Collision-Free 3-Axis Milling and Selection of Cutting-Tools

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The present paper deals with local and global conditions for collision-free 3-axis milling of sculptured surfaces and the selection of cutting-tools for a given surface. We describe local and global millability results whose proofs have been published in a previous paper. The theoretical background involves general offset surfaces. Here an algorithm is presented which after evaluation of the surface curvature yields a differential inequality for the meridian curve of the cutting-tool, which is fulfilled if and only if the cutting-tool is able to mill the entire surface. The choice or even design of the optimal tool then besides this inequality involves further characteristics of the tools, such as its shape and its size.

Keywords: 3-axis milling, collision avoidance, general offsets, indicatrix, isophotic line, global millability

Recently we have studied the problem of locally and globally collision-free milling of sculptured surfaces [9]. It turns out that if some conditions on the curvature of the surfaces involved are fulfilled, we can show that locally, and in certain cases also globally, no unwanted collision of the cutting-tool with the surface occurs.

The present paper deals with the optimal selection (or even design) of cutting tools in order to mill a given surface. We will present algorithms which allow to

(*i*) test whether or not a given cutter is able to mill a given surface,

(*ii*) select a given number of optimal cutters from a given set of available cutting-tools.

(*iii*) calculate the shape of an 'optimal' cutting-tool which will be able to mill the surface.

The third problem, however hypothetical its engineering applications, is included here because the mathematics which is necessary to solve it will be needed anyway, and also because it is an interesting geometric problem which independently deserves interest.

Local Properties of Smooth Surfaces

We are going to describe the background and mathematical foundations of the local millability test when restricted to smooth surfaces, which consist of C^2 patches with C^1 join. Non-smooth surfaces will be considered in the next section.

We denote the surface by X and the cutter by Σ. X is the boundary of a solid, and we speak of the solid as of the interior of X and will call the ambient space the exterior of X. The cutting-tool is, geometrically, always a convex body of rotational symmetry. The rotation of the cutter around its axis, however important for the mechanical engineering aspect of the problem, can be completely neglected from the geometric point of view. The actual cutter, while rotating, has a surface of revolution as envelope. It is this surface which we consider in this paper. We restrict ourselves to the case of convex cutters, that is, the line segment which joins any two points of Σ is completely contained in Σ. This will be sufficient for most applications. We further assume that Σ is strictly convex, which means that it does not contain parts of planes, cylinders or cones. This condition is often not fulfilled. But there is always a sequence of strictly convex cutters which

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converges to the actual cutter (where convergence is in the sense of the Hausdorff metric).

This is easily seen by replacing planar (cylindrical, conic) parts of the cutter by spherical (toric) parts which approximate them. Because collision checking involves, in principle, only the measurement of distances, and not of derivatives, a close enough approximation will fail the test for millability if the original cutter would have failed, and vice versa. Thus strict convexity is no essential restriction.

If we assume a coordinate system such that the z-axis coincides with the cutter axis a, then the cutter has a parametrization of the form

$$\Sigma : (r(z)\cos(\phi), r(z)\sin(\phi), z) \qquad (1)$$

The curve $r = r(z)$ is called the *meridian curve* of Σ (see Fig. 1).

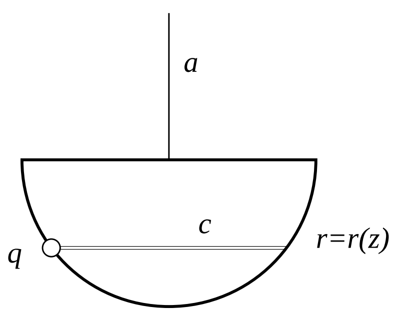

Figure 1: Cutting-tool Σ and meridian curve $r = r(z)$.

While milling the surface X, the surface Σ undergoes a translational motion such that the enveloping surface is just the given surface X. This translation is described by $\Sigma \mapsto \Sigma + s$, where s denotes the vector of the translation. For each position $\Sigma + g$ of Σ, there is a point $p \in X$ and a point $q \in \Sigma$ such that $\Sigma + g$ touches X in the point $p = q + g$. The tangent planes τ_p and τ_q are parallel. For a given p, there are at most two $q \in \Sigma$ with $\tau_p \| \tau_q$ (because Σ is strictly convex), and we are able to choose the right one by noting that X and $\Sigma + g$ lie on different sides of $\tau_p = \tau_q + g$. (see Fig. 2). Because the translation vector g depends on p we denote it by $g(p)$. The point q will be called corresponding to p, or short, $q = q(p)$. Evidently there is no collision if the intersection $(\Sigma + g) \cap X$ is just the point p.

