Configuration Space of Surface-Surface contact

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Abstract: We consider a space X of constant curvature (Euclidean, elliptic, and hyperbolic space, and the sphere). The set of isometries of X transforming a surface such that it touches a second surface is called the configuration space of surface-surface contact. We investigate when this subset of $\mathrm{Isom}(X)$ is an immersed submanifold, and characterize the absence of singularities by the principal curvatures of the surfaces involved.

1 Introduction

The set of positions of one object such that it touches another, has been studied from various different points of view. From the side of mechanical engineering comes the problem of milling a sculptured surface by a cutter, which is, geometrically, a convex surface of revolution. It moves such that it always touches a given surface, and it should touch only in one point in order to avoid undercuttings. Geometric aspects of the restricted problem of translational motions, which include self-intersections of generalized offset surfaces and special cases of the obstacle problem, have been studied in [14].

Integral-geometric kinematic formulas for submanifolds in spaces of constant curvature have been studied by E. Teufel [12, 13], where he introduces a measure on the aforementioned set of positions.

The aim of this paper is to determine when the set of positions of one submanifold of a space of constant curvature, such that it touches another submanifold, has the structure of an immersed submanifold of the isometry group of the space.

2 Definitions

Let X equal a simply connected space of constant curvature, i.e., $X = E^m$ $(n \geq 1)$, or $X = S^m$ $(n \geq 2)$, or $X = H^m$ $(n \geq 2)$, with E^m being the Euclidean space, S^m the sphere, and H^m the hyperbolic space of m dimensions. Then the group $G = \mathrm{Isom}_+(X)$ of orientation-preserving isometries acts with stabilizer $G_x \cong \mathrm{SO}_m$.

Let $\iota_1 : M_1 \to X$ and $\iota_2 : M_2 \to X$ be two $(m-1)$-dimensional immersed oriented differentiable submanifolds. The tangent space at $p \in M_i$ is denoted by T_pM_i, the oriented unit normal vector of $d\iota_i(T_{p_i}M_i)$ by $n_i(p)$.

We say that M_1 touches M_2 at (p_1, p_2) if $\iota_1(p_1) = \iota_2(p_2)$ and $n_1(p_1) = n_2(p_2)$, and that M_1 touches M_2 if there are points (p_1, p_2) such that the M_i touch at (p_1, p_2). If $g \in G$, by abuse of notation, we denote the g-transform of M_i by $g(M_i) = g \circ \iota_i$. Then the set

$$C = \{g \in G \mid g(M_1) \text{ touches } M_2\}. \tag{1}$$

is called the *configuration space of surface-surface contact* of M_1, M_2.

We consider the orthonormal frame bundles $O(M_i)$ $(i = 1, 2)$, the scalar product in T_pM_i being given by $\langle v, w\rangle = \langle d\iota_i(v), d\iota_i(w)\rangle$. Obviously every pair of frames (p_i, F_i) $(i = 1, 2)$ defines a unique $g \in C$: $g(M_2)$ touches M_1 in (p_1, p_2) such that $d\iota_2(F_2)$ is mapped onto $d\iota_1(F_1)$. For all orthonormal $(m-1)$-frames F and $h \in \mathrm{SO}_{m-1}$ the image $h(F)$ is defined. Thus SO_{m-1} acts on $O(M_1) \times O(M_2)$ by $h((p_1, F_1), (p_2, F_2)) = ((p_1, h(F_1)), (p_2, h(F_2)))$ and obviously for all $h \in \mathrm{SO}_{m-1}$ the pairs $h(p_i, F_i)$ determine the same $g \in G$ as the pairs (p_i, F_i). Therefore there is a differentiable mapping

$$\phi : (O(M_1) \times O(M_2))/\mathrm{SO}_{m-1} \to C, \tag{2}$$

which is onto. The factor space in (2) has dimension $2 \dim O(M_i) - \dim \mathrm{SO}_{m-1} = 2(\dim M_i + \dim \mathrm{SO}_{m-1}) - \dim \mathrm{SO}_{m-1} = (m+2)(m-1)/2$.

3 Euclidean space

We want to know when ϕ is an immersion, i.e., C is a $(m+2)(m-1)/2$-dimensional immersed submanifold of G. Let us first consider Euclidean space E^m. Our aim is to find linearly independent paths in C which pass through a given $g \in C$, which means paths $g_i : I \to C$ with $g_i(0) = g$ and $\{\frac{d}{dt}\big|_{t=0} g_i\}$ linearly independent.

