Geometric Contributions to 3-Axis Milling of Sculptured Surfaces

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Abstract: When we are trying to shape a surface X by 3-axis milling, we encounter a list of problems: First we have to decide if locally the milling tool Σ is able to move along the surface such that its envelope during the motion is the given surface. This is a question involving the curvatures of X and Σ. Second, we want to avoid that while milling in one part of X, Σ intersects another, already finished, part of the surface. This is a problem which involves global shape properties of the surface and can be successfully attacked by considering the general offset surface of X with respect to Σ. Third, in practice a cutting-tool is not able to perform a 2-dimensional motion along a surface. It has to trace out a finite number of piecewise smooth paths such that the resulting surface does not differ from X too much. This question again involves, in the limit case of very small error tolerance, only the curvatures of X and Σ. If we allow larger scallop heights, the path finding also requires the study of local and global properties.

Keywords: 3-axis milling, collision avoidance, general offsets, milling path, global millability

Recently we have studied the problem of locally and globally collision-free milling of sculptured surfaces [6, 14]. It turned out that if some conditions on the curvature of the surfaces involved are fulfilled, we can show that locally, and in certain cases also globally, no unwanted collisions of the cutting-tool with the surface occur. The present paper is summing up our previous results and also deals with the problem of determining tool paths such that the actually shaped surface is within some error tolerance from the given surface X.

If we are given a surface X, we have to do the following:

1. Test whether or not a given cutter is able to mill the given surface locally.
2. Test whether or not a given cutter is able to mill the given surface globally.
3. Select an optimal cutter Σ from a given set of available cutting tools.
4. Find curves $c_1, \ldots, c_r$ such that the cutter Σ, while moving along these curves, shapes a surface $\overline{X}$ which lies between X and its outer parallel surface at distance ε.

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The next sections contain a description of phenomena which occur in this context. We closely follow [6].

Local Properties of Smooth Surfaces

We are going to describe the background and the mathematical foundations of the local millability test when restricted to smooth surfaces, which consist of C^2 patches with C^1 join. Non-smooth surfaces will be considered in the next section.

We denote the surface by X and the cutter by Σ. X is the boundary of a solid, and we speak of the solid as of the interior of X and will call the ambient space the exterior of X. The cutting-tool is a convex body of rotational symmetry. The rotation of the cutter around its axis, however important for the mechanical engineering aspect of the problem, can be completely neglected from the geometric point of view. The actual cutter generates a surface of revolution by its rotation. It is this surface which we consider in this paper. We restrict ourselves to the case of convex cutters, that is, the line segment which joins any two points of Σ is completely contained in Σ. We additionally assume *strict convexity* which means that no line segments are part of the boundary. An actual cutter which contains cylindrical or planar parts is easily approximated by a strictly convex cutter. Because all our problems (collision, error tolerance) can be formulated in terms of *distance* only, and do not need derivatives, this is justified.

While milling the surface X, the surface Σ undergoes a translational motion such that the resulting envelope is just the given surface X. This translation is described by $\Sigma \mapsto \Sigma + g$, where g denotes the vector of the translation. For all $p \in X$ there is exactly one point q in Σ such that the oriented normal vector in p (pointing to the outside) equals the negative oriented normal vector (pointing to the inside) of q. The translation vector is given by $g(p) = q - p$. The new position $\Sigma + g(p)$ is denoted by $\Sigma(p)$.

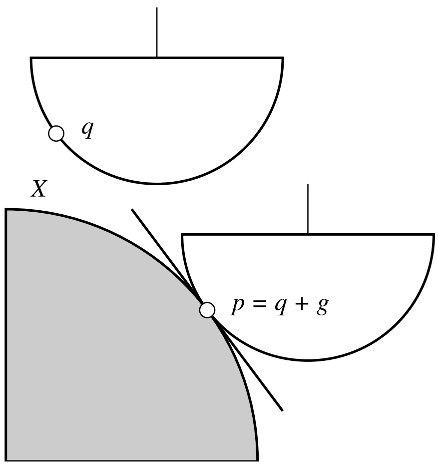

Figure 1: Cutting-tool Σ touching the surface X.

