Generalized Multiresolution Analysis for Arc Splines

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Abstract. In order to approximate a curve by another curve consisting of circular arcs ('arc spline'), we apply a generalized multiresolution analysis on base of trigonometric spline functions to the support function of this curve.

§1. Homogeneous B-Splines and Trigonometric Splines

In this section we recall some facts about the connection between trigonometric B-spline functions and homogeneous polynomial B-spline functions. First, we follow [10,3], then [9,12].

We assume that the reader is familiar with the definition of the space $\mathcal{T}_k(\phi, M)$ of trigonometric splines of order k, based on a knot sequence $\phi = (\phi_0, \ldots, \phi_{n-k})$ with $a = \phi_0 < \phi_1 < \cdots < \phi_{n-k-1} < \phi_{n-k} = b$, and a vector $M = (m_1, \ldots, m_{n-k-1})$ of integer multiplicities m_i which satisfy $1 \le m_i \le k+1$. It is of dimension n and is a special case of an L-spline space. We are going to demonstrate its connections with homogeneous polynomial B-splines.

The polar angle $\theta(u, v)$ of two nonzero vectors $u, v \in \mathbb{R}^2$ is defined to be the smallest $\theta \in \mathbb{R}$, $\theta \ge 0$, such that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot u = \lambda v$ ($\lambda > 0$). For $u, v \in \mathbb{R}^2 \setminus 0$ we define the closed interval $[u, v] := \{x \in \mathbb{R}^2 \setminus 0, \ \ \theta(u, x) + \theta(x, v) = \theta(u, v)\}$, and the half-open interval $[u, v) := \{x \in [u, v], \forall \lambda > 0 : x \neq \lambda v\}$. Let $k \ge 0$ and $t = (t_0, \ldots, t_{n+k})$ be a sequence of nonzero vectors $\in \mathbb{R}^2$ (knot sequence), such that for all $i = 0, \ldots, n-1$

$$\theta(t_i, t_{i+1}) + \theta(t_{i+1}, t_{i+2}) + \cdots + \theta(t_{i+k}, t_{i+k+1}) = \theta(t_i, t_{i+k+1}) < \pi \qquad (1)$$

holds, and we always have $\theta(t_i, t_{i+k+1}) > 0$. Because the sequence may wind itself around the origin more than once, we lift it to the universal cover $\widetilde{\mathbb{R}^2}$ of $\mathbb{R}^2 \setminus 0$; i.e. we tacitly assign to each vector of $\mathbb{R}^2 \setminus 0$ a winding number. This will help to avoid ambiguities.

Mathematical Methods for Curves and Surfaces II

Morten Dæhlen, Tom Lyche, and Larry L. Schumaker (eds.), pp. 537–544.

ISBN 0-8265-1315-8.

Definition. *The homogeneous polynomial B-spline basis function* $N_i^0 : \widetilde{\mathbb{R}^2} \to \mathbb{R}$ *of degree* 0 *is the characteristic function of the interval* $[t_i, t_{i+1})$*, and the homogeneous B-spline basis functions* $N_i^k : \widetilde{\mathbb{R}^2} \to \mathbb{R}$ *of degree* $k > 0$ *are defined recursively by*

$$N_i^k(x) = \frac{\det(x, t_i)}{\det(t_{i+k}, t_i)} N_i^{k-1}(x) + \frac{\det(t_{i+k+1}, x)}{\det(t_{i+k}, t_i)} N_{i+1}^{k-1}(x). \qquad (2)$$

It is well known that the trigonometric B-spline functions of order k can be expressed in terms of homogeneous B-spline functions of order k: For $u \in \mathbb{R}$, we define the unit vector $n(u) = (\cos u, \sin u) \in \widetilde{\mathbb{R}^2}$. The angle u determines the appropriate winding number. To a knot angle sequence ϕ with multiplicities M we assign the knot vector sequence

$$\underbrace{n(\phi_0/k), \ldots,}_{k+1 \text{ times}} \underbrace{n(\phi_1/k), \ldots,}_{m_1 \text{ times}} \underbrace{n(\phi_2/k), \ldots,}_{m_2 \text{ times}} \ldots, \underbrace{n(\phi_{n-k}/k), \ldots}_{k+1 \text{ times}}. \qquad (3)$$

Then the linear span of the restriction to the unit circle of the homogeneous polynomial B-spline basis functions $N_i^m|S^1$ equals the space of all functions $f(k \cdot u)$, where $f \in \mathcal{T}_k(\phi, M)$ if the knot sequence fulfills condition (1).