In Euclidean space $X = E^m$, the group G equals the semidirect product $G = T^m \rtimes_{\mathrm{id}} \mathrm{SO}_m$ of SO_m with the group $T^m = \mathbb{R}^m$ of translations. It is an $m(m+1)/2$-dimensional Lie group, with T^m as its m-dimensional normal Lie subgroup. If ϕ of (2) is actually an immersion then C has codimension 1. We consider a neighborhood of a $g \in C$ which belongs to a pair (p_i, F_i). The translation in direction of $n(p_1)$ never is in this neighborhood, so T^m is transverse to ϕ, which we denote by $T^m \pitchfork \phi$. This shows that locally $\phi^{-1}(T^m) = \phi^{-1}(T^m \cap C)$ is a $(n-1)$-dimensional submanifold. Having this in mind, we assume without loss of generality that $g = \mathrm{id}$ and M_i touch each other in (p_1, p_2). We are going to describe a number of linearly independent paths which span $T_{\mathrm{id}}C$. We let $n = n_1(p_1) = n_2(p_2)$ and identify the tangent spaces $\iota_i(T_{p_i}) = T_{n(p_i)}S^{m-1} = V$ by parallel translations.

Paths in C which leave the common normal fixed: It is easy to find linearly independent rotations which leave the common surface normal fixed: The stabilizer (denoted by H) in G of the tangent space V and the common surface normal is canonically isomorphic to the special orthogonal group $\mathrm{SO}(V)$ of the tangent space V. H is entirely contained in C, so there are $\dim(H) = (m-1)(m-2)/2$ linearly independent paths $h_i(t)$ in $H \subset C$ with $h_i(0) = \mathrm{id}$.

Other rotational paths in C: Next we want to find $m-1$ rotations $g_i(t)$ in C with $g_i(0) = \mathrm{id}$, which are linearly independent, and none of which leaves the common surface normal invariant. Then the union of all paths h_i and g_i will still be linearly independent.

Consider the spherical mappings $\sigma_i : M_i \to S^{m-1} \subset E^m$, $p \mapsto n_i(p)$, and assume that $\sigma_1 \pitchfork \sigma_2$ at p_1, p_2. Then there are submanifolds $M_i' \subset M_i$ containing p_i such that

$$\bigoplus_{i=1,2} d\sigma_i(T_{p_i} M_i') = V \tag{3}$$

and $\sigma_i | M_i'$ locally is regular.

Construct an orthonormal basis $B = (e_1, \ldots, e_m)$ of $\mathbb{R}^m$ with $e_m = n$ and consider the $m-1$ linearly independent one-parameter rotation subgroups $\rho_i(t)$ $(i = 1, \ldots, m-1)$ of SO_m which leave the linear span of $B \setminus e_i$ fixed. None of them leaves the direction of common surface normal n invariant.

The submanifolds M_i' still are transverse, if one of them is transformed by a $g \in G$ sufficiently close to the identity. A transverse intersection of submanifolds of complementary dimension locally is a single point, so by appropriately resizing M_i we have

$$\sigma_1(M_1') \cap \rho_i(t) \circ \sigma_2(M_2') = \{q_i(t)\} \tag{4}$$

for t sufficiently small. The intersection point $q_i(t)$ depends smoothly on t, because the differentiable submanifolds $\{(t, \rho_i(t) \circ \sigma_2(p_2) \mid -\varepsilon < t < \varepsilon, p_2 \in M_2'\}$ and $\{(t, \sigma_1(p_1) \mid -\varepsilon < t < \varepsilon, p_1 \in M_1'\}$ of $\mathbb{R} \times X$ intersect transversely in the smooth curve $(t, q_i(t))$, whose smooth image is the curve $q_i(t)$.

The points in M_j' corresponding to $q_i(t)$ are then given by

$$q_i(t) = \sigma_1 \circ p_1^{(i)}(t) = \rho_i(t) \circ \sigma_2 \circ p_2^{(i)}(t). \tag{5}$$

Thus $p_i^{(j)}$ are smooth paths in SO_m, and obviously

$$g_i = \left(p_1^{(i)} - \rho_i \circ p_2^{(i)}, \rho_i\right) \in T^m \rtimes_{\mathrm{id}} \mathrm{SO}_m \tag{6}$$

are $m-1$ linearly independent paths in C with $g_i(0) = \mathrm{id}$, none of which leaves the common surface normal n invariant.

Translational paths in C: At last we want to find $m-1$ linearly independent purely *translational* paths t_i in C with $t_i(0)=\mathrm{id}$. Their existence will depend on the difference of second fundamental forms of the surfaces M_i.

Assume again that $\sigma_1 \pitchfork \sigma_2$ at p_1, p_2. A regular *translational* path $u(t)$ in $T^m \cap C$ is an image of a path $((p_1(t), F_1(t)), (p_2(t), F_2(t))) \cdot \mathrm{SO}_{m-1}$ in $(O(M_1) \times O(M_2))/\mathrm{SO}_{m-1}$, therefore there are paths $p_i(t)$ of contact points. Moreover, $u = \iota_2 \circ p_2 - \iota_1 \circ p_1$. The surfaces ι_1 and $\iota_2 + u$ touch at p_1, p_2 if and only $n(p_1) = n(p_2)$. Two pairs p_i and p_i' define the same translation if and only if $\iota_1 p_1 - \iota_2 p_2 = \iota_1 p_1' - \iota_2 p_2'$.