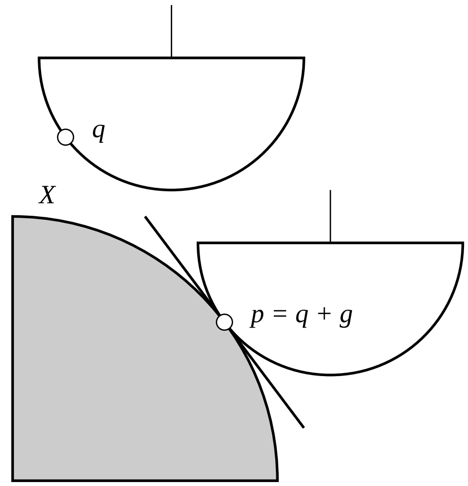

Figure 2: Cutting-tool Σ touching the surface X.

To examine the local behaviour in a neighbourhood of the touching point, we choose a reference plane such that locally both surfaces are graphs of real-valued functions f and s, respectively. Of course this does not mean that the surfaces X and Σ must be given as graphs of real-valued functions. If the tangent plane is not orthogonal to the base plane, it is always possible to re-parametrize both X and Σ locally such that they become graph surfaces.

Because the graphs touch each other, we have for all vectors v equality of first directional derivatives:

$$s_{,v} = f_{,v}. \qquad (2)$$

If additionally we have

$$s_{,vv} > f_{,vv} \text{ for all } v \qquad (3)$$

there is a neighbourhood U of p such that

$$s(x) \geq f(x) \text{ for all } x \in U. \qquad (4)$$

This means that the part of $\Sigma(p)$ which lies 'above' U then does not interfere with X. If there is a direction vector v such that

$$s_{,vv} < f_{,vv} \qquad (5)$$

then obviously Σ intersects the interior of the solid bounded by X. If $s_{,vv} = f_{,vv}$ we cannot tell the local behaviour from the infinitesimal one. The case that always $s_{,vv} \geq f_{,vv}$ but not always $s_{,vv} > f_{,vv}$ almost never occurs, if we have no line contact between the surface X and the cutting-tool. Thus from the engineering standpoint it is no loss of generality if we do not pursue it further. It is well known that $s_{,vv}$ (and analogously, $f_{,vv}$) can be expressed in terms of the Hessian

$$H_s = \begin{pmatrix} s_{,xx} & s_{,xy} \\ s_{,xy} & s_{,yy} \end{pmatrix} \quad (6)$$

as

$$s_{,vv} = v^T H_s v \quad (7)$$

and it is easy to see that [7, 9] Equ. 3 is equivalent to

$$\det(H_s - H_f) > 0 \text{ and } s_{,xx} > f_{,xx}. \quad (8)$$

Definition: The surface X is said to be *locally millable* by the cutter Σ, if Equ. 8 holds. In a point where several C^2 surface patches meet, Equ. 8 has to be satisfied for each of them.

From the considerations above we know that in our definition we excluded some 'boundary' cases between millability and non-millability. As a cutter will almost never be exactly at this boundary, this does not matter very much.

There is an equivalent condition in terms of the Euclidean curvature indicatrices of X and Σ. The indicatrices have the advantage that they are not dependent on the choice of a base plane. We repeat their definition here: Let $p \in X$ and let n be the line perpendicular to X in p. Fix a Cartesian Coordinate system in the tangent plane τ_p. A surface tangent $t(\phi)$ is now determined by its angle ϕ with the x-axis. The plane spanned by n and a surface tangent $t(\phi)$ intersects X in a curve $c(\phi)$ which has a curvature radius $\rho(\phi)$ at p. This radius is given a negative sign if the curve is locally beneath τ_p and a positive positive if it is locally above τ_p.

Definition: The diagram which in polar coordinates has the equation

$$(r, \phi) = (\sqrt{\rho(\phi)}, \phi), \quad (9)$$

whenever the square root is defined, is called the (signed) Euclidean indicatrix of curvature i_p of the surface X at the point p.