The functions $T_i^k(u) = N_i^k(n(\frac{u}{k}))$, $i = 1, \ldots, n$ corresponding to the homogeneous B-spline basis functions will be called trigonometric B-spline basis functions. It is well known that they form a basis of $\mathcal{T}_k(\phi, M)$.

Because we will need it later, we recall some facts about polar forms of polynomial functions: The polar form of a homogeneous polynomial function $p : \mathbb{R}^2 \to \mathbb{R}$ of degree n is the unique multilinear symmetric function in n variables $S_p : (\mathbb{R}^2)^n \to \mathbb{R}$ which has the property $S_p(t, \ldots, t) = p(t)$ for all $t \in \mathbb{R}^2$. Further, the following is true: Let $p = \sum c_i(t) N_i^r(t)$. If $[t_j, t_{j+1})$ is nonempty, let S_j be the polar form of the restriction $p|[t_j, t_{j+1})$. Then

$$c_i = S_i(t_{i+1}, \ldots t_{i+r}) = S_{i+1}(t_{i+1}, \ldots, t_{i+r}) = \cdots = S_{i+r}(t_{i+1}, \ldots, t_{i+r}), \quad (4)$$

whenever the polar forms S_j are defined. Thus the polar form can be used to calculate the coefficients c_i.

Lemma. *Let* $\alpha_i = (\phi_{i+1} - \phi_i)/2$*. The quadratic trigonometric spline functions possess the following* approximate convex hull property*:*

$$\sum c_i T_i^2(u) \in \text{c.h.}\left(\frac{c_{i-2}}{\cos\alpha_{i-1}}, \frac{c_{i-1}}{\cos\alpha_i}, \frac{c_i}{\cos\alpha_{i+1}}\right) \quad (\phi_i \le u \le \phi_{i+1}). \qquad (5)$$

Here all undefined terms are tacitly assumed to be absent.

Proof: The bivariate polynomial function $x^2 + y^2$ is constant when restricted to the unit circle. Its polar form is the euclidean scalar product. Thus (4) implies that the constant function 1 can be represented as a quadratic trigonometric B-spline function by $1 = \sum \langle t_{i+1}, t_{i+2} \rangle T_i^2(u) = \sum \cos\alpha_{i+1} T_i^2(u)$, where $t = (t_1, \ldots)$ is the knot vector sequence corresponding to ϕ and M according to (3). If $\phi_{i+k} - \phi_i < 2\pi$, we always have $0 \le T_i^k(u), \cos\alpha_i \le 1$. Therefore, the sum $\sum c_i T_i(u) = \sum \frac{c_i}{\cos\alpha_{i+1}} \cos\alpha_{i+1} T_i^2(u)$ is a convex combination. The assertion of the lemma now follows from the fact that for $u \in [\phi_i, \phi_{i+1}]$, at most three terms in the above sum are nonzero. □

§2. A Generalized Multiresolution Analysis with Trigonometric B-Splines

Definition. *Let $I = [a, b] \subset \mathbb{R}$. A* generalized multiresolution analysis *of $L^2(I)$ is a nested sequence $V_0 \subset V_1 \subset V_2 \subset \cdots$ of closed linear subspaces of $L^2(I)$ such that their union is dense in $L^2(I)$, together with Riesz bases of the spaces V_i and of complements W_i of V_i in V_{i+1}. The basis functions of W_i will be called (pre-)wavelets.*

We will show how to construct a generalized multiresolution analysis with trigonometric B-spline functions of order two. Let $(\phi^{(0)}, M^{(0)}) \leq (\phi^{(1)}, M^{(1)}) \leq \cdots$, be nested knot angle sequences. We choose the corresponding trigonometric B-spline functions as bases of $V_i = \mathcal{T}_2(\phi^{(i)}, M^{(i)})$. Let the spaces W_i be the orthogonal complements of V_i in V_{i+1} in the sense of L^2.

Let t and t' be knot vector sequences such that t' is a refinement of t. The homogeneous B-spline basis functions corresponding to t and t' will be denoted by N_i and N'_i. Then there are coefficients c_{ij} such that $N_i(x) = \sum c_{ij} N'_j(x)$.

