Self-Intersections and Smoothness of General Offset Surfaces

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A convex body N moves such that it touches a closed surface M. While doing this, it is undergoing a purely translational motion. A fixed point of N traces out the *general offset surface* γ during this motion. We study the connection between singularities and self-intersections of γ and the possible collisions of M with N during this motion and obtain some global results.

1 Introduction and Definitions

We have the following problem: Given is a solid M in space, which is piecewise curvature continuous, but may have curvature discontinuities along curves, and even edges. There is also given a strictly convex body N. We try to find conditions on M and N which characterize the situation that N can move in a purely translational manner such that it always touches M, the interiors of M and N are distinct, and such that the envelope of N during this two-parameter motion is precisely M? In [6] and [8] the problem is studied in the context of three-axis milling of sculptured surfaces.

We begin with a definition of 'strictly convex' which is slightly different from the usual one:

Definition: A *solid elliptic paraboloid* is a nonsingular affine image of the set $x_n \geq x_1^2 + \ldots + x_{m-1}^2$. A *strictly convex body* K is a compact convex subset of $\mathbb{R}^m$ which has the property that for all $p \in \partial K$ there is a *supporting paraboloid* P with $p \in \partial P$ and $K \subseteq P$. The boundary ∂K will be called *strictly convex surface*.

We want to study solids and their boundaries (surfaces) at the same time. We also want to include non-smooth surfaces. On the other hand we are studying *offset* surfaces and we know that concave edges always produce singularities in the offset surfaces (see Fig. 1). This leads to the definition of admissible solids below.

Recall that a *convex pyramid* P of $\mathbb{R}^m$ is a subset of $\mathbb{R}^m$ with the property that $x, y \in P$ implies $\lambda x + \mu y \in P$ for all $\lambda, \mu \geq 0$. We define:

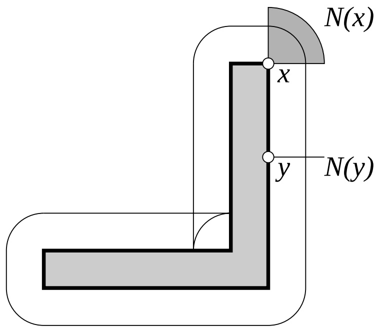

Figure 1: Offset curve with singularity at concave vertex and normal pyramids $N(x)$

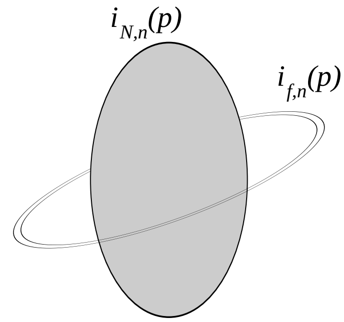

Figure 2: The indicatrix $i_{f,n}(p)$ is contained in the exterior of $i_{N,n}(p)$

Definition: An *admissible solid* M is a compact subset of $\mathbb{R}^m$ with the property that for all $x \in M$ there is a local C^1 diffeomorphism ϕ with $d\phi(x) = \mathrm{id}$, which takes M to a convex pyramid $T(x)$ with vertex x. This pyramid is called *tangent pyramid* of x. The points of the various d-dimensional faces of this pyramid correspond to points of the various d-dimensional *boundary surfaces* of M. The *normal pyramid* $N(x)$ of a point x of an admissible solid M is defined as the pyramid orthogonal to $T(x)$.

If $T(x)$ is entire $\mathbb{R}^m$, then x is an interior point, otherwise it is a boundary point. If $T(x)$ is a half-space, its boundary is the usual tangent plane at x.

Our definition of the normal pyramid assigns to an interior point the singleton $\{(x; 0)\}$, to an 'ordinary' boundary point the ray pointing to the outside of M, to a point of an 'ordinary' edge a wedge in the plane perpendicular to the edge, and so on (see Fig. 1).

In order to study offset surfaces, we parametrize the set of unit surface normals, which is, by definition, the union of the unit vectors in the normal pyramids $N(x)$. We exploit the fact that locally all normal pyramids $N(x)$ are diffeomorphic. This is true for all boundary surfaces because it is true for pyramids.

Definition: The set of unit vectors in $N(x)$ is denoted by $N_1(x)$. The set $\perp M = \bigcup_{p \in M} N(p)$ is called the *one-sided normal bundle* of M, and the set $\perp_1 M = \bigcup_{p \in M} N_1(p)$ is called the one-sided *unit* normal bundle of M.

