CURVATURES AND TOLERANCES IN THE EUCLIDEAN MOTION GROUP

HANS-PETER SCHRÖCKER AND JOHANNES WALLNER

ABSTRACT. We investigate the action of imprecisely defined affine and Euclidean transformations and compute tolerance zones of points and subspaces. Tolerance zones in the Euclidean motion group are analyzed by means of linearization and bounding the linearization error via the curvatures of that group with respect to an appropriate metric.

1. INTRODUCTION

The topic of this paper is the action of imprecisely defined affine and Euclidean transformations on $\mathbb{R}^d$. More precisely this means that we consider a set of transformations which has a small diameter in a well-defined sense (a *tolerance zone* of transformations) and apply all transformations contained in this set to a geometric object of $\mathbb{R}^d$.

The idea to deal with imprecisely defined data in the way indicated above, i.e., by replacing exactly defined geometric entities by their tolerance zones, has been used in the investigation of error propagation in Computer-Aided Design [1, 2, 3, 4, 5]. It has been called *worst case tolerancing*, because the use of tolerance zones means that instead of finding the result of a computation with given input data, we find all possible results, regardless of probability, for all possible input data.

This aspect of geometric transformations has been studied by [6, 7, 8] where subsets of the complex plane are multiplied. [9] deals with toleranced geometric transformations in the Euclidean plane, which are in turn defined by toleranced points and lines.

2. OVERVIEW

In Section 3 we introduce a Euclidean metric in the space $\mathbb{R}^{d\times d+d}$ of affine mappings acting on $\mathbb{R}^d$. This allows us to define the diameter of a set of affine transformations, and balls of affine transformations. Section 4 shows that the image of a fixed point of $\mathbb{R}^d$ under such a ball of transformations is a ball of points in the ordinary sense. We use this result to compute the image of lines and other subspaces as well.

Section 5 deals with Euclidean transformations. A tolerance zone of a given Euclidean transformation is defined as the set of those Euclidean transformations whose distance to the given one does not exceed a certain radius. An exact description of such

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a set would require to compute the intersection of the Euclidean motion group SE_d, which lies as a surface in $\mathbb{R}^{d\times d+d}$, with a ball in that space. In order to circumvent this problem, we replace SE_d by one of its tangent spaces and compute an upper bound for the linearization error we make in this process. The analysis of the latter requires an investigation of the curvature of curves which cover SE_d.

3. The space of affine mappings

An affine mapping γ of $\mathbb{R}^d$ into itself is given by

$$x \mapsto \gamma(x) = Gx + g, \tag{1}$$

where $G \in \mathbb{R}^{d\times d}$, and $g \in \mathbb{R}^d$. We denote the mapping γ also by the pair (G, g). In this way the set of affine mappings is identified with $\mathbb{R}^D$, $D = d^2 + d$. We embed both $\mathbb{R}^{d\times d}$ and $\mathbb{R}^d$ into $\mathbb{R}^D$ by mapping $G \mapsto (G, 0)$ and $g \mapsto (0, g)$. The action of the affine mapping γ on a point $x \in \mathbb{R}^d$ defines the mapping

$$\pi_x \colon \mathbb{R}^D \to \mathbb{R}^d \subset \mathbb{R}^D, \quad \pi_x(\gamma) = \gamma(x). \tag{2}$$

This is a *parallel projection*, which follows from linearity and the property $\pi_x \circ \pi_x = \pi_x$. In [10, 11, 12], geometric properties of π_x have been used for investigating affine and projective motions.

The Euclidean scalar product in $\mathbb{R}^d$ is denoted by $\langle x, y\rangle$. A positive Borel measure (*mass distribution*) μ can be used to define a scalar product in $\mathbb{R}^D$ by

$$\langle \gamma, \beta\rangle := \int \langle \gamma(x), \beta(x)\rangle \, d\mu(x) \tag{3}$$

(see for example [13, 14, 15]). This scalar product is *left-invariant* in the sense that for any $\alpha \in \mathrm{SE}_d$ and $\beta, \gamma \in \mathbb{R}^D$, we have $\|\beta - \gamma\| = \|\alpha \circ \beta - \alpha \circ \gamma\|$, i.e., multiplication by α from the left yields an affine isometry of $\mathbb{R}^D$.

Associated to the mass distribution μ is the *inertia matrix* $J \in \mathbb{R}^{d\times d}$ whose coefficients are $j_{kl} = \int x_k x_l \, d\mu(x)$. J is symmetric and positive definite, if μ is not concentrated in an affine subspace of dimension $< d$. We always assume that this is the case. An important special case is that μ is a finite sum of unit point masses located at points $p_1, \ldots, p_n$, in which case we have $J = \sum_{i=1}^n p_i {p_i}^T$.

The total mass of μ is denoted by $|\mu| := \int 1 \, d\mu$, and $b := |\mu|^{-1} \int x \, d\mu(x)$ is called the barycenter of μ. Of course, this makes sense only if all integrals above are finite. In the special case mentioned above, $|\mu| = n$ and $b = n^{-1} \sum p_i$. In a coordinate system with b as origin, the scalar product of two affine transformations is computed by the formula

$$\langle (G, g), (H, h)\rangle = \mathrm{tr}(G^T H J) + |\mu|\langle g, h\rangle. \tag{4}$$

If $\mu_1, \ldots, \mu_d$ are the eigenvalues of J and $e_1, \ldots, e_d$ is a corresponding orthonormal basis of eigenvectors, then it is elementary to verify that the scalar product (4), which is characterized by J and $|\mu|$, is generated by $2d$ points of mass $|\mu|/(2d)$ at positions $\pm\lambda_i e_i$, where $\mu_i = \lambda_i^2 |\mu|/d$ (see Figure 1, left).

Multiplying the measure μ by a constant factor will multiply the scalar product defined by μ with the same factor, but does not change the orthogonality relation. We thus normalize μ such that $\|(E, 0)\| = 1$. By (4), this is equivalent to

$$\mathrm{tr}(J) = 1. \tag{5}$$

Most of the time we will use $e_1, \dots, e_d$ as basis of an affine coordinate system whose origin is the barycenter. This results in $J = \operatorname{diag}(\mu_1, \dots, \mu_d)$. Without loss of generality, we assume that $0 < \mu_1 \le \cdots \le \mu_d$.