Lemma. *The coefficients c_{ij} are given by the following algorithm: Choose k such that $j \leq k \leq j+2$ and $[t'_k, t'_{k+1})$ is not empty. This interval is contained in an interval $[t_l, t_{l+1})$. Then we have $t'_{j+1} = \alpha t_l + \beta t_{l+1}$, $t'_{j+2} = \gamma t_l + \delta t_{l+1}$ and*

$$c_{ij} = \alpha\gamma N_i(t_l) + \beta\delta N_i(t_{l+1}) + (\alpha\delta + \beta\gamma)\delta_{p,i-1}. \tag{6}$$

Proof: Equation (4) implies that c_{ij} equals $S'_k(t'_{j+1}, t'_{j+2})$, where S'_k is the polar form of $N_i|[t_k, t_{k+1})$. Because of the multilinearity and symmetry of the polar form we have $c_{ij} = \alpha\gamma S'_k(t_l, t_l) + \beta\delta S'_k(t_{l+1}, t_{l+1}) + (\alpha\delta + \beta\gamma) S'_k(t_l, t_{l+1})$. Because of $[t'_k, t'_{k+1}) \subset [t_l, t_{l+1})$, the latter is not empty and we have $S_l = S'_k$, where S_l is the polar form of $N_i|[t_l, t_{l+1})$. It follows that $S'_k(t_l, t_l) = N_i(t_l)$ and $S'_k(t_{l+1}, t_{l+1}) = N_i(t_{l+1})$. Equation (4) implies that $S_l(t_l, t_{l+1})$ equals 1 if $i = l-1$ and 0 if not. The assertion follows. □

Let $\phi^{(i)}_{\max} = \max_j(\phi^{(i)}_{j+1} - \phi^{(i)}_j)$ and $\phi^{(i)}_{\min} = \min_j(\phi^{(i)}_{j+1} - \phi^{(i)}_j)$. Assume that $\lim_{i\to\infty} \phi^{(i)}_{\max} = 0$. Then the union of the V_i is dense in $L_2(I)$. This is proved in [3], where it is derived from a more general theorem of [10]. In our special case this also follows directly from the approximate convex hull property.

In [3] a basis of the space W_i is constructed which consists of functions ψ of the minimal possible support.

§4. Support Functions

4.1. (Locally) Convex Curves

For simplicity, we restrict ourselves to the case of piecewise C^2 curves. What we are going to do in this section can be done for continuous curves as well, because the restriction of (local) convexity is strong enough to eliminate possible degeneracies, but we use the curvature of the curve to simplify the discussion.

Definition. *Assume that $I = [a,b]$ and the curve $c : I \to \mathbb{R}^2$ is parametrized by arc length. We call c piecewise C^k if there is a discrete set T of parameter values such that for all intervals $(u,v) \subset I \setminus T$ there is a C^k extension of c to an interval $(u-\epsilon, v+\epsilon)$ with an $\epsilon > 0$.*

Definition. *A piecewise C^2 curve is called* locally convex *and of* nonnegative curvature *if for all $t \in I \setminus T$, the curvature is nonnegative, and for all $t \in T$ the turning angle $\theta(\dot{c}_-, \dot{c}_+)$ of the limit tangent vectors is nonnegative. A locally convex curve of nonpositive curvature is defined in the obvious way.*

A unit vector n is called oriented unit normal vector at $u = u_0 \notin T$ if it is perpendicular to $\dot{c}(u_0)$ and points in the direction of $\ddot{c}(u_0)$. If $u = u_1 \in T$ and the curve is of nonnegative curvature, we consider the left- and right-handed limit normal vectors n_1 and n_2 and define all vectors n with $\theta(n_1, n) + \theta(n, n_2) = \theta(n_1, n_2)$ as oriented unit normal vectors. After re-parametrizing the curve such that for a whole interval the point $c(u)$ rests in a point of tangent discontinuity, we can define the piecewise C^1 oriented normal vector field $n(u)$ and the function

$$d_c : I \to \mathbb{R}, \quad u \mapsto \langle c(u), n(u) \rangle. \tag{7}$$

At last we re-parametrize d_c such that its argument is the polar angle of the oriented normal vectors. We will always assume that d_c is parametrized in this way. The function d_c is thus piecewise C^1.