If a surface N is smooth, then for every p there is exactly one n such that $(p; n) \in \perp_1 N$. Thus $\perp_1 N$ and N can be identified.

$\perp M$ and $\perp_1 M$ are subsets of the tangent bundle of $\mathbb{R}^m$, which is identified with $\mathbb{R}^{2m}$. When we speak of differentiable mappings with values in $\perp M$ [or $\perp_1 M$], we mean differentiable mappings into $\mathbb{R}^{2m}$ whose image is contained in $\perp M$ [or $\perp_1 M$]. The topology and differentiable structure of $\perp M$ [or $\perp_1 M$] is that of subset of $\mathbb{R}^{2m}$. Let $f : K \to M$ be a compact boundary d-surface. The unit normal bundle over this boundary surface is a continuous 1-1 image of $K \times (S^{m-1} \cap N)$ where N is a model normal pyramid for the points of this boundary surface. Thus $\perp_1 M$ is *compact* and consists of pieces homeomorphic to the various $K \times (S^{m-1} \cap N)$ glued together (It can be shown that $\perp_1 M$ is in fact homeomorphic to M, which is trivial if M is smooth).

A compact C^r d-surface $f : K \to \mathbb{R}^m$ is a mapping of a compact $K \subset \mathbb{R}^d$ to $\mathbb{R}^m$ with the property that there is an open subset U with $K \subseteq U$ and a C^r mapping $\bar{f} : U \to \mathbb{R}^m$ with $\bar{f}|K = f$.

Definition: A compact C^r boundary d-surface of M is a compact C^r d-surface $f : K \to \mathbb{R}^m$ which parametrizes a part of a d-dimensional boundary surface of M. An *admissible piecewise C^r solid* M is an admissible solid all of whose finitely many boundary surfaces are compact C^r surfaces (boundary surfaces need not be edges of M, it is also possible that pieces of ∂M which are C^r are separated by a boundary surface, and their union is C^s with $1 \leq s < r$).

Let $f : K \to \mathbb{R}^m$ be a compact boundary d-surface of an admissible solid, and let N by a pyramid diffeomorphic to an $N(x)$, $x = f(u)$. A C^r *surface normal parametrization* is a C^r parametrization $n : K \times (N \cap S^{m-1}) \to \perp_1 f(K)$ which is one-to-one and onto.

Of course a surface which we only know to be C^1 need not have a C^1 surface normal parametrization. We will however encounter surfaces which are C^1 but have a piecewise C^1 surface normal parametrization. Such surfaces are in fact essentially C^2:

Lemma 1 *Let f be a regular C^1 hypersurface with C^1 surface normal parametrization. Then in the neighborhood of every point there is a local C^1 change of parameters which makes f a C^2 surface.*

Proof: Let $p_0 = f(x_0)$. There is a C^1 change of parameters $\phi : V \to U$ such that after a suitable choice of coordinate system in $\mathbb{R}^m$, we have

$$f \circ \phi(u_1, \ldots, u_{m-1}) = (u_1, \ldots, u_{m-1}, z(u_1, \ldots, u_{m-1})). \tag{1}$$

The unit normal vector at $f \circ \phi(u)$ then equals

$$\bar{n}_0(u) = \bar{n}(u)/\|\bar{n}(u)\| \text{ with } \bar{n}(u) = (-z_{,1}(u), \ldots, -z_{,n}(u), 1), \tag{2}$$

where $z_{,i} = \partial z/\partial u_i$. On the other hand there is the C^1 surface normal parametrization $n \circ \phi : V \to \mathbb{R}^m$, $u \mapsto (n_1(u), \ldots, n_n(u))$. Because there is only one surface normal, we have $n(u) = \lambda \bar{n}(u)$ and the C^1 function $n_j(u)/n_n(u)$ equals $-\lambda z_j(u)/\lambda z_n(u) = -z_j(u)$ for $j = 1, \ldots, m-1$, which shows that all partial derivatives $z_{,1}, \ldots, z_{,m-1}$ are C^1, and $f \circ \phi$ ist therefore C^2. □

2 Local Interference of Surfaces

We want to study the case where a strictly convex C^2 body N touches the admissible solid in a point p. The restriction to strictly convex bodies in the sense of our definition above is essential for the analytic apparatus, because we need that the spherical mapping of N is regular. The collision problem however is no 'differentiable' problem at all, and it is easy to approximate (in the sense of the Hausdorff metric) a convex body M by a strictly convex body M' whose collision behaviour is arbitrarily close to that of M.

