(st^{-1})^i x^i$. □

Lemma 2. *A Jordan block* $J_k(t,s) \in \mathbb{R}^{k\times k}$ *with* $t>0$ *has a square root of the form* (6) *with* $a_i = \binom{1/2}{i}(s/t)^i\sqrt{t}$. *All matrices* $R_{2k}(a,b,s) \in \mathbb{R}^{2k\times 2k}$ *with* $b \neq 0$ *have square roots of the following forms:*

$$\sqrt{R_2(a,b)} = R_2(b/2\zeta, \zeta), \quad \text{where } \ 4\zeta^4 + 4a\zeta^2 - b^2 = 0,$$

$$\sqrt{R_{2k}(a,b,s)} = \begin{bmatrix} A_0 & A_1 & \dots & A_{k-1} \\ & A_0 & \dots & A_{k-2} \\ & & \ddots & \vdots \\ & & & A_0 \end{bmatrix}, \text{ where } A_i = \binom{1/2}{i}\sqrt{R_2(a,b)}s^i R_2(a,b)^{-i}.$$

Proof: The statement about $J_k(t,s)$ follows directly from Lemma 1 if we let $R = \mathbb{R}$. The statement about $R_2(a,b)$ is easily verified. Observe that the quadratic equation $4\xi^2 + 4a\xi - b^2 = 0$ has the solutions $\xi = \frac{1}{2}(-a \pm \sqrt{a^2+b^2})$, at least one of which is positive because $b \neq 0$. This shows that ζ exists and is nonzero.

Now let R equal the subring of the matrix ring $\mathbb{R}^{2\times 2}$ which is generated by the matrix $\sqrt{R_2(a,b)}$. Note that the matrix ring $R^{k\times k}$ is embedded into the matrix ring $\mathbb{R}^{2k\times 2k}$, and that $J_k(R_2(a,b), sI_2) = R_{2k}(a,b,s)$. With these identifications, the statement about $\sqrt{R_{2k}(a,b,s)}$ follows directly from Lemma 1. □

Corollary 2. *Selfadjoint linear endomorphisms* l *in* $\mathbb{R}^{n+1}$ *whose real eigenvalues are positive possess selfadjoint square roots.*

Proof: We use the coordinate matrices A of l and H of $\langle\ ,\ \rangle$ given by Th. 1 which have a common block diagonal structure. It is sufficient to consider each block A_k and $H_k = \pm S_k$ separately. Square roots $\sqrt{A_k}$ are given by Lemma 2. As $\sqrt{A_k}^T(\pm S_k) = (\pm S_k)\sqrt{A_k}$ in each case, the mapping $\sqrt{l}$ constructed in this way is selfadjoint. □

Corollary 3. *Both the matrices* $J_k(t,s)$ *for* $t>0$ *and* $R_{2k}(a,b,s)$ *with* $a^2+b^2>0$ *have square roots which depend smoothly on* t,s *and on* a,b,s, *respectively. The derivatives commute with the matrices.*

Proof: The result concerning smoothness follows from the explicit formula for the square roots given by Lemma 2 — we use the fact that ζ equals $(\frac{1}{2}(-a+(a^2+b^2)^{\frac{1}{2}}))^{\frac{1}{2}}$. We observe that derivatives of matrices of the form (6) are again of that form. Now commutativity follows from (7), as multiplication of power series is commutative. □

2.4. **Dual quadrics.** Points $X = \mathbf{x}\mathbb{R}$, $Y = \mathbf{y}\mathbb{R}$ are called *conjugate* with respect to a nondegenerate quadric Φ or its defining bilinear form σ, if $\sigma(\mathbf{x},\mathbf{y}) = \langle \mathbf{x}, l(\mathbf{y})\rangle = 0$.

The set of points conjugate to X is the hyperplane with gradients $\lambda l(\mathbf{x})$. If points X, Y are contained in a quadric Φ, and they are conjugate with respect to Φ, then the line $X \vee Y$ is contained in Φ. Conversely, any two points of a subspace contained in Φ are conjugate.