Assume a pair of paths $(p_1(t), p_2(t))$ which gives rise to a path in $T \cap C$. Let $(v, w) = (d\iota_1 \dot{p}_1(0), d\iota_2 \dot{p}_2(0))$. We introduce the mappings

$$s_i : V \to V, \quad s_i(v) = d\sigma_i \circ d\iota_i^{-1}(v). \tag{7}$$

We now can write down the differential of the condition that ι_1 and $\iota_2 + u$ touch at points p_i, and that $\dot{u} \neq 0$: It is necessary that

$$s_1(v) = s_2(w), \quad \text{and} \quad v \neq w.$$

We are going to determine the number of linearly independent solutions of this equation of differentials:

Let $\psi : V \oplus V \to V$, $(v,w) \mapsto s_1(v) - s_2(w)$, and let $\Delta = \{(v,v)\}$. Then $\mathrm{rk}\,\psi = m-1$ because of $\sigma_1 \pitchfork \sigma_2$, which implies $\dim\ker\psi = m-1$. Further $\dim(\ker\psi \cap \Delta) = \dim\Delta - \mathrm{rk}(\psi|\Delta)$. Thus $\dim(\ker\psi/(\Delta \cap \ker\psi)) = \mathrm{rk}(\psi|\Delta) = \mathrm{rk}(\sigma_1 - \sigma_2)$, and the maximum number of linearly independent paths in $C \cap T$ equals $\mathrm{rk}(s_1 - s_2)$. If σ_1, σ_2 are not transverse but $\mathrm{codim}(s_1(V) + s_2(V)) = k$, then the same calculations show that the number is $\mathrm{rk}(s_1 - s_2) + k$.

If we can show that there is an $(m-1)$-dimensional submanifold of possible pairs of points (p_1, p_2) in $M_1 \times M_2$ with $\sigma_1(p_1) = \sigma_2(p_2)$, then we can can use the calculations above to deduce that we can actually choose $m-1$ linearly independent paths $p_{i,1}(t), p_{i,2}(t)$ of contact points which give rise to $m-1$ linearly independent translational paths $t_i(t) = p_{i,1}(t) - p_{i,2}(t)$ in configuration space.

This is done by considering the mapping $(\sigma_1, \sigma_2) : M_1 \times M_2 \to S^{m-1} \times S^{m-1}$, which is easily see to be transverse to the diagonal $D = \{(n,n) \mid n \in S^{m-1}\}$ if $\sigma_1 \pitchfork \sigma_2$. If $\mathrm{rk}(s_2 - s_1) = m-1$, then $(\sigma_1, \sigma_2)^{-1}(D)$ actually is an $(m-1)$-dimensional differentiable submanifold.

Theorem 1 *A point $g \in C$ of the configuration space of two surfaces $\iota_i : M_i \to E^m$ is regular if and only if the spherical mappings σ_1, σ_2 fulfill the condition $\mathrm{rk}(s_1 - s_2) = m-1$ at the point of contact.*

Proof: A point g in C is regular if and only if we can find $(m+2)(m-1)/2$ linearly independent paths in C which pass through g. In the discussion preceding the theorem it is shown that under the assumption that $\sigma_1 \pitchfork \sigma_2$, there is a linearly independent set of paths consisting of $(m-1)(m-2)/2$ rotations about the common surface normal, of $m-1$ other rotations, and of $\mathrm{rk}(s_2 - s_1)$ translations.

If σ_1, σ_2 are not transverse, then $\mathrm{rk}(s_1 - s_2) < m-1$, there are less then $(m-1)(m-2)/2 + m - 1$ rotations and the number of translations is less or equal $m-1$, so g is not regular in this case. □

Remark: In [12] a G-invariant contact density in C is defined as

$$dC(v_2, \ldots, v_{\dim G}) = dG(v_1, v_2, \ldots, v_{\dim G}),$$

where v_1 is the tangent vector of a translation of unit speed in direction of the contact normal of the two surfaces, and dG is the both right and left invariant density in G. After introducing an appropriate density dB in the factor space of (2), and orthonormal bases in V, dC can be expressed as

$$dC = k \cdot \det(s_1 - s_2) \cdot dB,$$

with a certain constant k, which is in accordance with our result.

Corollary 1 *If $\sigma_1 \pitchfork \sigma_2$ a point in configuration space is regular if and only if there are $m-1$ linearly independent translations in C.*

The translations in C have been investigated in [14], where also the connections with collision problems are studied.

We consider the set C_o of those $g \in C$ which leave a point $o \in X$ fixed. Again we want to determine when C_o is an immersed differentiable submanifold.

Theorem 2 *Suppose $g \in C$ is a regular point of C and $g(o) = o$. Then g is a regular point of C_o, and C_o locally is a $(m+1)(m-2)/2$-dimensional submanifold, if and only if the tangent plane at the point $p = \iota_1(p_1) = \iota_2(p_2)$ of contact is not orthogonal to $\overrightarrow{po}$.*

Proof: Consider the mapping $\psi : C \to E^m$, $g \mapsto g(o)$. Then $\psi^{-1}(o) = C_o \subset G_o \cong \mathrm{SO}_m$. Without loss of generality assume $g = \mathrm{id}$.