It is well known that the indicatrix is void (if $\rho(\phi)$ is negative or infinite for all ϕ), or a pair of lines, or a conic section. Note that all curvature radii are positive and all indicatrices of Σ are ellipses because Σ is convex. For algorithms concerning curve and surface curvature, see also Elber [4].

We define the *interior* of the indicatrix i_p as the star-shaped (with respect to the origin) domain, whose boundary is i_p. It may be the whole plane.

The proof of the following is an exercise in Differential Geometry [3]:

Proposition: A surface is locally millable if and only if for all corresponding points $p \in X$ and $q = q(p) \in \Sigma$ the indicatrix i_q is contained in the interior of the indicatrix i_q (see Fig. 3).

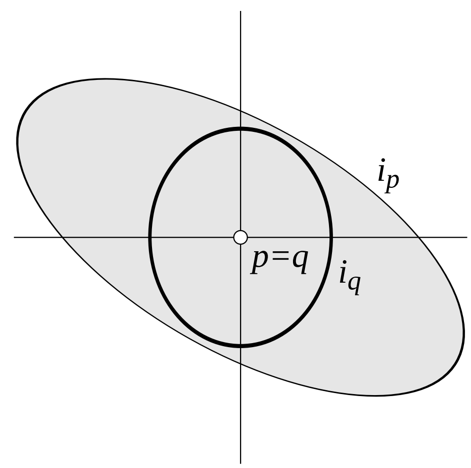

Figure 3: Indicatrix i_q contained in the interior of i_p.

Global Properties of Smooth Surfaces

In order to formulate *global* millability conditions we make use of the following

Definition: The surface Γ which is traced out by an arbitrary fixed point of Σ during the motion of Σ is called *general offset surface* [1, 2, 6, 8] of X with respect to Σ.

All possible general offset surfaces are translates of each other, and one candidate is the surface which is traced out by the translation vector g which we used above. This gives a parametrization $p \mapsto g(p)$ of Γ with X as the parameter domain.

Using elementary methods of geometric topology (degree of maps, homotopy, covering maps), it is possible to show [9] that the following is true:

Proposition: Let X be a smooth surface consisting of C^2 patches.

(i) If X is locally millable, the parametrization of Γ is regular.

(ii) If the surface Γ has no self-intersections, then X is globally millable by Σ.

(iii) Let X be such that it can be re-parametrized as the graph surface of a compactly supported smooth function defined in the entire plane. If X is locally millable, then also globally.

(iv) Let X be such that it can be re-parametrized as the graph surface of a smooth function defined in the entire plane. If Σ possesses steeper tangent planes than X (This is always the case if Σ has an equator circle), then the local millability of X implies the global millability.

(v) Let X be such that it can be seen as the graph surface over a piecewise smoothly bounded planar domain D. If the 'top view' of Σ is a closed symmetric convex domain S and the general outer parallel curve $D + S$ is free of self-intersections, then the local millability of X implies the global millability. (In most cases S is a disk and the condition is easily verified. See Fig. 4 for an example).

(vi) If X is strictly star-shaped with respect to an interior point, then local millability implies global millability.

(vii) In cases (iii) to (iv), the general offset surface Γ of X with respect to Σ is free of self-intersections.

In most applications we have one of the cases listed in the proposition, most frequently perhaps case (iv) (see Fig. 4).

This proposition, if slightly modified, will be later seen to hold also in the case of non-smooth surfaces.

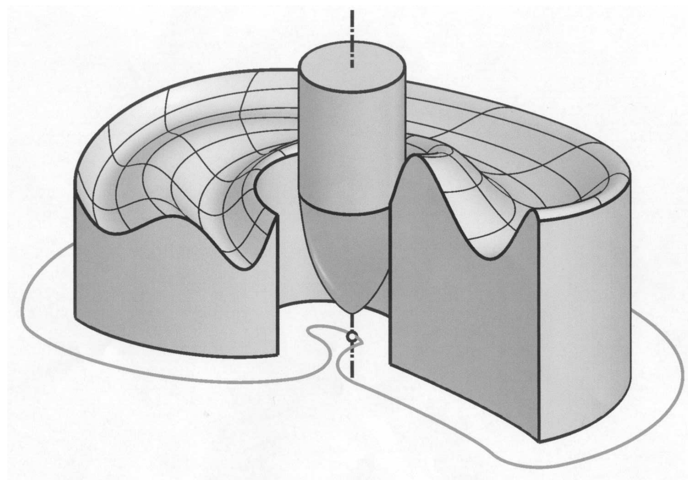

Figure 4: Milling a surface with boundary.