Definition. *The function d_c, parametrized by the polar angle of the oriented normal vectors, is called the* support function *of the (locally) convex curve c.*

If the support function d_c is given, the curve c can be reconstructed as the envelope of the lines $l(u) : \langle x, n(u) \rangle = d_c(u)$. If the envelope is not defined because d_c is not C^1, this was necessarily caused by a straight line segment contained in the curve c.

Not all piecewise C^1 functions, however, are support functions of piecewise C^2 locally convex curves. The following is well known:

Lemma. *The function d is the support function of a (locally) convex curve, if and only if the sign of $d + d''$ is constant.*

4.2 Filter Bank Decomposition of Piecewise Circular Curves

A circle with center $m = (m_1, m_2)$ can be parametrized by $c : \mathbb{R} \to \mathbb{R}^2$, $c(u) = m - rn(u)$. Then $n(u)$ is the oriented unit normal vector to c in the point $c(u)$. The circle's support function d_c is given by $d_c(u) = \langle m, n(u) \rangle - r = m_1 \cos u + m_2 \sin u - r$. This leads to the following definition and lemma:

Definition. *An* arc spline curve *is a C^1 curve which consists of discrete circular arcs.*

Lemma. *The locally convex curve c is an arc spline curve if and only if its support function d_c is a trigonometric B-spline function of order two whose knot vector has only knots of multiplicity one.*

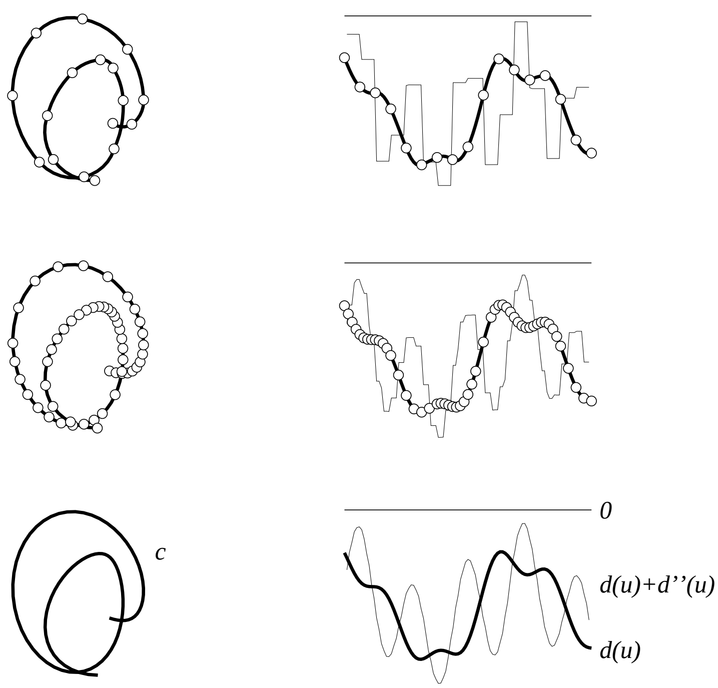

Fig. 1. Approximation of a locally convex curve by arc splines. Left: Curve and arc spline approximants. Right: Support functions d together with the curvature radius functions $d + d''$.

The filter bank algorithm defined above can now be used to define a wavelet transform of the arc spline c. This is defined as the wavelet transform of its support function, using the bases of the spaces W_i as wavelet functions: The orthogonal direct sum $V_{i+k} = V_i \oplus W_i \oplus W_{i+1} \oplus \cdots \oplus W_{i+k-1}$ leads in a natural way to projections $\pi_i : V_{i+k} \to V_i$ and $\rho_i : V_{i+k} \to W_i$. We will call the decomposition $x = \pi_i(x) + \sum_{j=0}^{k} \rho_{i+j}(x)$ $(x \in V_{i+k})$ the filter bank decomposition of x, and the sequence of coefficient vectors of the various projections in the bases selected above, its wavelet transform.

§5. Approximation of Curves by Arc Splines

There are many ways to approximate and interpolate curves and discrete point sets with arc splines. The interested reader is referred to [2,4,5,6,7,8] and the literature cited therein.