Consider all paths $g(t) : I \to C$ with $g(0) = \mathrm{id}$, which are also paths in T^m. The linear span of their tangent vectors $\frac{d}{dt}\big|_{t=0} g(t)(o)$ equals $V = T_{p_i}\iota_i$. Now consider the $m-1$ non-translational motions g_i in the discussion preceding Theorem 1. For at least one of them, $\frac{d}{dt}\big|_{t=0} g_i(t)(o) \notin V$, so $d\psi(d_g C) = \mathbb{R}^m$, i.e., $\psi \pitchfork \{o\}$, and therefore $C_o = \psi^{-1}(o)$ locally is a m-codimensional submanifold of C.

Comparison of dimensions shows that it is also a 1-codimensional submanifold of $G_o \cong \mathrm{SO}_{m-1}$. □

The following lemma is needed later:

Lemma 1 *With the notations of Th. 2, assume that* id *is a regular point of C, and choose w such that*

$$[w] = [\overrightarrow{po}, n] \cap [\overrightarrow{po}]^\perp,$$

where n is the common surface normal. Consider paths $g(t)$ in C_o with $g(0) = \mathrm{id}$. Then for all $x \notin p + [w, \overrightarrow{po}]$ the velocity vectors $\frac{d}{dt}|_{t=0}\, g(t)(x)$ of all paths in C_0 with $g(0) = \mathrm{id}$ span $[\overrightarrow{xo}]^\perp \subset T_x\mathbb{R}^m$.

Proof: Consider an orthogonal frame $b_1, \dots, b_m$ centered in o such that $b_1 = \overrightarrow{op}$ and $[b_2] = [w]$. Then $[b_3, \dots, b_n] = V \cap [po]^\perp$. Consider the frame's motion under $g(t)$. We have the differential equation $\frac{d}{dt}|_{t=0}\, g(t) b_j = \sum c_{ij} b_i$ with a skew-symmetric matrix c_{ij}, and $\frac{d}{dt}|_{t=0}\, g(t)(p) \in V$ shows that $c_{12} = -c_{21} = 0$. C_o has codimension 1 is C, so this is precisely the linear equations which defines that tangent space $T_{\mathrm{id}} C_o$.

The subspace of possible velocity vectors of x therefore spans entire $[\overrightarrow{xo}]^\perp$. □

4 Spaces of constant curvature

In order to prove that our results hold in other spaces of constant curvature as well, we embed them into a projective space and consider the group of isometries as a subgroup of PGL_m.

Real projective m-space $\mathbb{R}P^m$ is equipped with homogeneous coordinates $x_0 : \ldots : x_n$. Euclidean space E^m is the subset $x_0 \neq 0$ of $\mathbb{R}P^m$, and $\mathrm{Isom}(E^m)$ is the subgroup

$$\mathrm{Isom}(E^m) = \left\{ \mathbb{R} \left(\begin{array}{c|c} 1 & 0 \\ \hline t & M \end{array} \right) \;\Big|\; t \in \mathbb{R}^m, M \in \mathrm{O}_m \right\} \tag{8}$$

of PGL_m. Hyperbolic space is embedded into $\mathbb{R}P^m$ as follows:

$$H^m = \{\mathbb{R}x \mid \sum_{i>0} x_i^2 < x_0^2\} \tag{9}$$

and

$$\mathrm{Isom}(H^m) = \left\{ \mathbb{R}M | M^T J M = J, \right\}, \tag{10}$$

with $J = \mathrm{diag}(-1, 1, \dots, 1)$. Instead of the sphere we consider elliptic space $\widetilde{S}^m$ which is the factor space of S^m with respect to the two-element group of isometries consisting of the identity and the antipodal map. It is embedded into $\mathbb{R}P^m$ by:

$$\widetilde{S}^m = \mathbb{R}P^m, \tag{11}$$

and

$$\mathrm{Isom}(\widetilde{S}^m) = \left\{ \mathbb{R}M | M^T = M^{-1} \right\}. \tag{12}$$

We denote the connected component of the identity in $\mathrm{Isom}(X)$ by $\mathrm{Isom}_+(X)$. It consists of the orientation-preserving transformations, if the space is orientable.

The projective point $o = (1 : 0 : \ldots : 0)$ then is contained in $E^m \cap H^m \cap \widetilde{S}^m$. If X is any of the three spaces, and $G = \mathrm{Isom}_+(X)$, then

$$G_o = \{\mathbb{R}\,\mathrm{diag}(1, M) \mid M \in \mathrm{SO}_m\} \cong \mathrm{SO}_m. \tag{13}$$

If we are given two immersed surfaces as above, we again ask for C and C_o. As we can assume, without loss of generality, that $o = (1 : 0 : \ldots : 0)$, and G_o is the same for all three spaces, Th. 2 and Lemma 1 hold not only in E^n, but in $\widetilde{S}^m$ and H^m also. (The spherical mappings σ_i are always in the Euclidean sense).