Non-smooth Surfaces

If the surface is not smooth, but continuous and piecewise C^2, we are still able to given local and global millability conditions. The first method to overcome the non-smoothness is to consider instead of X an outer parallel surface $X + \varepsilon B$, where B is the Euclidean unit ball of Euclidean three-space. Because collision tests do not involve differentiation, the limit $\varepsilon \to 0$ gives the exact result.

Another possibility, which is better for computational purposes, is the following: An edge e where X is not smooth, is either a ridge or a valley. Valleys can never be milled exactly by smooth cutters, so we leave them aside. A ridge has in each of its points p a wedge $\mathcal{T}_p$ of admissible tangent planes τ_p. For $\tau_p \in \mathcal{T}_p$ there is a corresponding point $q(p, \tau_p)$ of Σ. Then obviously X can be milled at p if Equ. 8 or an analogous condition holds outside the edge and if the edge stays outside $\Sigma(p, \tau_p)$ with the exception of the point q itself.

In terms of curvatures of e and Σ this can be expressed as follows [3]: Denote the osculating plane of e at p by ε and the radius of curvature of c by R. The edge tangent at p will be denoted by t. For all admissible tangent planes τ_p do the following: Choose the coordinate system in τ_p such that the x-axis is horizontal. Let $\phi = \angle(x, t)$ and $\psi = \angle(\varepsilon, \tau_p)$. Draw the indicatrix i_q and the two points (in polar coordinates)

$$P_1\left(\sqrt{R/(\sin\psi)}, \phi\right), \quad P_2 = -P_1. \tag{10}$$

If P_1 and P_2 are outside i_q for all τ_p the surface is locally millable at p.

We define the region which is shown grey in Fig. 5 as a substitute 'interior' of i_p. It is easy to see that P_1 and P_2 lying outside i_q is equivalent to i_q lying in the substitute interior of i_p, because the rotational symmetry of Σ implies that the x- and y-axis are the major axes of i_q. The reader may ask why we did not choose the

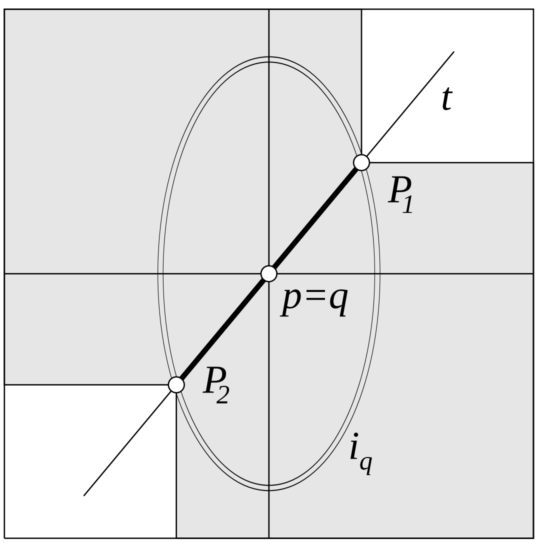

Figure 5: Indicatrix i_q and subsitute i_p in case of an edge with tangent t.

part of the line $[P_1, P_2]$ which is not between P_1 and P_2 as the indicatrix i_p, which would be the limit of the indicatrices when considering an outer parallel surface $X + \varepsilon B$ and letting ε tend to zero. The reason is that out version of the interior of i_p is better to implement (see later).

It can be shown [9] that here also the local millability implies the global one, in all cases listed in the proposition above.

Test for Millability of a Smooth Surface

In this section we are going to describe how to test whether a given cutting-tool is able to mill the surface X. The cutting-tool contains circles in parallel ('horizontal') planes, which will be called *parallel circles* (see Fig. 1, where a parallel circle, denoted by c, is shown). The tangent planes τ_p of the points q of such a circle c enclose the same angle

$$\psi = \angle(\tau_q, a) \tag{11}$$

with the axis a of the tool. The points of the surface X which will during the manufacturing process be in contact with the points of the circle c, are precisely the points on the *isophotic line* l_ψ which belongs to the angle ψ [5] and the direction of a. The curve l_ψ is defined as the set of points p of X whose tangent planes τ_p enclose the angle ψ with the axis a. It can be shown that the differentiability class of l_ψ is one less than the differentiability class of X, so the notion 'isophotic line' or 'curve' is justified. An example can be seen in Fig. 6.