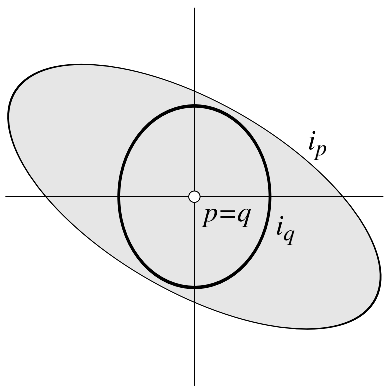

Figure 2: Indicatrix i_q contained in the interior of i_p.

To examine the local behavior in a neighborhood of the touching point, we choose a reference plane such that locally both surfaces are graphs of real-valued functions f and s, respectively. Of course this does not mean that the surfaces X and Σ must from the beginning be given as graphs of real-valued functions. If the tangent plane is not orthogonal to the base plane, it is always possible to re-parametrize both X and Σ locally such that they become graph surfaces.

Because the graphs of f and s touch each other, we have for all vectors v equality of first directional derivatives:

$$s_{,v} = f_{,v}. \tag{1}$$

We say that X is *locally millable* by Σ at p, if there is a neighborhood of p such that in this neighborhood the translates of Σ which touch X there locally do not interfere with the interior of X. It is easy to see [12, 14] that we have local millability, if the difference of the Hesse matrices

$$H_s - H_f = \begin{pmatrix} s_{xx} - f_{xx} & s_{xy} - f_{xy} \\ s_{xy} - f_{xy} & s_{yy} - f_{yy} \end{pmatrix} \tag{2}$$

is positive definite. In Equ. 2 the double subscripts denote second partial derivatives.

There is an equivalent condition in terms of the oriented Euclidean curvature indicatrices of X and Σ, which are preferable from the theoretical point of view. This is because they do not depend on a local parametrization as a graph surface. For the definition of the indicatrix, see any textbook of differential geometry, for example [4]. It may be seen as the limit of the suitably scaled intersection curve of a surface with a plane $\bar{\tau}$ parallel and close to a tangent plane τ. In our case we use *oriented* indicatrices: We translate τ in direction of the positively oriented normal vector of X and the negative normal vector of Σ.

We define the *interior* of the indicatrix i_p as the star-shaped (with respect to the origin) domain, whose boundary is i_p. It may be the whole plane.

The proof of the following is an exercise in differential geometry [4]:

Proposition: A surface is locally millable if and only if for all corresponding points $p \in X$ and $q = q(p) \in \Sigma$ the indicatrix i_q is contained in the interior of the indicatrix i_p (see Fig. 2).

Global Properties of Smooth Surfaces

We repeat a list of cases given in [6, 14]. In order to formulate *global* millability conditions we make use of the following

Definition: The surface Γ which is traced out by an arbitrary fixed point of Σ during the motion of Σ is called *general offset surface* [1, 2, 7, 6, 13, 14] of X with respect to Σ.

Using elementary methods of geometric topology (degree of maps, homotopy, covering maps), it is possible to show [14] that the following is true:

Proposition: Let X be a smooth surface consisting of C^2 patches.

1. If X is locally millable, the parametrization of Γ is regular and orientation preserving.
2. If the surface Γ has no self-intersections, then X is globally millable by Σ.
3. Let X be such that it can be re-parametrized as the graph surface of a compactly supported smooth function defined in the entire plane. If X is locally millable, then also globally.
4. Let X be such that it can be re-parametrized as the graph surface of a smooth function defined in the entire plane. If Σ possesses steeper tangent planes than X (This is always the case if Σ has an equator circle), then the local millability of X implies the global millability.
5. Let X be such that it can be seen as the graph surface over a piecewise smoothly bounded planar domain D. If the 'top view' of Σ is a closed symmetric convex domain S and the general outer parallel curve $D + S$ is free of self-intersections, then the local millability of X implies the global millability.
6. If X is strictly star-shaped with respect to an interior point, then local millability implies global millability.

In most applications we have one of the cases listed in the proposition, most frequently perhaps case 4. In most cases S is a disk and the condition is easily verified. (see Fig. 3).