Thus we are able to prove:

Lemma 2 *A point $g \in C$ of the configuration space of two surfaces $\iota_i : M_i \to X$, where X is any of $E^m, \widetilde{S}^m, H^m, S^m$ is regular if and only if the spherical mappings σ_1, σ_2 fulfill the condition $\mathrm{rk}(s_1 - s_2) = m - 1$.*

Proof: Assume that $X = E^m$, $X = H^m$ or $X = \widetilde{S}^m$. Then without loss of generality we may assume that $g = \mathrm{id}$. We denote the point of contact by p. Consider a point $o \in X$ such that op is not orthogonal to V. (The orthogonality is in the sense of X). Without loss of generality we may assume that $o = (1 : 0 : \ldots : 0)$. Then Th. 2 implies that C_o locally is an $(m+1)(m-2)/2$-dimensional submanifold of G_o.

Lemma 1 shows that for all points $x \in X \cap E^m$ except those in a plane, the paths in C_o give rise to velocity vectors which span a hyperplane orthogonal to $\overrightarrow{xb}$, where the orthogonality is both in the sense of E^m and in the sense of X. Thus an appropriate choice of points $o_1, \ldots, o_m$ and regular paths g_j in C_{o_j} with $g_j(0) = \mathrm{id}$ such that $\{\frac{d}{dt}|_{t=0}\, g_j(t)(o)\}$ is linearly independent shows that there are in in fact $(m+1)(m-2)/2 + m = (m+2)(m-1)/2$ linearly independent paths in C, and we have shown the result in the cases $X = E^m, H^m, \widetilde{S}^m$.

If $X = S^m$ the results also holds because it is of a local nature and S^m is a twofold covering of $\widetilde{S}^m$, which is locally isomorphic to $\widetilde{S}^m$ with respect to both its smooth structure and its metric. $\square$

5 Second order contact of Surfaces

We used the spherical mappings σ_i and their derivatives $d\sigma_i$ and s_i to characterize the situation that there are less than $m-1$ linearly independent translations in C. We want to interpret the condition $\mathrm{rk}(s_1 - s_2) < m - 1$ geometrically: Suppose the two regular surfaces $\iota_i : M_i \to E^m$ of Euclidean space locally touch each other along a regular curve, i.e., there are curves $c_i : I \to m_i$ with $\iota_1 c_1 = \iota_2 c_2$ and $n_1 c_1 = n_2 c_2$. Then it is easily seen that $\mathrm{rk}(s_1 - s_k) < m - 1$. On the other hand, if $\mathrm{rk}(s_1 - s_2) < m - 1$, then there is a regular surface $\iota_3 : M_3 \to E^m$ which *osculates* M_1 and touches M_2 along a regular curve.

Osculation of two surfaces is defined as follows: We use the affine tangent space $d\iota_i(T_{p_i}M_i)$ as coordinate chart and choose an appropriate Cartesian coordinate system in E^m such that we can write both ι_1 and ι_2 in the form

$$\bar{\iota}_i : u = (u_1, \ldots, u_m) \mapsto (u_1, \ldots, u_m, z_i(u_m)). \tag{14}$$

Then $\frac{\partial}{\partial u_j} z_i = 0$, and the two surfaces *osculate* if and only if $\frac{\partial^2}{\partial u_i \partial u_j}(z_1 - z_2) = 0$, for all i, j. Osculation is invariant with respect to diffeomorphisms, so it is well defined in any of $E^m, S^m, \widetilde{S}^m, H^m$, and is invariant with respect to the action of every $g \in \mathrm{Isom}(X)$. Thus what in Euclidean space is $\mathrm{rk}(s_2 - s_1) < m - 1$ is a well-defined notion of geometry of spaces of constant curvature, and we call it *second order contact* of surfaces.

6 Curvatures

In Euclidean space the mappings s_i are self-adjoint with respect to the Euclidean scalar product in V, so there are eigenvalues $\kappa_1^{(i)}, \ldots, \kappa_{m-1}^{(i)}$, which are the principal curvatures of ι_i at p_i. If X is not Euclidean space, we define the principal curvatures of ι_i at p_i as the principal curvatures of the then Euclidean surface $g \circ \iota_i$ where $g \in \mathrm{Isom}_+(X) = G$ is such that $g(\iota_i(p_i)) = (1 : 0 : \ldots : 0) = o$. Because $G_o \subset \mathrm{Isom}(E^m) \cap \mathrm{Isom}_+(X)$, they are well defined.