4. TOLERANCE ZONES FOR AFFINE MOTIONS

We consider a subset $\Gamma \subset \mathbb{R}^D$ and ask for the set of points obtained by applying the transformations of Γ to a point x or a subset X of $\mathbb{R}^d$. We use the notations

$$\Gamma(x) := \{\gamma(x) \mid \gamma \in \Gamma\} \quad \text{and} \quad \Gamma(X) := \{\gamma(x) \mid \gamma \in \Gamma, x \in X\} \tag{6}$$

4.1. **Tolerance zones of points.** According to (2), $\Gamma(x) = \pi_x(\Gamma)$ is an affine image of Γ, so $\Gamma(x)$ is an ellipsoid if Γ is a ball. We can show even more:

Theorem 1. *If $\Gamma \subset \mathbb{R}^D$ is a ball, then so is $\Gamma(x)$ for all $x \in \mathbb{R}^d$. The radii r_x and r of $\Gamma(x)$ and Γ are related via*

$$\varrho(x)^2 := \frac{r_x^2}{r^2} = \frac{1}{|\mu|} + \sum_{i=1}^{d} \frac{x_i^2}{\mu_i}, \quad \varrho(x) \ge 0. \tag{7}$$

Proof. The theorem is proved if we can show that all singular values of the projection π_x are equal to $\varrho(x)$. In order to compute the singular values, we construct an orthonormal basis of $\mathbb{R}^D$. By E_{ij} we denote the $d \times d$ matrix whose entries e_{kl} are zero with exception of $e_{ij} = 1$. We let

$$\varepsilon'_{i0} = (0, e_i) \ (i = 1, \dots, d), \quad \varepsilon'_{ij} = (E_{ij}, 0), \ (1 \le i, j \le d). \tag{8}$$

With (4) we compute $\|\varepsilon'_{i0}\|^2 = |\mu|$ and $\|\varepsilon'_{ij}\|^2 = \mu_j$ for $j \ne 0$. Furthermore, we have $\langle \varepsilon'_{ij}, \varepsilon'_{kl} \rangle = 0$ if $(i,j) \ne (k,l)$, so the affine mappings $\varepsilon_{ij} := \varepsilon'_{ij}/\|\varepsilon'_{ij}\|$ constitute an orthonormal basis of $\mathbb{R}^D$. The coordinate matrix P_x of π_x with respect to the basis $(\varepsilon_{10}, \varepsilon_{11}, \dots, \varepsilon_{1d}, \varepsilon_{20}, \dots, \varepsilon_{dd})$ has the form

$$P_x = \begin{bmatrix} \boxed{c_x} & & \\ & \ddots & \\ & & \boxed{c_x} \end{bmatrix}, \quad \text{where} \quad c_x = \Big(\frac{1}{\sqrt{|\mu|}}, \frac{x_1}{\sqrt{\mu_1}}, \dots, \frac{x_d}{\sqrt{\mu_d}} \Big). \tag{9}$$

The theorem now follows from the obvious fact that all singular values of P_x are equal to $\|c_x\| = \varrho(x)$. □

There are at least three arguments for using balls as tolerance zones for affine mappings: One is the spherical shape of $\Gamma(x)$ if Γ is a ball (Theorem 1). Another is the fact that for $\gamma \in \mathrm{SE}_d$, $\gamma \circ \Gamma$ is again a ball, by left-invariance of the metric. The third argument is computational simplicity. Thus we focus on spherical tolerance domains in the rest of this paper.

The ratio $\varrho(x) = r_x/r$ of the radii of $\Gamma(x)$ and Γ attains its minimum $|\mu|^{-1}$ in the barycenter of μ, and is obviously constant for all points of the ellipsoid with equation $E_t\colon \ \sum_{i=1}^d x_i^2/\mu_i = td/|\mu|, \ (t \ge 0)$. The distribution of $\varrho(x)$ is visualized in Figure 1. The point masses at positions $\pm\lambda_i e_i$, which define the same scalar product as the mass distribution μ, are contained in the ellipsoid E_1.

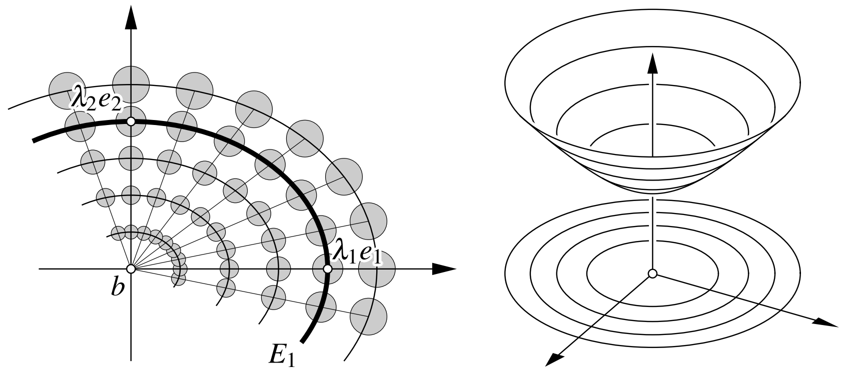

FIGURE 1. The distribution of $\varrho(x)$ and the points $\pm\lambda_i e_i$.

4.2. **Tolerance zones of subspaces.** In this section we describe the set $\Gamma(X)$, if Γ is a ball of affine mappings and X is k-dimensional affine subspace. In the generic case the boundary of $\Gamma(X)$ either will not exist or consist of a quadratic hypersurface (if the radius of Γ is "sufficiently" small). In view of Theorem 1, this surface has rotational symmetry with X as axis space.

We use a coordinate system with origin $a \in X$ and an orthonormal basis $B = (b_1, \ldots, b_d)$ such that the vectors $b_1, \ldots, b_k$ are parallel to X. Coordinates with respect to this system are denoted by $\xi, \eta \in \mathbb{R}^d$. Points in X have coordinates of the form $\xi = (\xi_1, \ldots, \xi_k, 0, \ldots, 0)$. We now assume that we have chosen a and B such that the function $\varrho|_X$ is represented in normal form:

$$\bar{\varrho}(\xi)^2 := \varrho(a + B\xi)^2 = \varrho_0^2 + \sum\nolimits_{i=1}^{k} \xi_i^2/\gamma_i^2, \tag{10}$$

where $1/\gamma_i^2$ are the eigenvalues of the matrix $B^T \cdot \operatorname{diag}(1/\mu_1, \ldots, 1/\mu_d) \cdot B$. By construction, the values γ_i obey the inequality

$$\mu_1 \le \gamma_1^2, \ldots, \gamma_k^2 \le \mu_d. \tag{11}$$

Lemma 2. *If the radius of the ball Γ equals r, then the envelope of the balls $\Gamma(x)$ for $x \in X$ is the quadratic surface*

$$\sum\nolimits_{\substack{i\le k\\ \gamma_i^2 \ne r^2}} \frac{\eta_i^2}{1-(\gamma_i/r)^2} + \sum\nolimits_{i>k} \eta_i^2 = (r\varrho_0)^2, \quad \eta_i = 0 \text{ if } \gamma_i^2 = r^2. \tag{12}$$