As discussed above, the condition for collision-free manufacturing of X can be expressed in terms of the Euclidean curvature indicatrices i_p and i_q of corresponding points $p \in \Sigma$ and $q = q(p) \in X$: The surface is locally (and hence, globally) millable if and only if the indicatrix i_q is contained in the interior of i_q.

The indicatrices of all points q of a parallel circle c are the same (up to rotation of Σ), so we can speak of *the* indicatrix $i_q = i_c$ of the points of c. The connection between the various tangent planes of c is given by the cutter's rotation around the axis a. The connection between the various tangent planes along the curve l_ψ is given by the condition that when moving along l_ψ, the horizontal line in the tangent plane stays horizontal.

This now makes it possible to re-formulate the condition that for all points $p \in l_\psi$ the interior i_p must contain i_c, as follows: We identify all tangent planes along l_ψ and intersect the interiors of the indicatrices i_p. This gives the region I_ψ.

$$I_\psi = \bigcap_{p \in l_\psi} \operatorname{int} i_p \tag{12}$$

Local millability is now equivalent to

$$i_c \subset I_\psi. \tag{13}$$

We are going to express this condition in terms of the curvature of the meridian curve $r = r(z)$ of the cutter. It is well known that the indicatrix $i_q = i_c$ is an ellipse whose major axes lie in the horizontal and in the gradient line through q. The length of the horizontal axis is

$$a = \sqrt{r / \cos\psi}. \tag{14}$$

The length of the other axis equals

$$b = \sqrt{\rho_m}, \tag{15}$$

where ρ_m is the radius of curvature of the meridian curve $r = r(z)$. Thus, for given r and ψ, we have an inequality

$$\rho_m \leq \rho_{\max}(r, \psi), \tag{16}$$

where $\rho_{\max}$ is determined such that $\sqrt{\rho_{\max}}$ is the largest 'vertical' major axis of an ellipse which is contained in I_ψ.

Test for Millability of a Non-Smooth Surface

First we have to say something about the isophotic lines in the presence of edges. For this purpose it is best to think of l_ψ as a set of surface elements (p, τ_p), where a surface element is a pair point – tangent plane. This makes it possible to simplify the notation in cases where a point has more than one tangent plane.

If p is situated in an edge e of X, there is a wedge $\mathcal{T}_p$ of admissible tangent planes. There are up to two planes in $\mathcal{T}_p$ which enclose the angle ψ with the axis a. Thus the isophotic line l_ψ can contain up to two surface elements (p, τ_p). For each of them we have already defined the substitute indicatrix $i_{p,\tau}$. Of course this has nothing to do with a Euclidean indicatrix of curvature except for the fact that it is still true that the indicatrix $i_{q,\tau}$ must be contained in the interior of $i_{p,\tau}$, if (q, τ) is the surface element of Σ which corresponds to the surface element (p, τ_p) of X.

The test for millability now runs in exactly the same way as in the smooth case. For all angles ψ we have to test if i_c is contained in I_ψ, where c is the parallel circle which belongs to the angle ψ, and I_ψ is defined as

$$I_\psi = \bigcap_{(p,\tau)\in l_\psi} \text{int}\, i_{p,\tau} \tag{17}$$

Implementation

In this section we are going to describe how to implement the intersection of the interiors of the various indicatrices along an isophotic line i_ψ.

All indicatrices $i_{p,\tau}$ are star-shaped curves and their interiors are star-shaped domains in the plane. To intersect them, we write $i_{p,\tau}$ in polar coordinates (r, ϕ) as in Equ. 9: $r = r_p(\phi)$. The interior of the indicatrix $i_{p,\tau}$ then is written as

$$\text{int}\, i_p : r < r_p(\phi) \tag{18}$$

and I_ψ has the equation

$$I_\psi : r(\phi) < \min_{(p,\tau)\in l_\psi} r_p(\phi). \tag{19}$$

This is easy to implement: We choose discrete rays emanating from the origin by prescribing a discrete set $\phi_0 < \phi_1 < \phi_2 < \ldots$ of angles and discrete set of surface elements $(p_0, \tau_0), (p_1, \tau_1), (p_2, \tau_2), \ldots$. For all i we are buffering the minimum value of $r_p(\phi_i)$. This is an efficient way to calculate the domain I_ψ. In Figures 6 and 7 you can see a discrete number of isophotic lines l_ψ together with the boundaries of the domains I_ψ.