The *tangent hyperplane* of Φ in $\mathbf{x}$ has gradients $\lambda l(\mathbf{x})$. Conversely, a hyperplane with gradient $\mathbf{v}$ is tangent to Φ if and only if $\widehat{\sigma}(\mathbf{v},\mathbf{v}) = \sigma(l^{-1}(\mathbf{v}), l^{-1}(\mathbf{v})\rangle = \langle l^{-1}(\mathbf{v}), \mathbf{v}\rangle = \langle \mathbf{v}, l^{-1}(\mathbf{v})\rangle = 0$. If $\mathbf{a}^*, \mathbf{b}^*$ have respective gradient vectors $\mathbf{v}, \mathbf{w}$, then we consider the bilinear form

$$\sigma^*(\mathbf{a}^*, \mathbf{b}^*) = \widehat{\sigma}(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, l^{-1}(\mathbf{w})\rangle. \tag{9}$$

Definition 2. *The quadric $\widehat{\Phi}$ in the dual space defined by* (9) *is called the dual of the original quadric Φ defined by σ.*

If we use coordinates with respect to some basis, we assume linear forms to be coordinatized with respect to the corresponding dual basis. If $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T H \mathbf{y}$, then a linear form $\mathbf{a}^*$ and its gradient $\mathbf{v}$ are connected via $\mathbf{a}^* = H\mathbf{v}$. If $\sigma(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T HA\mathbf{y}$, we have $\widehat{\sigma}(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T HA^{-1}\mathbf{w}$, and $\sigma^*(\mathbf{a}^*, \mathbf{b}^*) = \mathbf{a}^{*T}(HA)^{-1}\mathbf{b}^*$.

The dual quadric $\widehat{\Phi}$ may be seen as a set of hyperplanes (the zero sets of the corresponding linear forms). It is the image of Φ under the mapping "$X \mapsto$ hyperplane of points conjugate to X".

We have excluded singular quadrics and singular forms from this discussion about dual quadrics — they do not possess duals in the sense of Def. 2. There are however *singular dual quadrics* defined by singular symmetric bilinear forms in $\mathbb{R}^{(n+1)*}$.

We compute the image $\Phi_1 = k(\Phi_0)$ of a quadric Φ_0 under a nonsingular endomorphism k: We have $\mathbf{x} \in \Phi_0 \iff k(\mathbf{x}) \in \Phi_1$. Thus if Φ_0 is defined by the bilinear form $\sigma_0(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, l(\mathbf{y})\rangle$, then Φ_1 is defined by $\sigma_1(\mathbf{x}, \mathbf{y}) = \langle k^{-1}(\mathbf{x}), lk^{-1}(\mathbf{y})\rangle = \langle \mathbf{x}, (k^{-1})^* lk^{-1}(\mathbf{y})\rangle$. If we represent linear forms by their gradients, the respective duals of Φ_0 and Φ_1 are defined by bilinear forms $\langle \mathbf{v}, l^{-1}(\mathbf{w})\rangle$ and $\langle \mathbf{v}, kl^{-1}k^*(\mathbf{w})\rangle$.

The fact that the inverse of k does not appear in the form which defines $\widehat{\Phi}_1$ allows to extend these computations by the following definition:

Definition 3. *If k is a linear endomorphism and the quadric Φ is given by the endomorphism l, then we define the dual k-image of Φ to have the equation*

$$\widehat{\sigma}(\mathbf{v}, \mathbf{v}) = 0, \quad \textit{with} \quad \widehat{\sigma}(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, kl^{-1}k^*(\mathbf{w})\rangle.$$

$\widehat{\sigma}$ is understood to apply to gradients.

It so appears that singular linear mappings, when applied to nonsingular quadrics, still define meaningful *dual* image quadrics.

2.5. **Families of confocal forms and quadrics.** Assume that the quadrics Φ_0 and Φ_1 are defined by symmetric bilinear forms σ_0 and σ_1, or selfadjoint endomorphisms l_0 and l_1, respectively, and that their respective duals $\widehat{\Phi}_0$, $\widehat{\Phi}_1$ are represented by the symmetric bilinear forms $\widehat{\sigma}_0$ and $\widehat{\sigma}_1$ according to Def. 2.