Proof. We compute the envelope of the spheres $\Sigma(x)$ with center $x \in X$ and radius $r\varrho(x)$. By (10), $\Sigma(x)$ has the equation

$$\eta \in \Sigma(x) \iff \sum\nolimits_{i=1}^{d} (\eta_i - \xi_i)^2 - r^2\Big(\varrho_0^2 + \sum\nolimits_{i=1}^{k} \xi_i^2/\gamma_i^2\Big) = 0. \tag{13}$$

Differentiation of (13) with respect to $\xi_1, \ldots, \xi_k$ yields $\eta_i = 0$ if $\gamma_i^2 = r^2$, and $\eta_i = \xi_i(1 - r^2/\gamma_i^2)$ if $\gamma_i^2 \ne r^2$. We plug this into (13) and obtain an algebraic equation of the envelope. □

The graph surface of the function $r(x) = r\varrho(x)$ in $X \times \mathbb{R}$ is one sheet of a hyperboloid (Figure 1, right, Figure 2). If $\dim X = 1$, it is one branch of a hyperbola. The union of balls $\Gamma(x)$ for $x \in X$ is the interior of the surface (12) if the angle enclosed by the line X and the asymptotes of this hyperbola is less than $45°$ (see Figure 2, right),

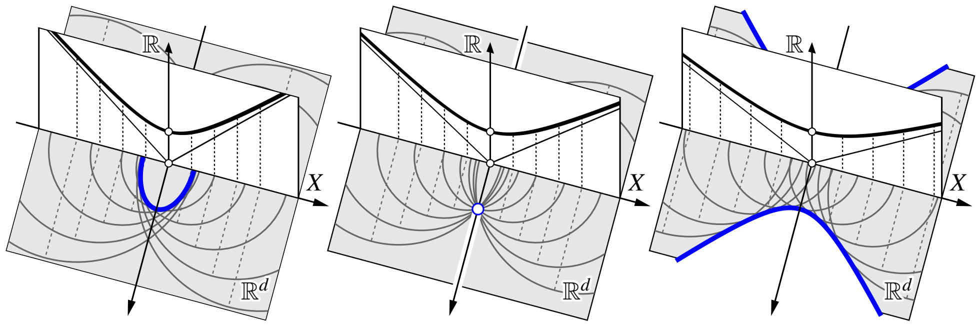

FIGURE 2. Different types of envelopes: ellipse ($\gamma_1^2 < r^2$), pair of points ($\gamma_1^2 = r^2$) and hyperbola ($\gamma_1^2 > r^2$).

and equals entire $\mathbb{R}^d$ if this angle exceeds $45°$ (see Figure 2, left). In the case the angle equals $45°$, the closure of $\Gamma(X)$ equals $\mathbb{R}^d$ (Figure 2, center). The term 'interior' refers to that component of the surface's complement which contains X.

For general X, $\Gamma(X)$ is the interior of the surface (12) if and only if for all lines $Y \subseteq X$ the above mentioned angle does not exceed $45°$. This leads to the following

Theorem 3. *Assume an affine subspace X and a corresponding coordinate system as above such that the function ϱ has the coordinate representation* (10)*. For a ball Γ of affine mappings of radius r, the set $\Gamma(X)$ equals the interior of the surface* (12) *if and only if $r^2 < \gamma_i^2$ for $i = 1, \ldots, k$. In the remaining cases, the closure of $\Gamma(X)$ equals $\mathbb{R}^d$.*

Proof. In view of the discussion preceding the theorem, we have to investigate hyperbolas which occur as graph of the function $r\varrho(x)$, when x ranges in a 1-dimensional subspace of X. By (10), this angle does not exceed $45°$, if and only if $r < \gamma_i$. □

Theorem 3 has the following implications:

– If $\Gamma(X)$ is the interior of the surface (12), then this is the case for all subspaces parallel to X, because they have the same γ_i's.

– If the radius r of $\Gamma(X)$ is less than $1/\mu_d$, the condition of Theorem 3 is fulfilled for all subspaces because of the inequalities (11).

In $\mathbb{R}^3$, the set $\Gamma(X)$ is the interior of a hyperboloid of revolution if X is a line. If X is a plane, the lack of rotations with X as axis means that $\Gamma(X)$ is bounded by a hyperboloid of two sheets which is symmetric with respect to X. Figure 3, left, shows an example of $\Gamma(X)$, where X consists of the edges of a cube.

Remark. The so-called cyclographic mapping (see e.g. [16, p. 366ff]) assigns an oriented sphere in $\mathbb{R}^d$ with center $(x_1, \ldots, x_d)$ and signed radius x_0 to the point $(x_0, \ldots, x_d) \in \mathbb{R}^{d+1}$. Obviously this mapping and the geometries related to it are the right setting to discuss further geometric properties of envelopes of spheres. In this paper we do not pursue this viewpoint.

5. TOLERANCE ZONES FOR EUCLIDEAN TRANSFORMATIONS

We turn our attention to imprecisely defined Euclidean transformations. They are more intricate than affine ones, because the exact intersection of SE_d with a ball Γ of

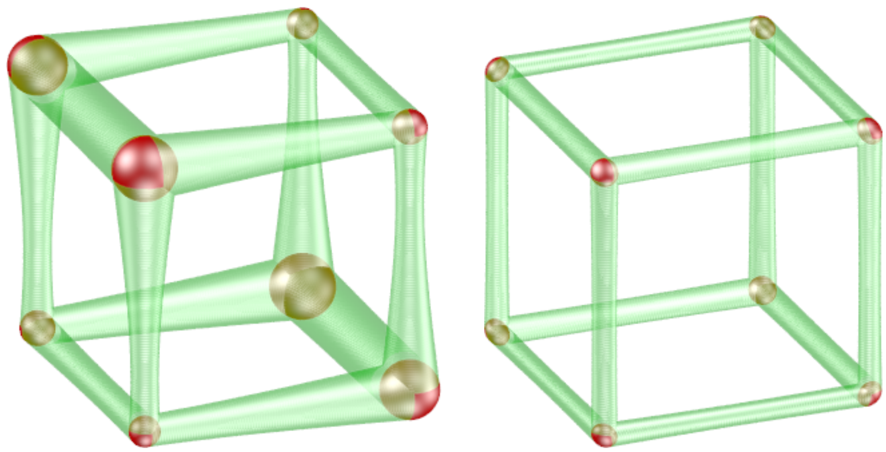

FIGURE 3. Tolerance zones of a cube with respect to a ball $\Gamma \subset \mathbb{R}^{12}$ (left) and linearization of Euclidean tolerance zones (right).

radius r centered in $\gamma \in \mathrm{SE}_d$ is not so easy to compute. We pursue a different approach and replace SE_d by its tangent space in the point γ. This means that we do not consider the intersection $\Gamma \cap \mathrm{SE}_d$, but the intersection S of Γ with the tangent space $T_\gamma \mathrm{SE}_d$ of SE_d in γ (cf. Figure 3, right and Figure 4). The linearization error we make in this process will be estimated by means of the curvatures of SE_d.