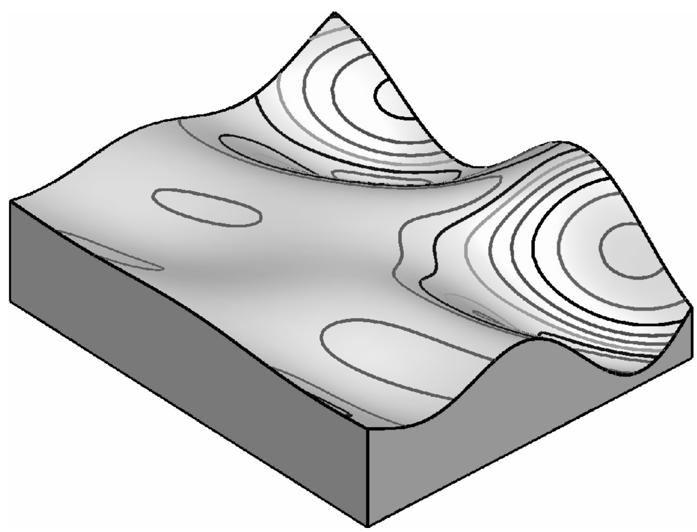

Figure 6: Isophotic lines l_{ψ_i} of the surface X.

To test a given cutter-surface pair for local millability (and, in the cases listed in the proposition above, also global millability), there is the following

Algorithm: (i) Choose a discrete set of angles $\psi_0, \psi_1, \ldots$ such that the corresponding parallel circles $c_0, c_1, \ldots$ (if $q \in c_i$, then $\angle(\tau_q, a) = \psi_i$) are distributed evenly on c_i.

(ii) Choose a discrete set of angles $\phi_0, \phi_1, \ldots$ such that the rays having angle ϕ_i with the x-axis in a planar

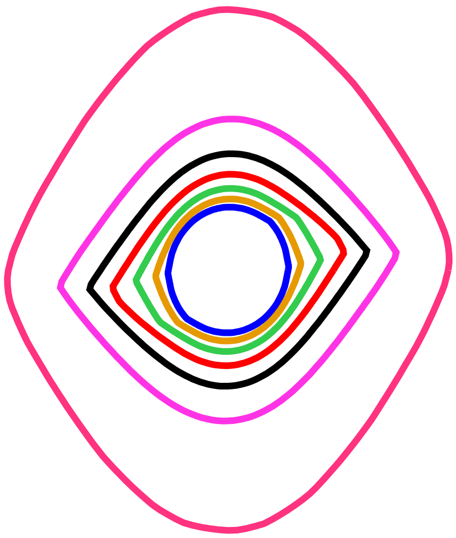

Figure 7: Boundary curves of the sets I_{ψ_i} which belong to the isophotic lines of Fig. 6

coordinate system are distributed evenly in the plane.

(*iii*) For all i, choose surface elements (p, τ_p) evenly distributed on the isophotic line l_{ψ_i}.

(*iv*) For all (p, τ_p), calculate the indicatrix of curvature. Formulae are given above and can be found in many differential geometry textbooks [3].

(*v*) For all i, calculate the domain I_{ψ_i} and the indicatrix i_{c_i} of an arbitrary point of c_i. Test whether or not i_{c_i} is in I_{ψ_i}. If this is the case for all i, the cutter is able to mill the surface locally, and in the cases listed in the above proposition, also globally.

Suppose we have given a set of cutters and we have to choose one of them to mill the given surface X. This choice depends on the properties of the cutters which are, after performing the test above, now known to be able to perform the manufacturing process. Depending on the material, the shape, the size, and other properties of the cutters, we can assign to each cutter a number signifying how 'good' or 'bad' it would be to use this cutter as a tool.

A factor which contributes the the goodness or badness is the required surface quality of X: The smaller the curvature radii of the cutter, the denser the curves must be which it is moving along. Thus one would prefer the 'larger' cutters, but on the other hand they more often fail to mill the whole surface without collisions.