In coordinates, we let $l_i(\mathbf{x}) = A_i\mathbf{x}$ and $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T H\mathbf{y}$. Then $\sigma_i(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T Q_i \mathbf{y}$ with $Q_i = HA_i$, and $\sigma_i^*(\mathbf{a}^*, \mathbf{b}^*) = \mathbf{a}^{*T} Q_i^{-1}\mathbf{b}^*$.

Definition 4. *We use the notations defined immediately above.* Φ_0 *and* Φ_1 *are said to be confocal (or homofocal), if one of the following equivalent conditions holds true:*

(i) *the bilinear forms* $\widehat{\sigma}_0$, $\widehat{\sigma}_1$, $\widehat{\langle\ ,\ \rangle} = \langle\ ,\ \rangle$ *are linearly dependent,*
(ii) *the linear endomorphisms* l_0^{-1}, l_1^{-1}, id *are linearly dependent,*
(iii) *the coordinate matrices* Q_1^{-1}, Q_2^{-1}, *and* H^{-1} *are linearly dependent.*

The family of quadrics Φ *confocal to* Φ_0 *is defined by endomorphisms* l *which satisfy* $l^{-1} = \lambda l_0^{-1} + \mu\mathrm{id}$, $(\lambda, \mu) \in \mathbb{R}^2$, $\lambda \neq 0$.

Hence the dual quadrics $\widehat{\Phi}$ together with the dual $\widehat{\Omega}$ of the absolute quadric form a linear system of algebraic hypersurfaces in the dual projective space. Any absolute tangent hyperplane of Φ is also tangent to Φ_0, and vice versa.

Remark. This definition extends the usual definition of confocality in Euclidean geometry, which fits into this framework after introducing homogeneous coordinates and defining a singular symmetric bilinear form $\widehat{\langle\ ,\ \rangle}$, cf. [5, 11, 4, 3].

In the Euclidean plane, confocal conics (see Fig. 1) may be of different type, i.e., one of them an ellipse, and another a hyperbola. The obvious fact that there is no continuous transition from an ellipse to a hyperbola is the motivation for the following definition.

Definition 5. *Within the family of confocal bilinear forms spanned by* l_0, *the connected components of* $\{(\lambda, \mu) \mid \lambda(l_0^{-1} + \mu\mathrm{id})$ *nonsingular*$\}$ *correspond to quadrics of different types.*

Lemma 3. *In the* n*-dimensional elliptic or hyperbolic space* ($n > 1$) *all confocal families possess at least two types of quadrics.*

Proof: By Def. 5 it is sufficient to show that l_0^{-1} has an eigenvalue. As l_0^{-1} is selfadjoint, we apply Cor. 1 and the fact that $n + 1 \geq 3$. □

The following is well known:

Lemma 4. *If confocal quadrics* Φ_0 *and* Φ_λ *intersect, they do so orthogonally.*

Proof: Assume that g_0 and g_λ are endomorphisms which define confocal quadrics, such that $g_\lambda^{-1} = g_0^{-1} + \lambda\mathrm{id}$ ($\lambda \neq 0$), and that $\mathbf{v} = g_0(\mathbf{x})$ and $\mathbf{w} = g_\lambda(\mathbf{x})$ are gradient vectors of tangent hyperplanes in $\mathbf{x} \in \Phi_0 \cap \Phi_\lambda$.

From the relation $0 = \langle \mathbf{x}, g_0(\mathbf{x})\rangle = \langle \mathbf{x}, \mathbf{v}\rangle = \langle g_\lambda^{-1}(\mathbf{w}), \mathbf{v}\rangle$ and the analogous relation $0 = \langle g_0^{-1}(\mathbf{w}), \mathbf{v}\rangle$ we get $0 = \langle \mathbf{v}, \mathbf{w}\rangle$ by linear combination. □

3. The Ivory property and selfadjoint endomorphisms

3.1. **Ivory maps are linear.** We show that Ivory's Theorem for two quadrics Φ_0, Φ_1 is always related to the existence of a selfadjoint linear endomorphism l with $\Phi_1 = l(\Phi_0)$.