Figure 4 shows the set S and the orbits $S(x)$ and $\Gamma(x)$. Because of $S(x) = \pi_x(S)$, $S(x)$ is an ellipsoid, whose dimensions can be computed from the singular values of $\pi_x|_{T_\gamma \mathrm{SE}_d}$. It turns out that the position of one of its axes is special and the corresponding semi-axis length is independent of x:

Theorem 4. *Let $\Gamma \subset \mathbb{R}^D$ be a ball with center γ and radius r, and denote by $S(x) = \pi_x(\Gamma \cap T_\gamma \mathrm{SE}_d)$ the corresponding linearized tolerance zone of a point $x \in \mathbb{R}^d$. Then $S(x)$ is an ellipsoid one of whose axes contains the γ-image of the barycenter b of the measure μ. The corresponding semi-axis length is $r|\mu|^{-1/2}$.*

Proof. Without loss of generality we can assume $\gamma = E$, i.e., $\gamma(b) = b = 0$. As $T_E \mathrm{SO}_d$ consists of the skew-symmetric matrices, $\pi_x(T_E \mathrm{SO}_d)$ is a hyperplane H_x incident with x and at the same time orthogonal to x.

We decompose $T_E \mathrm{SE}_d \cap \Gamma$ into a family of balls $\Gamma_{\gamma'}$ with centers $\gamma' \in T_E \mathrm{SO}_d$, contained in the subspace $\gamma' \times \mathbb{R}^d$, and having radius $\sqrt{r^2 - \mathrm{dist}(\gamma, \gamma')^2}$. As $\Gamma \cap T_\gamma \mathrm{SE}_d$ is a sphere, its π_x-image $S(x)$ is an ellipsoid with center $\gamma(x)$. The restriction of π_x to $\gamma' \times \mathbb{R}^d$ is a similarity transformation with factor $|\mu|^{-1/2}$. Thus, each $\pi_x(\Gamma_{\gamma'})$ is symmetric with respect to H_x and so is $S(x)$. Consequently, one of the axis of $S(x)$ contains b and the corresponding semi-axis length is $r|\mu|^{-1/2}$. □

5.1. **The distance of curves from their tangents.** We present an auxiliary result from differential geometry concerning the distance of a curve C with bounded curvature from any of its tangents. In this paper, curvatures of curves are understood in the Euclidean sense, with respect to a previously defined scalar product. All curvatures are nonnegative, i.e., we do not assign a sign to curvatures of curves in $\mathbb{R}^2$.

Lemma 5. *Assume that* $I \subseteq (-r\pi/2, r\pi/2)$ *is an interval with* $0 \in I$ *and* $c\colon I \to \mathbb{R}^d$ *is an arc length parametrization of a* C^2 *curve* C *with curvature* $\varkappa \leq 1/r$*. Then there is a parametrization* $\bar{c}(u) = ue_1 + f(u) \cdot n(u)$ *of* C *such that* $e_1 = \dot{c}(0)$*,* $n(u)$ *is a unit vector orthogonal to* e_1*,* $|u(t)| \geq r\sin(t/r)$ *and* $|f(u)| \leq r - \sqrt{r^2 - u^2}$*.*

Proof. It is no loss of generality to assume $r = 1$. We observe $\|\dot{c}\| = 1$ and $\ddot{c}(t) = \varkappa(t)c_2(t)$, where $\varkappa$ is the curvature and c_2 is a unit vector orthogonal to $\dot{c}$. Let $\varphi(t) = \sphericalangle(e_1, \dot{c}(t))$, $\psi(t) = \sphericalangle(e_1, c_2(t))$ such that $\varphi, \psi \geq 0$. Then

$$-\dot{\varphi}\sin\varphi = \frac{d}{dt}\cos\varphi = \frac{d}{dt}\langle e_1, \dot{c}\rangle = \varkappa\langle e_1, c_2\rangle = \varkappa\cos\psi. \tag{14}$$

We assume that the curve is defined in an interval such that $\varphi \leq \pi/2$. Later we will see that this inequality is true for all $t \in I$. Because of $\sphericalangle(\dot{c}, c_2) = \pi/2$ and the spherical triangle inequality we have $\pi/2 - \varphi \leq \psi \leq \pi/2 + \varphi$. Together with $|\varkappa| \leq 1$, $0 \leq \varphi \leq \pi/2$ and Equation (14) this implies

$$|\dot{\varphi}\sin\varphi| = \varkappa|\cos\psi| \leq |\cos(\pi/2 - \varphi)| = |\sin\varphi|. \tag{15}$$

If $\sin\varphi \neq 0$, it follows that $|\dot{\varphi}| \leq 1$. By continuity, this is true for all t in the closure of the set $\{t \mid \varphi(t) \neq 0\}$. If however $\varphi(t) = 0$ in an entire interval, then we have $\dot{\varphi} = 0$ anyway. So in all cases, $\dot{\varphi} \in [-1, 1]$ and consequently

$$|\varphi(t)| \leq |t|. \tag{16}$$

In particular, the assumption that $\varphi \leq \pi/2$ in entire I is now justified.

We use (16) for estimating the distance $\Delta(t)$ of the point $c(t)$ from the tangent at $t = 0$. Because of $\dot{\Delta} = \sin\varphi$, we have

$$\Delta(t) = \textstyle\int_0^t \sin\varphi(\tau)\, d\tau \leq \int_0^t \sin\tau\, d\tau = 1 - \cos t. \tag{17}$$

We write $c(t) = c(0) + u(t)e_1 + \Delta(t)\bar{n}(t)$, where $\bar{n}(t)$ is a unit vector orthogonal to e_1. As $\dot{u}(t) = \cos\varphi(t)$, the function $u(t)$ is strictly increasing in I. It follows that we may use $u(t)$ as a parameter transform and C may be parametrized by $\bar{c}(u) = c(0) + ue_1 + f(u) \cdot n(u)$ as stated in the theorem. The required lower bound for $u(t)$ is computed by

$$u(t) = \textstyle\int_0^t \cos\varphi(\tau)d\tau \geq \int_0^t \cos\tau d\tau = \sin t. \tag{18}$$