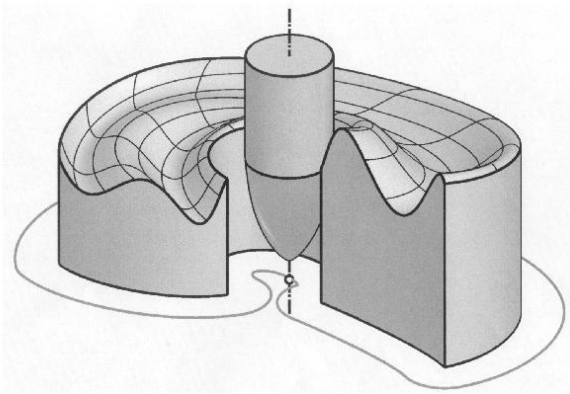

Figure 3: Milling a surface with boundary.

Non-smooth Surfaces

If the surface is not smooth, but continuous and piecewise C^2, we are still able to give local and global millability conditions. The first possibility to overcome the non-smoothness is to consider instead of X an outer parallel surface at distance ε. This surface is G^1 and the previous proposition applies. Because collision tests involve distance information only, the limit $\varepsilon \to 0$ gives the exact result.

Another method, which is better for computational purposes, is the following: An edge e where X is not smooth, is either a ravine or a ridge. Ravines can never be milled exactly by smooth cutters, so we leave them aside. For a ridge this is possible. It is easy to derive the following criterion for local millability in the neighborhood of a point p which is situated on an edge c: Write X and Σ as graph surfaces. The 'top view' of the edge c is denoted by c'. It has a parametrization $c' : t \mapsto (c_1(t), c_2(t))$. The edge itself has a parametrization of the form $c : t \mapsto (c_1(t), c_2(t), c_3(t))$. Then Σ locally does not interfere with the edge if the inequality

$$(\dot{c}')^T H_s(q)\dot{c}' + \text{grad}(s(q)) \cdot \ddot{c}' > \ddot{c}_3 \tag{3}$$

holds for all points q which have the property that Σ can be translated such that q is translated to p and $\Sigma(p)$ touches X at p.

It can be shown [14] that here also the local millability implies the global one, in all cases listed in the proposition above.

Test for Millability of a Smooth Surface

In this section we are going to describe how to test whether a given cutting-tool is able to mill the surface X. The cutting-tool contains circles in parallel ('horizontal') planes, which will be called *parallel circles*. The tangent planes τ_p of the points q of such a circle c enclose the same angle

$$\psi = \angle(\tau_q, a) \tag{4}$$

with the axis a of the tool. The points of the surface X which will during the manufacturing process be in contact with the points of the circle c, are precisely the points on the *isophotic line* l_ψ which belongs to the angle ψ and the direction of a [10]. The curve l_ψ is defined as the set of points p of X whose tangent planes τ_p enclose the angle ψ with the axis a.

As discussed above, the condition for collision-free manufacturing of X can be expressed in terms of the Euclidean curvature indicatrices i_p and i_q of corresponding points $p \in \Sigma$ and $q = q(p) \in X$: The surface is locally (and hence, globally) millable if and only if the indicatrix i_q is contained in the interior of i_p.

The indicatrices of all points q of a parallel circle c are the same (up to rotation of Σ), so we can speak of *the* indicatrix $i_q = i_c$ of the points of c. The connection between the various tangent planes of c is given by the cutter's rotation around the axis a. The connection between the various tangent planes along the curve l_ψ is given by the condition that when moving along l_ψ, the horizontal line in the tangent plane stays horizontal.

This now makes it possible to re-formulate the condition that for all points $p \in l_\psi$ the interior i_p must contain i_c: We identify all tangent planes along l_ψ

and intersect the interiors of the indicatrices i_p. This gives the region I_ψ. Local millability is now equivalent to $i_c \subset I_\psi$.

This is implemented easily if we exploit the fact that all I_ψ are the intersection of domains which are star-shaped with respect to the same point.

Test for Millability of a Non-Smooth Surface

First we have to say something about the isophotic lines in the presence of edges. For this purpose it is best to think of l_ψ as a set of surface elements (p, τ_p), where a surface element is a pair (point, tangent plane). This makes it possible to simplify the notation in cases where a point has more than one tangent plane.