Lemma 5. *Assume that* Φ *is a quadric, possibly singular but not contained in a hyperplane, and that there is a mapping* $\mathbf{x} \mapsto \mathbf{x}'$ *such that*

$$\langle \mathbf{x}_1', \mathbf{x}_2\rangle = \langle \mathbf{x}_1, \mathbf{x}_2'\rangle \quad \textit{for all} \quad \mathbf{x}_1, \mathbf{x}_2 \in \Phi,$$

then there is a selfadjoint linear endomorphism l of $\mathbb{R}^{n+1}$ such that $\mathbf{x}' = l(\mathbf{x})$ for all $\mathbf{x} \in \Phi$.

Proof: Choose linearly independent vectors $\mathbf{x}_0, \dots, \mathbf{x}_n \in \Phi$ and define a linear endomorphism l by $l(\mathbf{x}_i) = \mathbf{x}'_i$ $(i = 0, \dots, n)$. By our assumption, we have

$$\langle \mathbf{x}_i, l^*(\mathbf{y}) \rangle = \langle l(\mathbf{x}_i), \mathbf{y} \rangle = \langle \mathbf{x}'_i, \mathbf{y} \rangle = \langle \mathbf{x}_i, \mathbf{y}' \rangle \quad (i = 0, \dots, n).$$

By linear independence of the vectors $\mathbf{x}_i$, we conclude that $\mathbf{y}' = l^*(\mathbf{y})$. This holds for all $\mathbf{y}$, so we especially have $l^*(\mathbf{x}_i) = \mathbf{x}'_i$, which shows that $l = l^*$. □

3.2. **The quadrics with equation $\langle \mathbf{x}, \mathbf{x} \rangle - \langle l(\mathbf{x}), l(\mathbf{x}) \rangle = 0$.**

Lemma 6. *If the linear endomorphism l is selfadjoint, then the quadric*

$$\Phi_0 : \sigma(\mathbf{x}, \mathbf{x}) := \langle \mathbf{x}, \mathbf{x} \rangle - \langle l(\mathbf{x}), l(\mathbf{x}) \rangle = 0 \tag{10}$$

together with its l-image Φ_1 has the Ivory property

$$\delta(l(\mathbf{x}), \mathbf{y}) = \delta(\mathbf{x}, l(\mathbf{y})) \quad \textit{for all} \quad \mathbf{x}, \mathbf{y} \in \Phi \ \textit{with} \ \langle \mathbf{x}, \mathbf{x} \rangle, \langle \mathbf{y}, \mathbf{y} \rangle \neq 0.$$

The restriction of l to any linear subspace contained in Φ_0 is isometric in the sense of δ.

Proof: We have to show that

$$\frac{\langle \mathbf{x}, l(\mathbf{y}) \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle l(\mathbf{y}), l(\mathbf{y}) \rangle}} = \frac{\langle l(\mathbf{x}), \mathbf{y} \rangle}{\sqrt{\langle l(\mathbf{x}), l(\mathbf{x}) \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}$$

The denominators of these fractions are equal due to (10), whereas the numerators are equal because l is selfadjoint.

As to the second statement, the distance δ is preserved for $\mathbf{x}, \mathbf{y} \in \Phi_0$ if and only if $\langle \mathbf{x}, \mathbf{y} \rangle = \langle l(\mathbf{x}), l(\mathbf{y}) \rangle$. This equation characterizes conjugacy with respect to Φ_0, and it follows that the restriction of l to any subspace contained in Φ_0 is isometric. □

Lemma 7. *Assume that l is selfadjoint and that the quadric Φ_0 given by* (10) *is regular. Then Φ_0 and $\Phi_1 = l(\Phi_0)$ are confocal (We apply Def. 3 if necessary).*

Proof: We rewrite the equation of Φ_0:

$$\sigma_0(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle - \langle l(\mathbf{x}), l(\mathbf{y}) \rangle = \langle \mathbf{x}, (\mathrm{id} - l^2)(\mathbf{y}) \rangle.$$