5.1. Approximation of Locally Convex Curves

Let a locally convex curve $c : I \to \mathbb{R}^2$ be given. We can approximate its support function $d_c(u)$ by a trigonometric spline function of order two. An obvious choice is the closest approximation in the sense of L^2. Points of tangent discontinuity can be reproduced by choosing appropriate greater multiplicities.

Applying the wavelet transform to $\overline{d}$ and setting all coefficients below some threshold to zero gives an approximation $\widetilde{d}$ of $\overline{d}$ and d_c which is the support function of an arc spline $\widetilde{c}$, if the condition $\widetilde{d} + \widetilde{d}''$ is fulfilled. The points of curvature discontinuity of $\widetilde{c}$ correspond to the points of curvature discontinuity of $\widetilde{d}$ and are contained the union of the points of curvature discontinuity of all basis and wavelet functions which actually contribute to $\widetilde{d}$. The more coefficients of higher index we set to zero, the fewer points of curvature discontinuity the resulting arc spline will have, i.e., the fewer circular arcs it will consist of. Fig. 1 shows an example. The small circles indicate the points of curvature discontinuity.

5.2 Estimates

Definition. *The distance $d(c_1, c_2)$ between locally convex curves $c_i : I \to \mathbb{R}^2$ is the minimal ϵ such that c_1 lies in a closed ϵ-neighborhood of the union of c_2 and its initial and final tangent ray, and vice versa.*

Theorem. *Let $c : I \to \mathbb{R}^2$ be a locally convex piecewise C^2 curve and d_c its support function, and let $\mathcal{T}_2(\phi, M)$ be a trigonometric spline space. Then there exists an arc spline curve $\widetilde{c}$ with support function $d_{\widetilde{c}} \in \mathcal{T}_2(\phi, M)$ such that $d(c, \widetilde{c}) \leq C_1 \phi_{\max}^3 \|d_c\|_\infty + C_2 \omega_\infty^3(d_c, \phi_{\max})$. If d_c is in C^3, we have $d(c, \widetilde{c}) \leq C_3 \phi_{\max}^3(\|d_c\|_\infty + \|D^3 d_c\|_\infty)$. The constants C_i depend on the parameter interval, but not on c, ϕ or M. If $\phi_{\max}/\phi_{\min}$ is bounded, for all knot vectors ϕ with $\phi_{\max}$ small enough, the curve $\widetilde{c}$ does not have points of regression.*

Proof: Let L be the differential operator of degree m which defines the spaces $\mathcal{T}_{m-1}(\phi, M)$. For $m = 3$, we have $L = D(D^2 + 1)$. Theorem 10.24 of [10] implies that for $j = 0, 1, \ldots, m-1$ there exists a constant C, such that for all admissible (ϕ, M) and all $f \in C^m[a, b]$, the inequality

$$\|D^j(f - Qf)\|_\infty \leq C\|Lf\|_\infty \cdot (\phi_{\max}^m / \phi_{\min}^j) \tag{8}$$

holds. Q is a projection onto $\mathcal{T}_{m-1}(\phi, M)$, which is introduced in [10]. By Theorem 10.1 of [10], there is a constant C independent on f, such that $\|Lf\|_\infty \leq C(\|f\|_\infty + \|D^m f\|_\infty)$. Letting $j = 0$ in (8), this implies

$$\|f - Qf\|_\infty \leq C\phi_{\max}^m(\|f\|_\infty + \|D^m f\|_\infty), \tag{9}$$

with a constant independent of the knot sequence and of f. Because (9) holds for all $\phi_{\max} > 0$, Theorem 2.68 of [10] allows us to conclude that there are constants C_1, C_2 such that for all $f \in C[a, b]$ and all admissible (ϕ, M), there is an $\overline{f} \in \mathcal{T}_{m-1}(\phi, M)$ such that $\|f - \overline{f}\|_\infty \leq C_1 \phi_{\max}^m \|f\|_\infty + C_2 \omega_\infty^m(f, \phi_{\max})$. If $f \in C^m[a, b]$, the modulus of smoothness can be replaced by $\phi_{\max}^m \|D^m f\|_\infty$. Now assume that always $\phi_{\min} \geq k\phi_{\max}$. Letting $j = 2$ and $m = 3$ in (8), for all $f \in C^3([a, b])$ we have

$$\|(1 + D^2)(f - Qf)\|_\infty \leq C\phi_{\max}(\|f\|_\infty + \|D^3 f\|_\infty) \tag{10}$$

for all knot sequences with $\phi_{\max}$ small enough, with a constant C independent on f and the knot sequence. $\square$