This brute force method to *define* curvatures on the one hand is well suited for our method to describe non-euclidean isometries by Euclidean ones, which allows to use the more elementary geometry of one-parameter subgroups of the Euclidean isometry group, instead of the respective spherical and hyperbolic ones. On the other hand curvatures are already defined for submanifolds of Riemannian manifolds, so it is necessary to convince ourselves that these notions actually coincide:

Lemma 3 *A curve $c : I \to \mathbb{R}P^n$ with $c(0) = (1 : 0 : \ldots : 0)$ has curvatures κ_e, κ_s, and κ_h with respect to the Euclidean, elliptic and hyperbolic metrics. All these curvatures are equal: $\kappa_e = \kappa_s = \kappa_h$.*

Proof: It is sufficient to consider *circles* as curves. They are contained in planes, so it is sufficient to consider the two-sphere and the hyperbolic plane. We embed the former in Euclidean 3-space as the unit sphere $\bar{X}$, and likewise the latter in pseudo-euclidean 3-space as the unit sphere $\bar{X} : z^2 - x^2 - y^2 = 1, z > 0$. A distance circle $\bar{c}$ of radius r then is the intersection of X with a cone $\Delta = \{x \in X, \angle(x, m) = r\}$, where the angle is in the respective metric. The models X of (9) and (11) are the result of central projection of the spatial models with center 0 onto the plane $\eta : z = 1$. The projection of the distance circle $\bar{c}$ then equals $c = \Delta \cap \eta$.

$\bar{c}$ is a circle in the sense of (pseudo)-euclidean geometry, so its axis of curvature is known. Assume that $o = (0,0,1) \in \Delta$. Meusnier's theorem implies that the Euclidean center of curvature of c at o equals $[m] \cap \eta$. Thus the Euclidean curvature of c at o equals $1/\tan r$ for $\widetilde{S}^2$ and $1/\tanh r$ for H^2.

On the other hand the formulas for perimeter $p(r)$ and area $A(r)$ of distance circles in the elliptic/spherical and hyperbolic geometries, which are given by $p(r) = 2\pi \sin r$ and $p(r) = 2\pi \sinh r$, $A(r) = 2\pi(1-\cos r)$ and $A(r) = 2\pi(\cosh r - 1)$, together with the Gauss-Bonnet theorem $\pm A(r) + \kappa p(r) = 2\pi$ imply that the geodesic curvatures of distance circles are given by $1/\tan(r)$ and $1/\tanh r$, respectively.

Thus we have shown the theorem in elliptic/spherical geometry. In hyperbolic geometry we have shown it for all circles with $\kappa > 1$. The rest follows from linearity. $\square$

Thus we have justified our mixing of Euclidean and Non-Euclidean geometries, and we are able to specify the following theorem without having to state which geometry the curvatures refer to:

Theorem 3 *In any of the spaces $X = E^m$, S^m, $\widetilde{S}^m$, H^m the configuration space of surface-surface contact has no singularity at the points p_1, p_2 if and only if the principal curvatures of the surfaces $\iota_i : M_i \to X$ lie completely separated (see Fig. 1). If this holds for all pairs $p_i \in M_i$, then C is an immersed $(m+2)(m-1)/2$-dimensional submanifold of $\mathrm{Isom}_+(X)$.*

Proof: We assume that ι_i touch each other in points p_i. We ask the question: is it possible to rotate ι_2 about the common surface normal such that $\mathrm{rk}(s_1 - s_2) < m-1$? If it is, then there is a singular point in C with contact points p_i. Denote the common tangent space by V. Let $h \in \mathrm{SO}(V)$ and consider its extension $\bar{h}$ to G. If M_2 is transformed by $\bar{h}$, the mapping s_2 transforms by $h^{-1}s_2h$ or $h^T s_2 h$, depending on our interpretation of s_2 as linear mapping or as bilinear form. Because $h^T = h^{-1}$ in a Cartesian coordinate system, this makes no difference. The eigenvalues of s_i are the principal curvatures $\kappa_j^{(i)}$ at p_i.

Now we can use the result in [15] where it is shown that there is an $h \in \mathrm{SO}(V)$ such that $\mathrm{rk}(s_1 - h^T s_2 h) < m-1$ if and only if the sets $\{\kappa_1^{(1)}, \ldots, \kappa_{m-1}^{(1)}\}$ and $\{\kappa_1^{(2)}, \ldots, \kappa_{m-1}^{(2)}\}$ are completely separated on the projective line $\mathbb{R} \cup \{\infty = -\infty\}$ (see Fig. 1). $\square$

Different from the usual terminology, we call a surface *strictly convex*, if it is convex and has only elliptic surface points.

Corollary 2 *Assume that $X = E^m$ or $X = H^m$. If neither $\iota_1(M_1)$ nor $\iota_2(M_2)$ is a strictly convex surface, then the configuration space has singularities.*

Proof: All smooth surfaces have elliptic surface points. If they have only elliptic surface points, then they are convex. Thus a surface which is not strictly convex

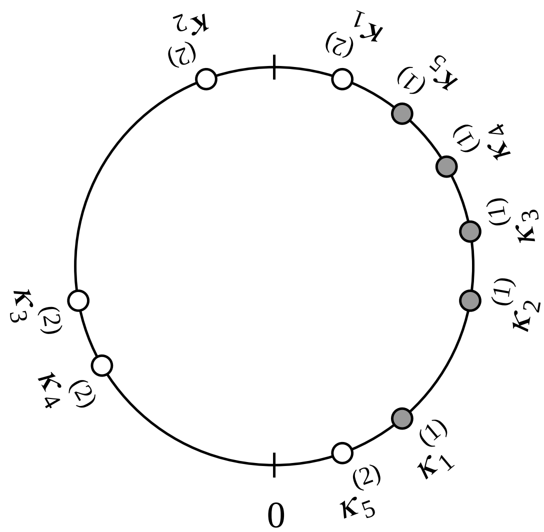

Figure 1: Principal curvatures such that $\mathrm{rk}(s_1 - s_2) = m - 1$.