Its dual is represented by

$$\widehat{\sigma}_0(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, (\mathrm{id} - l^2)^{-1}(\mathbf{w}) \rangle.$$

The dual l-image of Φ_0 is, according to Def. 3, defined by

$$\widehat{\sigma}_1(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, l(\mathrm{id} - l^2)^{-1} l(\mathbf{w}) \rangle.$$

We show that

$$\widehat{\sigma}_0 - \widehat{\sigma}_1 = \langle \ , \ \rangle \tag{11}$$

by using $l(\mathrm{id} - l^2) = (\mathrm{id} - l^2) l$ and thus verifying that $(\mathrm{id} - l^2)^{-1} - l(\mathrm{id} - l^2)^{-1} l = \mathrm{id}$. □

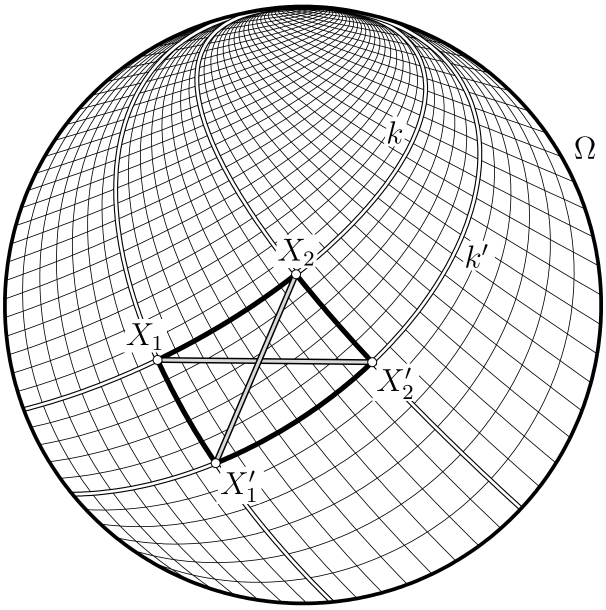

FIGURE 2. Ivory's Theorem in the hyperbolic plane $\mathbb{H}^2$ — case (iv) of Th. 1 which leads to conics without center and axis.

Lemma 8. *If l is selfadjoint, then in most cases the quadric Φ_0 as defined by (10) is of the same type as $\Phi_1 = l(\Phi_0)$ provided both are regular. Different types are only possible when the normal form of l contains a block matrix $R_2(0,b)$ or $R_{2k}(0,b,1)$.*

Proof: The quadric Φ_i is the zero set of $\sigma_i(\mathbf{x},\mathbf{x}) = \langle \mathbf{x}, g_i(\mathbf{x})\rangle$, with selfadjoint g_i $(i = 0,1)$. The proof of Lemma 7 shows that $\widehat{\sigma}_0 - \widehat{\sigma}_1 = \langle\ ,\ \rangle$, i.e., $g_0^{-1} - \mathrm{id} = g_1^{-1}$. We ask if

$$g_\lambda^{-1} = (1-\lambda)g_0^{-1} + \lambda g_1^{-1} = g_0^{-1} - \lambda \mathrm{id} \quad (0 \le \lambda \le 1)$$

can be singular — if it is not for any $\lambda \in [0,1]$, then according to Def. 5, Φ_0 and Φ_1 are of the same type (if it is, then the notation g_λ^{-1} of course does not make sense).

By construction, we have $g_0^{-1} = (\mathrm{id} - l^2)^{-1}$. We use coordinates according to Th. 1. Because of the block matrix structure of (4) it is sufficient to assume that the coordinate matrix A of l equals either $J_k(t,1)$ or $R_{2k}(a,b,1)$. In the first case, the coordinate matrix of g_λ^{-1} reads

$$(I_k - A^2)^{-1} - \lambda I_k = \begin{bmatrix} (1-t^2)^{-1} - \lambda & & * \\ & \ddots & \\ & & (1-t^2)^{-1} - \lambda \end{bmatrix}.$$