The inverse of the function $u(t)$ consequently obeys the inequality $t(u) \leq \arcsin(u)$ for $u \in [-1, 1]$. By construction, $f(u(t)) = \Delta(t)$, so

$$|f(u)| = \Delta(t(u)) \leq 1 - \cos(t(u)) \leq 1 - \cos\arcsin u = 1 - \sqrt{1 - u^2}. \tag{19}$$

The proof is complete. □

Lemma 6. *Consider a surface* $M \subseteq \mathbb{R}^d$*, a point* $p \in M$*, and assume that there exists a family* $\mathcal{C}$ *of curves such that*

– *every curve of* $\mathcal{C}$ *contains* p*;*
– *the curvature of the curves of* $\mathcal{C}$ *does not exceed* $1/r$*; and*
– *every point* $q \in M$ *with* $\|q - p\| \leq \varrho$ *lies on a curve* $C \in \mathcal{C}$ *such that the arc length between* p *and* q *is less than* $r\pi/2$*.*

If $q \in M$ *with* $\|p - q\| \leq \varrho$*, then* $\operatorname{dist}(q, T_pM) \leq r - \sqrt{r^2 - \varrho^2}$*.*

Proof. We consider a curve C which connects p and q, and we assume that its parametrization is $\bar{c}(u)$ as described by Lemma 5. As $u \le \|\bar{c}(u) - p\|$, we have $u \le \varrho$, and consequently the distance of $c(u)$ from C's tangent at p does not exceed $r - \sqrt{r^2 - \varrho^2}$. □

We will apply Lemma 6 to SO_d and use the result for an upper bound of the distance of SO_d from its tangent spaces. For that purpose we will construct two families of curves on SO_d: The one-parameter subgroups with their cosets, and the geodesics. The similar problem for SE_d is easily reduced to the case of SO_d.

5.2. **The curvature of one-parameter subgroups in SO_d.** We introduce the positive square root of the inertia matrix J and denote it by K: If $J = P^{-1}\operatorname{diag}(\mu_1, \dots, \mu_d)P$ with $P \in \mathrm{SO}_d$, then $K = P^{-1}\operatorname{diag}(\sqrt{\mu_1}, \dots, \sqrt{\mu_d})P$.

Recall that the Frobenius norm $\|A\|_F$ of a matrix A has a definition similar to the norm $\|A\|$ defined by (4). A relation between these two norms is given by

$$\|A\|_F = \operatorname{tr}(A^TA), \quad \|A\| = \operatorname{tr}(A^TAJ) = \operatorname{tr}((AK)^TAK) = \|AK\|_F. \tag{20}$$

Recall further that for all A, B we have $\|AB\|_F \le \|A\|_F \cdot \|B\|_F$. We let

$$\Theta := \|K^{-1}\|_F = \big(\textstyle\sum_{i=1}^d \mu_i^{-1}\big)^{1/2}. \tag{21}$$

Lemma 7. *For A, B we have*

$$\|A\|_F \le \Theta \cdot \|A\| \quad \textit{and} \quad \|AB\| \le \Theta \cdot \|A\| \cdot \|B\|. \tag{22}$$

Proof. By (20), $\|A\|_F = \|AKK^{-1}\|_F \le \Theta \cdot \|AK\|_F = \Theta \cdot \|A\|$. Analogously, $\|AB\| = \|AKK^{-1}BK\|_F \le \|A\| \cdot \Theta \cdot \|B\|$. □

Theorem 8. *One-parameter cosets of SO_d have curvature $\le \Theta$.*

Proof. A one-parameter coset of SO_d can be parameterized by $c(t) = Me^{tX}$, where $M \in \mathrm{SO}_d$ and X is skew-symmetric. By left-invariance of the metric, it is sufficient to consider $M = E$. Derivatives of c are then given by $\dot{c} = cX$ and $\ddot{c} = cX^2$. Left-invariance of the metric implies that $\|\dot{c}(t)\| = \|X\|$ and $\|\ddot{c}(t)\| = \|X^2\|$. An appropriate scaling of X makes c an arc length parametrization, i.e., $\|\dot{c}\| = \|X\| = 1$, so that the curvature $\varkappa = \|\ddot{c}\|$. By Lemma 7, $\varkappa = \|X^2\| \le \Theta \cdot \|X\|^2 = \Theta$. □

5.3. **The curvature of the geodesics in SO_d and SE_d.** The curvature of the geodesics in a surface M is bounded by the maximal normal curvature of M. The normal curvature associated with a unit tangent vector T and a unit normal vector N in a point $x \in M$ can be computed by the formula

$$\varkappa_n(T, N) = \langle \mathrm{II}_x(T, T), N\rangle, \tag{23}$$

where $\mathrm{II}_x\colon T_xM \times T_xM \to \perp_x M$ is the second fundamental form and $\perp_x M$ denotes the orthogonal space of M at x. If $V^\perp$ denotes the orthogonal projection of a vector V onto $\perp_x M$, and $x(s,t)$ is an M-valued function with $x(0,0) = x$, $\frac{\partial x}{\partial s}(0,0) = V$, $\frac{\partial x}{\partial t}(0,0) = W$, then $\big(\frac{\partial^2 x}{\partial s \partial t}(0,0)\big)^\perp = \mathrm{II}_x(V, W)$ (cf. [17]). Equation (23) implies that normal curvatures do not exceed any upper bound for $\|\mathrm{II}_x(T, T)\|$.

Theorem 9. *The normal curvatures $\varkappa_n$ of both SO_d and SE_d, and therefore the curvature of their geodesics, are bounded by Θ.*

Proof. Since SE_d is a cylinder with normal section SO_d, it is sufficient to consider SO_d, and by left-invariance of the metric, it is sufficient to compute normal curvatures in the point E. So we consider $\mathrm{II}_E\colon \mathrm{so}_d \times \mathrm{so}_d \to \mathrm{so}_d^{\perp}$, where so_d and $\mathrm{so}_d^{\perp}$ denote the tangent and normal spaces in the point E of the surface SO_d in $\mathbb{R}^{d\times d}$, respectively. SO_d can be parameterized by

$$\mathrm{SO}_d\colon S(t,X) = \exp(tX), \quad \text{where } X^T = -X \text{ and } \|X\| = 1. \tag{24}$$

We compute $\mathrm{II}_E(X,X) = \big(\frac{\partial^2}{\partial t \partial s} e^{tX+sX}\big|_{t=s=0}\big)^{\perp} = (X^2)^{\perp}$ and hence, $\|\mathrm{II}_E(X,X)\| \leq \Theta \cdot \|X\|^2 = \Theta$. □

Theorems 8 and 9 show that Θ is an upper bound for the curvature of both the one-parameter subgroups and the geodesics. As to the geodesics, we give an alternative upper bound which is better if the dimension d is low.