in our sense has parabolic points. This means that there are points $p_i \in M_i$ such that at least one principal curvature is zero at p_1 and p_2. Now Th. 3 shows that there is a singularity at p_1, p_2. $\square$

Theorem 4 *Assume that M_i is compact, that ι_i is an embedding and that $X = E^3$ or $X = H^3$. There are the following possibilities concerning the singularities of C:*

1. *If both $\chi(M_1), \chi(M_2) \neq 2$, then C has singularities.*
2. *If $\chi(M_1) = 2$ and $\chi(M_2) \neq 0$, denote the minimum and maximum principal curvatures of $\iota_i(M_i)$ by k_i' and k_i''. Then C has no singularities if and only if $[k_1', k_1''] \cap [k_2', k_2''] = \emptyset$.*
3. *If $\chi(M_1) = 2$ and $\chi(M_2) = 0$, then the set of all principal curvatures of M_2 is either an interval $K_2 = [k_2', k_2'']$ or a union of two intervals $K_2 = [k_2', k_2'''] \cup [k_2''', k_2'']$. C has no singularities if and only if $[k_1, k_1'] \cap K_2 = \emptyset$.*

Proof: We consider the set $R^2_{\mathrm{sym}} = \mathbb{R}^2/S^2$ of unordered pairs of real numbers where S^2 denotes the symmetric group of two elements. It is well known (cf. [2]) that the mapping $K_i : M_i \to \mathbb{R}^2_{\mathrm{sym}}$, $p_i \mapsto (\kappa_1^{(i)}, \kappa_2^{(i)})$ is continuous. In Euclidean three-space, every surface with nonzero Euler characteristic has an umbilic, so $K_i(M_i)$ is connected, if $\chi(M_i) \neq 0$.

If $K_i(M_i) \subset \mathbb{R}^+$, then $\iota_i(M_i)$ is strictly convex in our sense, and $\chi(M_i) = 2$. All surfaces have elliptic points. If $\chi(M_i) \neq 2$, or if $\iota_i(M_i)$ is not strictly convex in our sense, then it has parabolic points also, which means $0 \in K_i(M_i)$.

Thus if both $\chi(M_1), \chi(M_2) \neq 2$, then C has a singularity according to Cor. 2, and we have shown part (i).

If $\chi(M_i) \neq 0$, then $\iota_i(M_i)$ has at least one umbilic point p_i, i.e., $\kappa_1^{(i)}(p_i) = \kappa_2^{(i)}(p_i)$. The set of all principal curvatures of $\iota_i(M_i)$ on the real axis is the union of the two projections of $K_i(M_i)$ onto the x_1- and the x_2-axis in $\mathbb{R}^2_{\text{sym}}$. It has therefore at most two connected components. If there is an umbilic, it has one component, and equals therefore the interval $[k_i', k_i'']$. If $\chi(M_i) = 0$, it is possible that it equals the union of two intervals.

Theorem 3 now shows the second and third assertion of the theorem. $\square$

7 *C* as an embedded submanifold

If we know some global shape properties of the surfaces involved, we are able show that the mapping ϕ of Equ. (2) is an embedding in some cases. We will use the results of [14], where analogous results have been obtained in the restricted case of purely translational motions.

Theorem 5 *Assume that $X = E^m$, M_1, M_2 are compact. If both are non-convex, C is not an embedded submanifold. If $\iota_2(M_2)$ is a strictly convex surface, then there is a $\lambda_0 > 0$ such that for all surfaces $\lambda\iota_2 : M_2 \to X$, $p \mapsto \lambda\iota_2(p)$ with $\lambda < \lambda_0$ the configuration space is an embedded submanifold of G.*

Proof: If both surfaces are not strictly convex, C has singularities according to Cor. 2. Assume now that $\iota_2(M_2)$ is a strictly convex surface. For all $g \in G$ we consider the set $T_g \subset (O(M_1) \times O(M_2))/\mathrm{SO}_{m-1}$ which is defined by $x \in T_g$ if and only if $\phi(x)g^{-1}$ is a translation. $\phi(T_g)$ is the set of all $h \in G$ which transform M_2 in a position where it touches M_1, but such that h differs from g only by a translation.

Clearly, if $\phi(x_1) = \phi(x_2) = g$, then $x_1, x_2 \in T_g$ and $\phi|T_g$ is not injective. The injectivity of this mapping has been studied in [14]. It is shown that for all $g \in G$ there is a λ_g such that for all $\lambda < \lambda_g$ the mapping ϕ_g, defined by surfaces ι_1 and $\lambda\iota_2$, is an embedding. By compactness of $(O(M_1) \times O(M_2))/\mathrm{SO}_{m-1}$, there is a positive minimum λ_1 of the λ_g's.