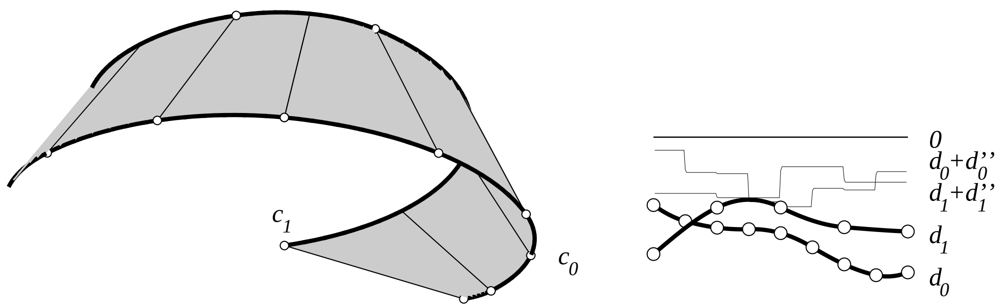

Fig. 2. Left: Approximation of a developable surface by segments of quadratic cones. Right: Support functions d_i of contour curves c_i with their curvature radius functions $d_i + d_i''$.

5.3 Approximation of Curves with Interpolation of Line Elements

We want to approximate the given support function d_c such that both $d_c(a)$ and $d_c'(a)$ are reproduced. This can be done as follows:

Assume that the spaces V_i and W_i are constructed as above on the interval $[a, b]$. Fix an index i and denote the basis functions of V_i by $c_{0i}, c_{1i}, \ldots, c_{n_i i}$. Equation (2) shows that $c_{ki}(a) = 0$ for $j \geq 1$ and $c_{ki}'(a) = 0$ for $k \geq 2$. Let

$$d^* = d_c - (d_c(a)/c_{00}(a))c_{00} \text{ and } d^{**} = d^* - (d^{*\prime}(a)/c_{10}'(a))c_{10} \tag{11}$$

Let $\widetilde{V}_i = \text{span}(c_{2i}, \ldots, c_{n_i i})$ and let $\widetilde{W}_i$ equal the orthogonal complement of $\widetilde{V}_i$ in $\widetilde{V}_{i+1}$. It is easy to find a basis of $\widetilde{W}_i$ The decomposition $\widetilde{V}_n = \widetilde{V}_0 \oplus \widetilde{W}_0 \oplus \cdots \oplus \widetilde{W}_{n-1}$ defines a multiresolution analysis for functions f with $f(a) = f'(a) = 0$.

We therefore approximate d^{**} by a trigonometric spline function in $\widetilde{V}_n$, apply the modified wavelet transform and set all coefficients below some threshold to zero. This gives an approximation $\bar{d}^*$ to d^{**} and

$$\bar{d} = \bar{d}^* + (d_c(a)/c_{00}(a))c_{00} + (d^{*\prime}(a)/c_{10}'(a))c_{10} \tag{12}$$

is an approximation to d_c with the property that $d_c(a) = \bar{d}(a)$ and $d_c'(a) = \bar{d}'(a)$. The algorithm can be modified in an obvious way, if one wants to reproduce the line element at the endpoint b of the interval also.

This can be used to approximate curves with inflection points: Assume that, after some pre-smoothing process, the curve has a discrete set of inflection points which we want to be reproduced after an approximation by arc splines. Now approximate each of the curve's maximal (locally) convex segments separately. In order to fit together, the single arc spline segments have to approximated in such a way that the initial and final line elements are reproduced exactly.

§6. Final Remarks

It should be remarked that the procedure can be applied to dual focal splines as well, because they are defined in terms of trigonometric spline functions.

This makes it possible to define a multiresolution analysis for the special class of rational curves with rational offsets which is studied in [9].

There is also an application to *surfaces*: Two locally convex curves c_1, c_2 in parallel ("horizontal") planes define a developable surface if we join points possessing parallel tangents with straight lines. The multiresolution analysis defined in this paper can be applied to both curves separately and gives a multiresolution analysis for the surface. The approximant will be a developable surface which consists of pieces of quadratic cones all of whose contour lines in horizontal planes are circles. Fig. 2 shows an example.

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