Theorem 10. *The normal curvatures* $\varkappa_n$ *of* SO_d *and* SE_d *are bounded by*

$$\varkappa_n^2 \leq \frac{d(d+1)}{2} \max\Big\{\frac{\mu_j}{(\mu_i+\mu_j)^2}, \frac{\mu_i\mu_k}{(\mu_i+\mu_j)(\mu_j+\mu_k)(\mu_k+\mu_i)}\Big\}, \tag{25}$$

where $1 \leq i < j \leq d$ *and* $1 \leq k \leq d$*.*

The proof of Theorem 10 is spread over two subsections. It will be finished by the end of Section 5.3.2. Readers who are not interested in the details of the computations may continue with Section 5.4. As in the proof of Theorem 9, we consider SO_d only.

5.3.1. *Orthonormal bases of* so_d *and* $\mathrm{so}_d^{\perp}$. We compute a coordinate representation of the second fundamental form II_E with respect to orthonormal bases of the tangent space so_d (which consists of the skew-symmetric matrices) and the normal space $\mathrm{so}_d^{\perp}$ of SO_d in E. We consider SO_d as a surface in $\mathbb{R}^{d\times d}$, which is equipped with the scalar product $\langle A, B\rangle = \mathrm{tr}(A^T B J)$. Without loss of generality we assume that $J = \mathrm{diag}(\mu_1, \dots, \mu_d)$.

We use the matrices E_{ij} as in the proof of Theorem 1 and define $T'_{ij} := E_{ij} - E_{ji}$, $N'_{i0} := E_{ii}$ and $N'_{jk} := \mu_j E_{jk} + \mu_k E_{kj}$ for $k > 0$. Further, we let

$$T_{jk} := \frac{1}{\sqrt{\mu_j + \mu_k}} T'_{jk} \quad (1 \leq j < k \leq d). \tag{26}$$

It is elementary to verify that the matrices T_{jk} constitute an orthonormal basis of so_d, and so do the matrices

$$N_{i0} := N_{i0}/\|N_{i0}\|,\ N_{jk} := N_{jk}/\|N_{jk}\| \quad (1 \leq i \leq d,\ 1 \leq j < k \leq d) \tag{27}$$

for $\mathrm{so}_d^{\perp}$. The norms of N'_{ij} are given by

$$\|N'_{i0}\|^2 = \mu_i, \quad \|N'_{ij}\|^2 = \mu_i\mu_j(\mu_i + \mu_j). \tag{28}$$

5.3.2. *The second fundamental form.* We use the parametrization

$$X(t_{10}, \dots, t_{d-1,d}) = \exp\big(\textstyle\sum_{j<k} t_{jk} T_{jk}\big), \tag{29}$$

of SO_d in order to compute a coordinate representation of II_E with respect to the bases (26) and (27) of tangent space and normal space, respectively:

$$\mathrm{II}_E(T_{ij}, T_{kl}) = \Big(\frac{\partial^2 X}{\partial t_{ij} t_{kl}}\Big|_{t_{ij},t_{kl}=0}\Big)^{\perp} = \frac{1}{2}(T_{ij}T_{kl} + T_{kl}T_{ij})^{\perp}, \tag{30}$$

where $T_{ij}T_{kl} = \frac{\delta_{jk}E_{il}-\delta_{jl}E_{ik}-\delta_{ik}E_{jl}+\delta_{il}E_{jk}}{\sqrt{\mu_i+\mu_j}\cdot\sqrt{\mu_k+\mu_l}}$. By splitting E_{ij} into a tangential and a normal component according to

$$E_{ii} = \sqrt{\mu_i}N_{i0}, \quad E_{ij} = \frac{\mu_j}{\sqrt{\mu_i+\mu_j}}T_{ij} + \frac{\sqrt{\mu_i\mu_j}}{\sqrt{\mu_i+\mu_j}}N_{ij} \ (i \neq j) \tag{31}$$

one verifies that $(T_{ij}T_{kl})^\perp = (T_{kl}T_{ij})^\perp$. With $\widetilde{\mu}_{ik} = \sqrt{\mu_i\mu_k}/\sqrt{\mu_i+\mu_k}$, it follows that

$$\mathrm{II}_E(T_{ij}, T_{kl}) = (T_{ij}T_{kl})^\perp = ((\mu_i+\mu_j)(\mu_k+\mu_l))^{-1/2} \cdot \Big(-\delta_{ik}\delta_{jl}(\sqrt{\mu_i}N_{i0} + \sqrt{\mu_j}N_{j0})$$
$$+\delta_{jk}(1-\delta_{il})\widetilde{\mu}_{il}N_{il} - \delta_{jl}(1-\delta_{ik})\widetilde{\mu}_{ik}N_{ik} - \delta_{ik}(1-\delta_{jl})\widetilde{\mu}_{jl}N_{jl} + \delta_{il}(1-\delta_{jk})\widetilde{\mu}_{jk}N_{jk}\Big).$$

If τ_k ($k = 1, \dots, d(d-1)/2$) are coordinates of $T \in \mathrm{so}_d$ with respect to the basis T_{ij} and ν_k ($k = 1, \dots, d(d+1)/2$) are coordinates of $N \in \mathrm{so}_d^\perp$ with respect to the basis N_{ij}, then a coordinate representation of II_E has the form $\nu_i = \sum_{j,k}\xi_{ijk}\tau_j\tau_k$. Note that at most one of the five terms in (5.3.2) is different from zero, which implies that the entries of the symmetric matrix $M_i := (\xi_{ikl})_{k,l=1}^{d'}$, $d' = d(d-1)/2$, are zero except for two elements — either off the diagonal and equal, or on the diagonal. Such a matrix has the property that for any unit vector x the inequality $|x^T \cdot M_i \cdot x| \leq \max_{jk}|\xi_{ijk}|$ holds. It follows immediately that any unit vector $T \in \mathrm{so}_d$ has the property

$$\|\mathrm{II}_E(T,T)\| \leq \sqrt{\dim(\mathrm{so}_d^\perp)}\max_i \nu_i \leq \sqrt{\dim(\mathrm{so}_d^\perp)}\max_i(\max_{jk}|\xi_{ijk}|). \tag{32}$$

This is the value given in (25), so the proof of Theorem 9 is complete. □

With Theorems 9 and 10, we have found two upper bounds for the curvature of geodesics in SO_d, one of which equals the upper bound for the curvature of one-parameter subgroups. This shows that for geodesics we may have smaller curvatures and consequently a smaller linearization error. The reason why we discuss subgroups and their left cosets at all, is that they have a simpler parametrization and consequently more of their properties are known. In Theorem 14 we use subgroups to show a result concerning geodesics.