The point of contact is an interior point of exactly one d-dimensional boundary surface of M and a boundary point of other $(d+k)$-dimensional boundary surfaces (in order to avoid complicated formulations, we call the only element of a one-point boundary surface its interior point).

The *second fundamental form* $\mathrm{II}_n(v,w)$ of a C^2 d-surface $f : U \to \mathbb{R}^m$ with respect to a normal vector n is defined by the scalar product $\mathrm{II}_n(v,w) = f_{,vw} \cdot n$. If $n : U \to \mathbb{R}^m$ is a normal vector field containing n, then differentiating $f_{,v} \cdot n = 0$ with respect to a tangent vector w gives

$$\mathrm{II}_n(v,w) = -f_{,v} \cdot n_{,w}. \tag{3}$$

If f is not C^2, then (3) serves as a *definition* of II_n. There is the following lemma:

Lemma 2 *If f is a C^1 surface with C^1 normal vector parametrization, then (3) defines a symmetric bilinear form.*

Proof: If f is a C^1 hypersurface of dimension $m-1$, there is a local diffeomorphism ϕ such that $f \circ \phi$ is C^2. Denote its second fundamental form by $\overline{\mathrm{II}}_n$. Obviously $f_{,d\phi(v)} = (f \circ \phi)_{,v}$ and therefore $\overline{\mathrm{II}}_n(v,w) = \mathrm{II}_n(d\phi(v), d\phi(w))$. $d\phi$ is an isomorphism and therefore II is symmetric.

If f is a C^1 d-surface $U \to \mathbb{R}^m$ with C^1 surface normal parametrization, let $n : U \to \mathbb{R}^m$ be a C^1 vector field of f which contains the given vector n. Define a hypersurface $\bar{f} : U \times \mathbb{R}^{m-d-1} \to \mathbb{R}^m$ by

$$\bar{f}(u_1, \ldots, u_{m-1}) = f(u_1, \ldots, u_d) + \sum_{d<i<m} u_i n_i(u_1, \ldots, u_d), \tag{4}$$

where $n_i(u)$ $(d < i < m)$ are linearly independent C^1 vectors fields orthogonal to $[f_{,1}(u), \ldots, f_{,d}(u), n]$. They are found by Gram-Schmidt orthonormalization applied to $B(u) = \{f_{,1}(u), \ldots, f_{,d}(u), n(u), n^{(0)}_{d+1}, \ldots, n^{(0)}_{m-1}\}$, where the $n^{(0)}_i$ are constant. Then the C^1 unit normal vector field of $\bar{f}$ coincides with n when restricted to $U \times 0$ and obviously the second fundamental forms II and $\overline{\mathrm{II}}$ of f and $\bar{f}$, respectively, have the property that $\overline{\mathrm{II}}_n(v,w) = \mathrm{II}_n(v,w)$ for all v, w tangent to f. Thus also II_n is symmetric. □

Definition: Let f be a d-surface in $\mathbb{R}^m$ such that its second fundamental form II is defined, let p a surface point and n a unit normal vector at p. For all tangent vectors $v \in T_pf \setminus 0$, we calculate the normal curvature radius $\rho_n(v) = \mathrm{I}(v)/\mathrm{II}_n(v)$ with respect to n, and consider the set

$$i_{f,n}(p) = \{\pm\sqrt{\rho_n(v)}/\|v\| \cdot v \mid v \neq 0, \quad 0 < \rho_n(v) < \infty\} \subset T_pf. \tag{5}$$

This set is called the *signed indicatrix of curvature* of f at p and n.

Consider a translate of N which touches M in a point of contact p with common normal vector n pointing to the inside of N. We consider the signed indicatrix $i_{N,n}(p)$ and the indicatrices $i_{f,n}(p)$ for all boundary surface patches f whose image contains p.

Lemma 3 *In a touching position with common surface normal n at the common point p, the two solids M and N do not interfere locally, if all indicatrices $i_{f,n}(p)$ are contained in the exterior of the indicatrix $i_{N,n}(p)$ (see Fig. 2). Here f ranges over all boundary surfaces of M whose image contains p.*

Note that the empty set is contained in any other set, and that $i_{N,n}$ is always a non-void oval quadric, because N is strictly convex.