Furthermore, scaling by λ scales the curvatures by the same factor, so there is a $\lambda_2 > 0$ such that for all $\lambda < \lambda_2$ the surfaces ι_1 and $\lambda\iota_2$ satisfy the conditions of Th. 3. Now we can take $\lambda_0 = \min(\lambda_1, \lambda_2)$. $\square$

If we know more about the shape of $\iota_i(M_i)$, then we can say more about self-intersections of C. We call a closed embedded surface positively oriented, if its positive normals point to its outside. The *convex core* of a star-shaped surface is the set of points whith to respect to which the surface is star-shaped. It is easily seen to be convex.

Theorem 6 *Assume that M_i are compact and that $X = E^m$. Denote the minimum and maximum principal curvature of $\iota_i(M_i)$ by k_i' and k_i'' (the sign of the curvatures is determined by the positively oriented normal vector). In the following cases the configuration space is an embedded submanifold:*

1. *if $\iota_i(M_i)$ are convex and oppositely oriented, and at least one of them is strictly convex;*
2. *if $\iota_i(M_i)$ are strictly convex, equally oriented, and $[k_1', k_1'']$, $[k_2', k_2'']$ are disjoint;*
3. *if $\iota_i(M_i)$ are oppositely oriented, one is star-shaped and the other is strictly convex, and $[k_1', k_1'']$, $[k_2', k_2'']$ are disjoint,*
4. *if $\iota_i(M_i)$ are equally oriented, $\iota_1(M_1)$ is star-shaped and $\iota_2(M_2)$ is strictly convex, $[k_1', k_1'']$, $[k_2', k_2'']$ are disjoint, and $\iota_2(M_2)$ is freely movable in the convex core of $\iota_1(M_1)$.*

Proof: Two oppositely oriented convex surfaces one of which is strictly convex touch in at most one point, so ϕ of (2) is injective. They satisfy also the conditions of Th. 3, because one surface has strictly positive principal curvatures, and the other one nonpositive ones, or vice versa, which proves the first statement of the theorem.

A strictly convex surface M with minimal and maximal principal curvatures k', k'' is freely movable in a sphere of radius $1/k''$ and a sphere of radius $1/k'$ is freely movable in M (see [5, 7, 10]). If M_i satisfy the conditions of statement 2, we assume without loss of generality that $k_1' < k_2'$. A sphere of radius $\frac{1}{2}(1/k_1'' + 1/k_2')$ shows that $\iota_1(M_2')$ is freely movable in the interior of $\iota_2(M_1')$ and it touches in at most one point. Thus ϕ is injective. The assumption on k_i', k_i'' further ensures that ϕ has no singularities, so we have shown the second statement.

As to the third statement, we may without loss of generality assume that $\iota_1(M_1)$ is star-shaped and not convex, and that $\iota_2(M_2)$ is strictly convex. We may also assume that $k_2', k_2'' > 0$. Then $k_1' < 0, k_2' > 0$, because a non-convex surface must have at least one elliptic and at least one hyperbolic point. Analogous to the proof of Th. 5 we assume that ϕ is not injective, i.e., $\phi(x) = \phi(x') = g$. Then $\phi|T_g$ is not injective. In [14] it is shown that if a strictly convex surface touches a star-shaped surface from the outside (which is the case here), and the difference of second fundamental forms in points of contact is positive definite (which is the case here, because all principal curvatures of $\iota_2(M_2)$ are greater than all principal curvatures of $\iota_1(M_1)$), then $h \circ g \circ \iota_2(M_2)$ touches $\iota_1(M_1)$ in at most one point for all translations h. This implies that $\phi|T_g$ is injective, and so is ϕ. Th. 3 shows again that ϕ is nonsingular, so we have shown the third statement.

The proof of the fourth statement is completely analogous: We use the result in [14] which says that $\phi|T_g$ is injective, if $h \circ g \circ \iota_2(M_2)$ is contained in the convex core of $\iota_1(M_1)$ for some translation h. $\square$

Example: At last we show an example where the configuration space is an embedded submanifold even if $k_1' < k_2' < k_2'' < k_1''$: Assume that M_1 is a torus in Euclidean three-space with radii r, R $(r < R)$ and M_2 is a sphere with radius ρ such that $r < \rho < R - r$, and $iota_i$ are the obvious embeddings. Then the principal curvatures fulfill

$$\kappa_1^{(1)} = 1/r, \qquad 1/(R+r) < \kappa_2^{(1)} < 1/(R-r),$$
$$\kappa_1^{(2)} = \kappa_2^{(2)} = 1/\rho.$$

All spheres touching M_1 in two points either have radius $\geq R - r$ or radius r, so ϕ of (2) is injective. Th. 3 shows that ϕ is also nonsingular, so C is an embedded 5-dimensional submanifold of G. This result holds also for surfaces, which are 'not very different' from these M_i with respect to their shape, first and second derivatives.

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