If p is situated on an edge e of X, there is a wedge $\mathcal{T}_p$ of admissible tangent planes. There are up to two planes in $\mathcal{T}_p$ which enclose the angle ψ with the axis a. Thus the isophotic line l_ψ can contain up to two surface elements (p, τ_p). For each of them it is possible to define a substitute indicatrix of curvature $i_{p,\tau}$ (see [6]) such that the test for millability now runs in exactly the same way as in the smooth case. For all angles ψ we have to test if i_c is contained in I_ψ, where c is the parallel circle which belongs to the angle ψ, and I_ψ is defined as $I_\psi = \bigcap_{(p,\tau)\in l_\psi} \operatorname{int} i_{p,\tau}$

Cutter Paths

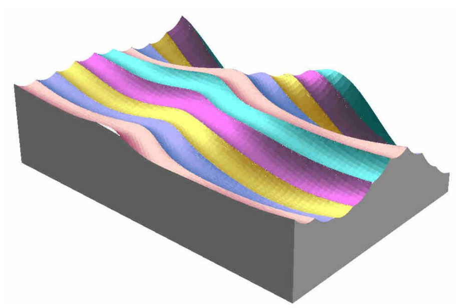

Figure 4: Surface produced by actual cutter (false proportions)

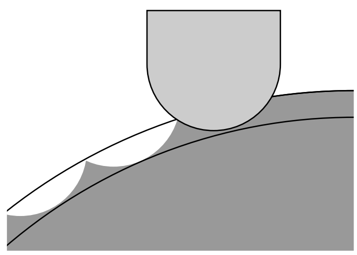

Figure 5: Surface X, cutter Σ, outer parallel surface, and actually shaped surface (dark grey).

Having shown that the milling-tool Σ is able to shape the surface during a two-parameter motion, we have now the problem that in practice Σ cannot trace out the entire surface in finite time, but only a finite number of piecewise smooth curves.

There are many publications concerning these tools paths, see for example [3, 5, 8, 9, 11, 15, 16]. What we want to do, is to show how to find the best tool paths locally, and how to find new or to modify existing global tool path schemes. We do this by specifying curves on the surface, which will be the locus of points where the cutter touches the surface X. The actual position of the cutter is easily calculated from these curves.

We specify a small error tolerance ε and require that Σ is to move along piecewise smooth curves, thereby shaping an approximate surface $\overline{X}$ which has to be contained between X and its outer parallel surface at distance ε. (see Fig. 5).

We will consider the *geometry* of the tool trajectories only, not the parametrization of the tool paths or the velocities of the tool. For example, the difference between 'isoparametric' and 'non-isoparametric' tool paths is not a geometric or shape property, it is a property of the parametrization.

Also, we do not study the possible limitations of actual CAM systems, such as the unability of using curves other than straight lines or memory restrictions. For instance, we do not think of discretizing our curves. These problems have to be solved by those who wish to implement new or use existing software.

In a neighborhood of a point $p \in X$ we can always write both X and $\Sigma(p)$ as graph surfaces:

$$X : z = f(x, y), \quad \Sigma(p) : z = s(x, y). \tag{5}$$

Also the outer parallel surface can be written as a graph surface locally. A good approximation is

$$\overline{X} : z \approx f(x, y) + \varepsilon / \cos \alpha(x, y), \tag{6}$$

where $\alpha(x, y)$ is the angle which the tangent plane at p encloses with the $z = 0$ plane:

$$\cos \alpha(x, y) = 1 / \sqrt{f_x^2 + f_y^2 + 1}. \tag{7}$$

The 'top view' of the intersection curve of $\Sigma(p)$ with the outer parallel surface then is approximated by the top view of the intersection curve of the difference graph (see Fig. 7) $z = s(x, y) - f(x, y) =: d(x, y)$ with the surface $z = \varepsilon / \cos \alpha(x, y)$. For small ε we can use the Taylor expansions of Σ and X to avoid the surface-surface intersection: If we intersect the Taylor expansion of $d(x, y)$ with the plane $z = \varepsilon / \cos \alpha(0, 0)$ we get the region

$$i(p) : (x, y)(H_s - H_f)(x, y)^T \leq \varepsilon / \cos \alpha(0, 0), \tag{8}$$

which is the interior of an ellipse. This ellipse is the 'top view' of a certain region $I(p)$ on the surface X which is a first order approximation to the actual intersection of the cutter and the outer parallel surface at distance ε.