Remark. In case of $d = 2$, it is not necessary to estimate an upper bound of the normal curvature: Since SO_2 is a unit circle, all curvatures (also that of the essentially unique one-parameter subgroup) equal 1.

Remark. For $d = 3$ it is possible to show that the upper bound given by (25) is always smaller than the upper bound Θ of Theorem 9. It is easy to see that $\mu_1 \leq \mu_2 \leq \mu_3$ implies

$$\Theta^2 = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} \geq \frac{6\mu_1\mu_2}{(\mu_1+\mu_2)(\mu_2+\mu_3)(\mu_3+\mu_1)}. \tag{33}$$

For large dimensions the factor $d(d+1)/2$ in (25) dominates and Theorem 9 yields a better bound. This can already happen in case of $d = 4$. If all values μ_i are equal, this happens for $d > 15$.

5.4. **Connecting transformations by geodesics or subgroups.** We are going to verify the third property needed in Lemma 6 for the geodesics and the one-parameter cosets in SO_d, i.e., either family of curves covers SO_d. A more precise statement is given by Lemma 13 and Theorem 14 below.

A one-parameter subgroup C of SO_d can be parameterized by $c(t) = \exp(tX)$ with a skew-symmetric matrix X. After a suitable change of coordinates in $\mathbb{R}^d$, X is of the shape

$$X = \mathrm{diag}(Y_1, \ldots, Y_r, 0, \ldots, 0), \quad \text{where} \quad Y_i = \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}, \tag{34}$$

which leads to

$$c(t) = \mathrm{diag}(R_1, \ldots, R_r, 1, \ldots, 1), \quad R_i = \begin{bmatrix} \cos t\omega_i & -\sin t\omega_i \\ \sin t\omega_i & \cos t\omega_i \end{bmatrix}. \tag{35}$$

It is no restriction to assume that $\omega_1 \geq \cdots \geq \omega_r > 0$. We use the left-invariance of the metric to compute

(36)

$$\|\dot{c}(t)\|^2 = \mathrm{tr}((Xe^{tX})^T(Xe^{tX})J) = \mathrm{tr}(-X^2 J) = \sum\nolimits_{i=1}^{r} \omega_i^2 (j_{2i-1,2i-1} + j_{2i,2i}).$$

By multiplying X with a suitable factor we can achieve $\|\dot{c}\| = 1$, which means that $c(t)$ then is an arc length parametrization.

Lemma 11. *If* (35) *is an arc length parametrization, then* $\omega_1 \geq 1$.

Proof. $\|\dot{c}\| = 1$ in (36) implies that $\omega_1^2 \,\mathrm{tr}(J) \geq 1$. Recall $\mathrm{tr}(J) = 1$. □

When using a coordinate system such that X has the normal form (34), we can no longer without loss of generality assume that J is a diagonal matrix. We therefore need the following inequality which compares the diagonal elements of J with its eigenvalues $\mu_1, \ldots, \mu_d$:

$$\mu_1 \leq j_{k,k} \leq \mu_d. \tag{37}$$

Lemma 12. *If* $\|\dot{c}(t)\| = 1$ *in* (35) *and* $|t| \leq \pi/\omega_1$, *then* $\|E - c(t)\|^2 \geq 4\mu_1(1 - \cos t)$.

Proof. It is sufficient to consider the case $t \geq 0$, and it is easy to see that

$$\|E - c(t)\|^2 = 2\sum\nolimits_{i=1}^{r} (1 - \cos t\omega_i)(j_{2i-1,2i-1} + j_{2i,2i}). \tag{38}$$

This expression is greater or equal $2(1 - \cos t\omega_1)(j_{1,1} + j_{2,2})$. The statement of the lemma now follows immediately from Lemma 11, Equation (37) and the fact that the function $1 - \cos t\omega_1$ is nondecreasing in the interval $[0, \pi/\omega_1]$. □

Lemma 13. *The group* SO_d *is covered by segments of one-parameter subgroups* $c(t)$ *with* $\|\dot{c}\| = 1$, *where the parameter value* t *ranges in* $[0, \pi/\omega_1]$.

Proof. With respect to a suitable coordinate system in $\mathbb{R}^d$ every orientation-preserving isometry has a matrix representation of the shape

(39)

$$M = \mathrm{diag}(P_1, \ldots, P_r, 1, \ldots, 1), \quad P_i = \begin{bmatrix} \cos\varphi_i & -\sin\varphi_i \\ \sin\varphi_i & \cos\varphi_i \end{bmatrix}, \quad \varphi_i \in (0, \pi].$$

We use this same coordinate system and define a one-parameter subgroup $c(t)$ of the form (35) with $\omega_i = \lambda\varphi_i$, choose λ such that $\|\dot{c}(t)\| = 1$, and let $t_0 = 1/\lambda$. Obviously, $M = c(t_0)$ and $t_0 = \varphi_1/\omega_1 \leq \pi/\omega_1$. □

We collect the results of this section in

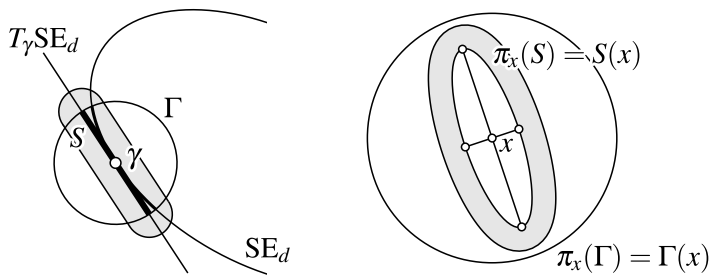

FIGURE 4. Linear approximation of $\Gamma \cap \mathrm{SE}_d$ (left) and corresponding orbit of a point (right).

Theorem 14. *If A is a ball with center $M \in \mathrm{SO}_d$ and radius ϱ such that $\varrho \le 4\mu_1(1 - \cos t)$, then segments of one-parameter cosets of length $\le t$ emanating in M cover $A \cap \mathrm{SO}_d$. The same is true for geodesic segments.*

Proof. By left-invariance of the metric, it is sufficient to show this for the case $M = E$, i.e., for the case of one-parameter subgroups. By Lemma 13 segments of length $\ge \pi/\omega_1$ cover entire SO_d, so the statement is true in this case. In the case $t \le \pi/\omega_1$, Lemmas 12 and 13 show that the part of SO_d whose distance from E does not exceed ϱ, is indeed covered by subgroup segments as described above.