If it is cheap (in the sense of time and cost) to change the cutting-tool it may be desirable to use two or more cutters to mill the surface X. By testing in a finite number of points we are able to estimate how many percent of X the given cutters are able to mill. Thus it is possible to choose the cutting tools such that first one does most of the surface, and the second one finishes in the parts of X which the first one could not reach without collision with X.

If we choose the region which the first cutter has mill such that it fulfills the condition of case (*iv*) of the proposition about global millability results, we are again able to apply our results and have therefore avoided the constant collision checking, which otherwise would be necessary.

If X or Σ is rotated, translated, or scaled, the indicatrices undergo the same transformation. This can be used when one has to test a set of cutters which are just differently scaled versions of each other. Essentially it is only necessary to test one of them.

Cutter design

In this section we are finally going to describe how to find an *optimal cutter* in the sense that it is able to mill the surface X, but has the largest possible curvature radii. This is done as follows: The cutter is described by the meridian function $r = r(z)$. The active region of Σ, which lies below its equator circle, can equivalently be described by

$$z = z(r). \tag{20}$$

Because Σ is convex, we can use the tangent slope

$$k = \frac{dz}{dr} \tag{21}$$

as a parameter and describe Σ by the functions

$$r = r(k) \text{ and } k = k(r). \tag{22}$$

The function $z = z(r)$ is recovered from $k = k(r)$ by integration. Because of

$$\frac{d^2z}{dr^2} = \frac{dk}{dr}, \tag{23}$$

the first derivative of the function $k(r)$ already determines the curvature of the meridian curve.

The radius of curvature of the meridian curve can be expressed, as is well known, by the first and second derivative of $z = z(r)$ as follows:

$$\rho = \frac{\sqrt{(1 + (dz/dr)^2)^3}}{d^2z/dr^2} = \frac{\sqrt{(1 + k^2)^3}}{dk/dr} \qquad (24)$$

The tangent slope k of a point q and the angle $\psi = \angle(\tau_q, z)$ are connected by

$$\tan\psi = \frac{1}{k}. \qquad (25)$$

If both k and r are given, there is a maximal radius $\rho_{\max}$ (see Equ. 16) such that the indicatrix, which is an ellipse with horizontal major axis $\sqrt{r/\cos\phi}$ and vertical major axis $\sqrt{\rho_{\max}}$, touches the boundary of I_ψ. The admissible curvature radii ρ of the meridian curve therefore have to fulfill the inequality

$$\rho(k) \leq \rho_{\max}(k, r(k)) \qquad (26)$$

This gives the first order differential inequality

$$\frac{dk}{dr} \leq \frac{\rho_{\max}(k, r(k))}{\sqrt{(1 + k^2)^3}} =: F(k, r), \qquad (27)$$

If we choose, for all k, the maximal possible curvature radius, we get the differential equation

$$\frac{dk}{dr} = F(k, r), \qquad (28)$$

whose solution $k = k(r)$ gives the meridian curve $z = z(r)$ of the *optimal* cutting-tool. Fig. 8 shows an example. In this case the global millability is guaranteed, because the conditions of the proposition above are fulfilled.

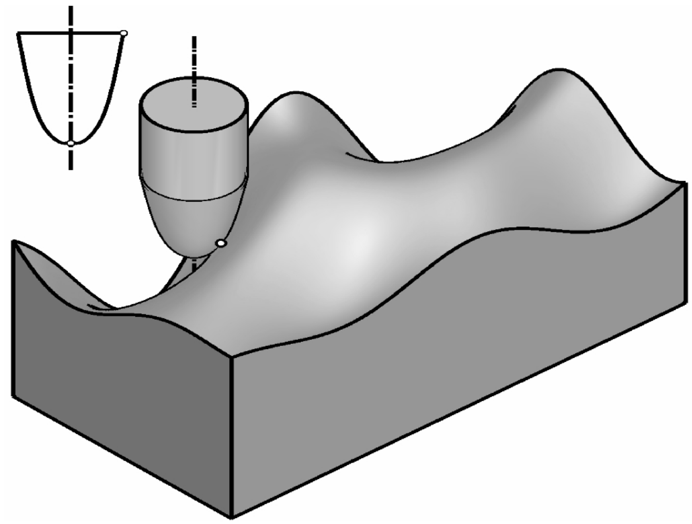

Figure 8: Surface and optimal cutting tool.

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