As $(1-t^2)^{-1}$ does not assume values in $[0,1]$, g_λ^{-1} is nonsingular. In the second case $A = R_{2k}(a,b,1)$ with $b \neq 0$ we use the abbreviation $U = (I_2 - R_2(a,b)^2)^{-1}$

and get

$$(I_k - A^2)^{-1} - \lambda I_{2k} = \begin{bmatrix} U - \lambda I_2 & & * \\ & \ddots & \\ & & U - \lambda I_2 \end{bmatrix}, \quad \text{with}$$

$$U = \frac{1}{(a^2+b^2+1)^2 - 4a^2} \begin{bmatrix} 1 - a^2 + b^2 & 2ab \\ -2ab & 1 - a^2 + b^2 \end{bmatrix} = R_2(a', b').$$

The eigenvalues of U are $a' \pm ib'$. They are nonreal if $b' \neq 0$, i.e., $a \neq 0$. So also in this case g_λ^{-1} is nonsingular.

In the case $a = 0$ the number $1/(b^2+1)$ is the only eigenvalue of U, and it is contained in the interval $[0, 1]$. Thus in this case there exists $\lambda \in (0, 1)$ such that g_λ is not defined. □

4. Ivory's Theorem

4.1. **A representation theorem.** In previous lemmas we have enumerated properties of quadrics with special equations. Now we are going to show that this case is actually the general case we need for Ivory's Theorem.

Lemma 9. *Consider two regular confocal quadrics Φ_0, Φ_1 which are of the same type. Then there is a selfadjoint endomorphism l such that $\Phi_1 = l(\Phi_0)$ and the equation of Φ_0 is given by $\langle \mathbf{x}, \mathbf{x} \rangle - \langle l(\mathbf{x}), l(\mathbf{x}) \rangle = 0$.*

Proof: We assume that Φ_i is determined by the selfadjoint endomorphism g_i $(i = 0, 1)$. By confocality, $g_1^{-1} = \lambda g_0^{-1} + \mu \mathrm{id}$. Without changing the quadrics we multiply g_0 and g_1 with real numbers such that

$$g_0^{-1} - g_1^{-1} = \mathrm{id}. \tag{12}$$

We first show that there exists l such that $g_0 = \mathrm{id} - l^2$, i.e., there exists a square root of $\mathrm{id} - g_0$. We assume coordinate matrices A and H of g_0 and $\langle\ ,\ \rangle$ as given by (4) and (5), respectively. By the block matrix structure of these matrices, it is sufficient to assume that $A = J_k(t, 1)$ or that $A = R_{2k}(a, b, 1)$. In the first case, g_1 has the coordinate matrix

$$(A^{-1} - I_k)^{-1} = \begin{bmatrix} (t^{-1}-1)^{-1} & & * \\ & \ddots & \\ & & (t^{-1}-1)^{-1} \end{bmatrix}$$

By our assumption on the type, t and $(t^{-1} - 1)^{-1}$ have the same sign. A positive sign implies that $t \in (0, 1)$ and a negative one implies that $t < 0$. In both cases $1 - t > 0$, so $\mathrm{id} - g_0$ has positive eigenvalues.

In the case that $A = R_{2k}(a, b, 1)$ the matrix $I_{2k} - A$ has the form $R_{2k}(1 - a, -b, -1)$ with $b \neq 0$, and $\mathrm{id} - g_0$ has no real eigenvalues. Thus in all cases we are able to apply Cor. 2, and there is a selfadjoint endomorphism l with $l^2 = \mathrm{id} - g_0$. It remains to show that indeed $l(\Phi_0) = \Phi_1$. We have just shown that Φ_0 has an equation of the form required by Lemma 7, and by (11), $l(\Phi_0)$ is defined by an endomorphism $\bar{g}_1$ with the property that

$$g_0^{-1} - \bar{g}_1^{-1} = \mathrm{id}.$$

Our assumption (12) now shows $\bar{g}_1 = g_1$ and the proof is complete. □

Lemma 10. *We use the notation of the proof of Lemma 9. There is $\delta > 0$ such that* $\mathrm{id} - \lambda g_0$ *has a square root which smoothly depends on λ, for $-\delta < \lambda < 1 + \delta$.*

Proof: We again look at the proof of Lemma 9. In the case that $A = J_k(t, 1)$, we have $t \in (0, 1)$ or $t < 0$, which implies that there is δ such that $1 - \lambda t > 0$ for $\lambda \in (-\delta, 1 + \delta)$. It follows that $\mathrm{id} - \lambda g_0$ has a square root which according to Cor. 3 smoothly depends on λ.