The lemma remains true if we replace subgroups by geodesics: By the Hopf-Rinow theorem [17, chapter 7], for all $M \in \mathrm{SO}_d$ there is a shortest curve connecting M with E, and this shortest curve is a geodesic. Thus its length does not exceed the length of the shortest one-parameter subgroup. □

5.5. **Toleranced Euclidean transformations.** We are now ready to study tolerance zones of points subject to toleranced Euclidean transformations. Assume that $\gamma \in \mathrm{SE}_d$ and consider a ball Γ with center γ. For a point $x \in \mathbb{R}^d$ we want to find a subset of $\mathbb{R}^d$ which contains $\Gamma(x)$. As $\Gamma(x) = \pi_x(\Gamma \cap \mathrm{SE}_d)$ is not so easy to compute, we replace SE_d by its tangent space and consider the intersection $S := \Gamma \cap T_\gamma \mathrm{SE}_d$ instead of $\Gamma \cap \mathrm{SE}_d$. The ellipsoid $\pi_x(S)$ is a linear approximation of $\Gamma(x)$, as illustrated by Figure 4. The next lemma is used to determine how much $\pi_x(S)$ differs from $\Gamma(x)$.

Lemma 15. *Assume that k is an upper bound for the normal curvatures of SO_d (obtained either from Theorem 9 or Theorem 10—whichever is smaller), $r = 1/k$, $t_0 \le r\pi/2$ and $\varrho \le 4\mu_1(1 - \cos t_0)$. Consider a ball Γ in $\mathbb{R}^D$ with center $\gamma \in \mathrm{SE}_d$ and radius ϱ. The distance of $\mathrm{SE}_d \cap \Gamma$ from the tangent space $T_\gamma \mathrm{SE}_d$ does not exceed $r - \sqrt{r^2 - \varrho^2}$.*

Proof. We consider $\beta \in \Gamma$, with $\beta = (B, b)$. Let $(C, c) = \gamma$. Orthogonal projection of $\mathbb{R}^D$ onto $\mathbb{R}^{d \times d}$ maps the cylinder SE_d to its basis SO_d, β to B, γ to C, $T_\gamma \mathrm{SE}_d$ onto $T_C \mathrm{SO}_d$, and the ball Γ to a ball Γ' of the same size and center C. It follows that

$$\operatorname{dist}(C, B) \le \operatorname{dist}(\gamma, \beta) \quad \text{and} \quad \operatorname{dist}(\beta, T_\gamma \mathrm{SE}_d) = \operatorname{dist}(B, T_C \mathrm{SO}_d), \tag{40}$$

and it is sufficient to give an upper bound for the distance of $\mathrm{SO}_d \cap \Gamma'$ from $T_C \mathrm{SO}_d$. By Theorem 14, there is a geodesic segment of length $\le t_0$ which connects C and B. By Lemma 6, the distance of B from $T_C \mathrm{SO}_d$ is bounded by $r - \sqrt{r^2 - \varrho^2}$. □

In the following theorem, we use the symbols B_D and B_d for the unit balls in $\mathbb{R}^D$ and $\mathbb{R}^d$, respectively.

Theorem 16. *Assume that $\Gamma \subset \mathbb{R}^D$ is a ball of radius ϱ with center $\gamma \in \mathrm{SE}_d$, and that r, ϱ fulfill the conditions of Lemma 15. Then*

$$\mathrm{SE}_d \cap \Gamma \ \subseteq \ \big(T_\gamma \mathrm{SE}_d \cap \Gamma\big) + (r - \sqrt{r^2 - \varrho^2}) B_D, \tag{41}$$

i.e., the exact intersection is contained in an offset body of the linearized intersection. The action of such motions on a point of $x \in \mathbb{R}^d$ fulfills

$$\pi_x(\mathrm{SE}_d \cap \Gamma) \ \subseteq \ \pi_x\big(T_\gamma \mathrm{SE}_d \cap \Gamma\big) + \varrho(x)(r - \sqrt{r^2 - \varrho^2}) B_d, \tag{42}$$

with $\rho(x)$ from (7). Thus the exact orbit of x is contained in an offset of the linearized orbit.

Proof. This follows directly from Lemma 15 and Theorem 1. □

We should mention that the obvious inclusion $\pi_x(\mathrm{SE}_d \cap \Gamma) \subseteq \pi_x(\Gamma)$ might be used to sharpen (42). Theorem 16 is illustrated in Figure 4, right.

5.6. **A numerical example.** We present a numerical example for toleranced Euclidean transformations. We assume a mass distribution in the Euclidean three-space such that $\mu_1 = 0.4$, $\mu_2 = 0.35$, $\mu_3 = 0.25$ and $|\mu| = 1$. An approximate upper bound of 3.06 for the curvature of the one-parameter cosets in SO_d is furnished by Theorem 8, and an upper bound of approximate value $k \approx 2.04$ for the geodesics is given by Theorem 10. In order to illustrate how Theorem 16 works, we choose $r = 1/k$, $t_0 = 0.5 < r\pi/2 \approx 0.77$ and $\varrho = 4\mu_1(1 - \cos t_0) \approx 0.20$ as radius of a rather big tolerance ball $\Gamma \subset \mathbb{R}^6$. According to Lemma 15, the linearization error is bounded by $r - \sqrt{r^2 - \varrho^2} \approx 0.04$.

6. Conclusion

We have developed a concept for dealing with toleranced affine and Euclidean transformations. The tolerance zone of a transformation is defined in the kinematic space $\mathbb{R}^D$ of affine mappings, by means of a certain Euclidean metric for affine mappings. Tolerance zones of Euclidean transformations are shown to be contained in thin offset bodies of linearized toleranced zones. The method has the following properties:

- The metric in the space of affine mappings $\mathbb{R}^D$ is defined by a mass distribution in $\mathbb{R}^d$. This allows adapting the metric to specific applications. In particular, it is possible to define a domain of interest where tolerancing results are particularly useful, i.e., tolerance zones of points, where toleranced transformations act on, are small.
- The set $\gamma(x)$, where x is a point of $\mathbb{R}^d$ and the affine transformation γ ranges in a ball of $\mathbb{R}^D$ is again a sphere. This fact is very convenient for applications.
- The method of linearization and giving upper bounds for the linearization error works comparatively well for the surface SE_d, owing to the global bounds on its curvatures. This is in contrast to the linearization errors for implicit constraints [5].
- The majority of applications involves basic geometric shapes like subspaces. The action of toleranced affine and Euclidean transformation on such geometric entities is easily described in view of the simple shape of tolerance zones of points.

There remains work to be done, such as a more detailed investigation of tolerance zones of simple geometric shapes, for both the affine and Euclidean cases. The dependence of imprecisely defined transformations on input data such as occurs in CAD applications is another important topic.

From the theoretical point of view, estimating the deviation of a curved surface from its tangent space via a family of curves of bounded curvature which covers this surface can be expected to be useful in other applications as well. The authors do not know whether the inequalities given by Lemmas 5 and 6 are new, because they are rather elementary.

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