Our aim is to use as few milling paths as possible to cover the entire surface. Locally, this is done as follows: We start at the point p and move along a curve c, thus defining the *approximate machining strip*

$$\bigcup_{p \in c} I(p) \tag{9}$$

on the surface which is formed by the union of all regions $I(p)$ for all $p \in c$. The meaning of the approximate machining strip is the following: For all points inside the strip, the enveloping surface of the cutter moving along the path differs from the given surface X not more than the given value ε, up to errors of second order. This means that inside the strip the surface is already approximated well enough. We now want to determine the curve c such that the machining strip has *maximum width* (see [8]).

It is not so easy to write down the actual width of the machining strip, because the boundary of the sets $I(p)$ may, in principle, touch their envelope, i.e., the boundary of the machining strip, in up to four points. Even this does not happen in practice, it indicates that the equation behind it is of order four. Thus we are satisfied with a first order approximation: We measure the dimensions of the ellipse $I'(p)$ in the tangent plane of p, whose top view coincides with $i(p)$. If the curve c is such that its tangents coincide with the direction of the minor axis of $I'(p)$, then the approximate machining strip will have maximum width (more precisely: our approximation of the width will reach its maximum).

In the parameter plane, the directions of the major and minor axes are given by the eigenvectors of the matrix $G^{-1}(H_s - H_f)$, where

$$G = \begin{pmatrix} 1+f_x^2 & f_x f_y \\ f_x f_y & 1+f_y^2 \end{pmatrix} \tag{10}$$

is the matrix of the first fundamental form of the surface $z = f(x, y)$.

If to each point we add a tangent vector pointing in the direction of the minor axis, then integrating this vector field gives a curve with maximum machining strip width, which obviously is our first choice for the cutter path.

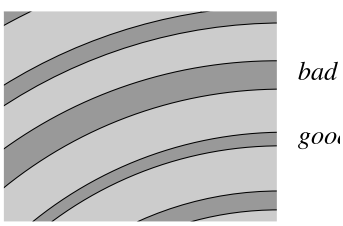

Figure 6: Overlapping machining strips.

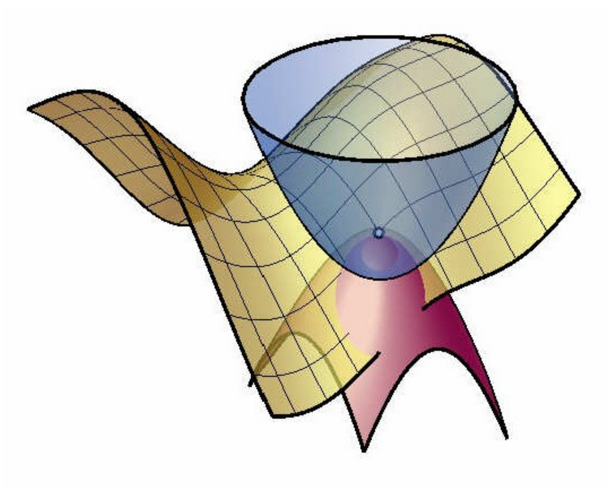

Figure 7: Cutter surface Σ (top), surface X, negative difference surface (bottom)

Having chosen the first path, we choose the neighboring paths under the following side-conditions: The machining strips defined by the paths shall cover the entire surface, although the strips should not overlap each other too much (Fig. 6). Also the direction of the strips should differ not too much from the direction of the optimal path.

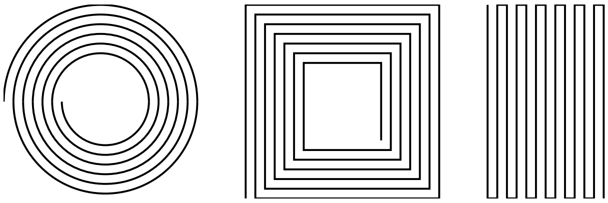

Figure 8: Milling path schemes.

If it is no longer possible to adjoin a cutter path to an already calculated sequence of cutter paths such that the width of the machining strip is satisfying, we start again with a new cutter path found by integrating the vector field of minor axes, and repeat the process. Thus eventually the whole surface will be covered by different regions of 'parallel' cutter paths, one of which is optimal.

If we want to use a prescribed global scheme for the cutter paths, such as shown in Fig. 8, we are still able to make use of our results: In every point we can evaluate the necessary width of the machining strip and thus determine the optimal distance of a tool path to its immediate neighbor (which may be an earlier part of the same tool path), thus optimizing the given scheme. More detailed investigations and the actual design of algorithms are a topic of future research.

Acknowledgements

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