In the case that $A = R_{2k}(a, b, 1)$, the endomorphism $\mathrm{id} - \lambda g_0$ has the coordinate matrix $R_{2k}(1 - \lambda a, -\lambda b, -\lambda)$. As there is no λ such that $1 - \lambda a = \lambda b = 0$, we may apply Cor. 3 and conclude that there exists a square root of $\mathrm{id} - \lambda g_0$ which smoothly depends on λ. □

Remark. In a neighbourhood of $\lambda = 0$ we can also apply the implicit function theorem to deduce that there is a selfajoint square root of $\mathrm{id} - \lambda g_0$ which smoothly depends on λ.

4.2. **Orthogonal trajectories of confocal families.** We want to generalize the Euclidean theorem that *corresponding* points X_i, X_i' of confocal conics k, k' are located on another confocal conic which intersects the two original ones orthogonally (see Fig. 1).

Lemma 11. *Suppose that Φ_0, Φ_1, g_0, g_1, l are as in Lemma 9 and its proof. Then there is a smooth family l_λ of endomorphisms with* $l_0 = \mathrm{id}$ *and $l_1 = l$, such that the quadric $\Phi_\lambda = l_\lambda(\Phi_0)$ is defined by the endomorphism g_λ with*

$$g_\lambda^{-1} = g_0^{-1} - \lambda \mathrm{id}. \tag{13}$$

All quadrics Φ_λ are confocal with Φ_0. They orthogonally intersect the path $l_\lambda(\mathbf{x})\mathbb{R}$ of a point $\mathbf{x}\mathbb{R} \in \Phi_0$.

Proof: We have $g_0 = \mathrm{id} - l^2$ and $g_1^{-1} = g_0^{-1} - \mathrm{id}$. We consider λg_0 instead of g_0 and define l_λ by

$$\lambda g_0 = \mathrm{id} - l_\lambda^2.$$

By Lemma 10, l_λ exists and depends smoothly on λ. We observe that l_λ and g_0 commute, which for $\lambda = 0$ is obvious and otherwise follows from $l_\lambda g_0 = l_\lambda \lambda^{-1}(\mathrm{id} - l_\lambda^2) = \lambda^{-1}(\mathrm{id} - l_\lambda^2) l_\lambda = g_0 l_\lambda$. Now we compute the endomorphism g_λ^{-1} which defines the dual of $l_\lambda(\Phi_0)$ according to Def. 3:

$$g_\lambda^{-1} = l_\lambda g_0^{-1} l_\lambda = l_\lambda^2 g_0^{-1} = (\mathrm{id} - \lambda g_0) g_0^{-1} = g_0^{-1} - \lambda \mathrm{id}.$$

So we have shown (13). Confocality of Φ_0 and $l_\lambda(\Phi_0)$ follows from Lemma 7 by construction. We differentiate the relation $\lambda g_0 = \mathrm{id} - l_\lambda l_\lambda$ and get

$$g_0 = -(\dot{l}_\lambda l_\lambda + l_\lambda \dot{l}_\lambda) = -2\dot{l}_\lambda l_\lambda \implies \dot{l}_\lambda = -\frac{1}{2} g_0 l_\lambda^{-1} = -\frac{1}{2} g_\lambda l_\lambda. \tag{14}$$

We have used $g_\lambda = g_0 l_\lambda^{-2}$ and the fact that l_λ and $\dot{l}_\lambda$ commute, which follows from Th. 1 and Cor. 3.

The tangent hyperplane of Φ_λ in $l_\lambda(\mathbf{x})$ has the gradient vector $g_\lambda l_\lambda(\mathbf{x})$. (14) implies that the tangent point $\dot{l}_\lambda(\mathbf{x})\mathbb{R}$ is conjugate to this tangent hyperplane with respect to the absolute quadric Ω. □

Lemma 12. *We use the notations of Lemma 11 and consider the quadrics Φ_λ, defined by endomorphisms g_λ. If Ψ is confocal with Φ_0, but different from Φ_0, and $\mathbf{x} \in \Phi_0 \cap \Psi$, then also $l_\lambda(\mathbf{x}) \in \Psi$.*

Proof: We have $\mathbf{x} \in \Phi_0 \iff \langle \mathbf{x}, g_0(\mathbf{x})\rangle = 0$, and we assume that $\mathbf{x} \in \Psi \iff \langle \mathbf{x}, g_\mu(\mathbf{x})\rangle = 0$, with g_μ selfadjoint. By definition of confocality, $g_\mu^{-1} = g_0^{-1} - \mu\mathrm{id}$ with $\mu \neq 0$. We are going to show below that

$$\lambda g_0 - \mu l_\lambda g_\mu l_\lambda - (\lambda - \mu) g_\mu = 0. \tag{15}$$

$\mathbf{x} \in \Phi_0 \cap \Psi$ together with (15) imply that $l_\lambda(\mathbf{x}) \in \Psi$, because

$$\mu\langle l_\lambda(\mathbf{x}), g_\mu l_\lambda(\mathbf{x})\rangle = \mu\langle \mathbf{x}, l_\lambda g_\mu l_\lambda(\mathbf{x})\rangle = \lambda\langle \mathbf{x}, g_0(\mathbf{x})\rangle - (\lambda-\mu)\langle \mathbf{x}, g_\mu(\mathbf{x})\rangle = 0.$$

We know (see the proof of Lemma 11) that l_λ and g_0 commute, therefore so do l_λ and g_μ^{-1}. We verify (15) by multiplying its left hand side by g_μ^{-1} from the right:

$$\begin{array}{rllll} & \lambda g_0(g_0^{-1} - \mu\mathrm{id}) & -\mu l_\lambda g_\mu l_\lambda g_\mu^{-1} & -(\lambda-\mu)\mathrm{id} & \\ = & \lambda\mathrm{id} - \lambda\mu g_0 & -\mu l_\lambda g_\mu g_\mu^{-1} l_\lambda & -(\lambda-\mu)\mathrm{id} & \\ = & \lambda\mathrm{id} - \lambda\mu g_0 & -\mu(\mathrm{id} - \lambda g_0) & -(\lambda-\mu)\mathrm{id} & = 0. \end{array}$$

□

Remark. There are up to n confocal quadrics meeting in a non-absolute point of $\mathbb{P}^n$ (more are not possible because of their mutual orthogonality).

4.3. **Ivory's Theorem.**

Theorem 2. *(Generalization of Ivory's Theorem) Assume that in a projective space $\mathbb{P}^n$ with metric* (3) *two regular quadrics Φ_0 and Φ_1 are confocal and of the same type. Then there is a smooth family $\Phi_\lambda = l_\lambda(\Phi_0)$ $(0 \le \lambda \le 1)$ of quadrics confocal with Φ_0 and $\Phi_1 = l_1(\Phi_0)$, such that l_λ is selfadjoint and has the Ivory property:*

$$\delta(\mathbf{x}, l_\lambda(\mathbf{y})) = \delta(l(_\lambda\mathbf{x}), \mathbf{y}) \quad \textit{for all} \quad \mathbf{x}, \mathbf{y} \in \Phi_0. \tag{16}$$

Any further quadric Ψ confocal with Φ_0 which contains a point $\mathbf{x} \in \Phi_0$ contains the entire path $l_\lambda(\mathbf{x})$, which intersects all quadrics Φ_λ orthogonally.

Proof: By Lemma 9, there exists l such that Φ_0 is defined by $\mathrm{id} - l^2$ and $\Phi_1 = l(\Phi_0)$. Lemma 11 shows the existence of Φ_λ and l_λ. By Lemma 6, l_λ has the

Ivory property (16). At last, Lemma 12 shows the statement about the quadric Ψ, if it exists. □

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