Proof: If the boundary surface f in question is a hypersurface, Lemma 1 shows that we can write both f and N as graphs of C^2 functions z_f, z_N, respectively, over the common tangent plane. Consider the difference surface defined by the graph of $z = z_N - z_f$. The assumption on the indicatrices implies that the quadratic form $\mathrm{II}^N - \mathrm{II}^f$, whose matrix in local coordinates $u_1, \ldots, u_{m-1}$ is the matrix $z_{,ij}$, is positive definite. The Lemma of Morse [7] implies that $z(u_1, \ldots, u_{m-1}) = \sum_{i,j=1}^{m-1} u_i u_j h_{ij}(u_1, \ldots, u_{m-1})$ with continuous functions h_{ij}, which have the property that $h_{ij}(0) = z_{,ij}$ and $h_{ij}(u) = h_{ji}(x)$.

All principal minors of the matrix $h_{ij}(0)$ are positive, so there is an $\varepsilon > 0$ such that all principal minors of $h_{ij}(u) > 0$ for all u with $\|u\| < \varepsilon$. Thus $z(u) > 0$ if $\|u\| < \varepsilon$, $u \neq 0$, and f does not interfere with N locally.

If $f : U \to \mathbb{R}^m$ is a d-surface with $d < m - 1$ then analogously to the proof of Lemma 2 we choose a normal vector field $n : U \to \mathbb{R}^m$ of f which contains n and extend f to a hypersurface $\bar{f} : U \times \mathbb{R}^{m-1-d}$, $\bar{f}(u) = f(u_1, \ldots, u_d) + \sum_{d<i<m} u_i n_i(u_1, \ldots, u_d) - n(u_1, \ldots, u_d) \cdot C \sum_{d<i<m} u_i^2$. Then $\bar{f}|U \times 0 = f$, $\bar{f}$'s second fundamental form coincides with that of f, and if C is large enough, the difference form above is still positive definite. □

Definition: If M is a C^1 admissible solid with piecewise C^1 surface normal parametrization, and the conditions of Lemma 3 are fulfilled for all pairs (p, q), $p \in M$, $q \in N$ with a common tangent plane, and for all boundary surfaces meeting at p and q, then we call the solid M *locally millable* by the convex body N.

Lemma 4 *If the local millability criterion is fulfilled in p, then also in a neighborhood of p.*

Proof: This is clear from the fact that the indicatrices $i_{f,n}(p)$ vary continuously with p, as all surfaces involved are C^1 with C^1 surface normal parametrization. □

3 General Offset Surfaces and their Smoothness

The mapping $\pi_1 : \perp M \to M$ denotes the mapping which maps a normal vector to its footpoint, and the the *spherical mapping* σ_M maps a unit normal vector $(p; n) \in \perp_1 M$ to $n \in S^{m-1}$. The mapping σ_N is defined analogously, but here we can identify $\perp_1 N$ with N.

Definition: The *relative* spherical mapping is defined by

$$\sigma_{M,N} : \perp_1 M \to N, \quad \sigma_{M,N} = \sigma_N^{-1} \circ \sigma_M. \tag{6}$$

and the mapping

$$\gamma_{M,N} : \perp_1 M \to \mathbb{R}^m, (p; n) \mapsto \pi_1(p; n) - \sigma_{M,N}(p; n) \tag{7}$$

is called the *general offset surface* of M with respect to N.

Obviously $\sigma_{M,N}$ is C^1 when restricted to surfaces with a C^1 surface normal parametrization.

The meaning of $\gamma_{M,N}$ is the following: Because N is strictly convex, for all $(p; n) \in \perp_1 M$ there is a unique $a \in \mathbb{R}^m$ such that $a + N$ touches M at $(p; n)$. The vector a is given by $\gamma_{M,N}(p; n)$ and the surface $\gamma_{M,N}(M)$ is the $(m-1)$-parameter set of positions of the origin during a motion of N such that it touches M during this motion.

Theorem 1 *If M and N fulfill the local millability condition, then the parametrization of $\gamma_{M,N}$ is regular and orientation-preserving when restricted to boundary d-surfaces $f : U \to \mathbb{R}^m$. Further it is locally injective, and the tangent space $T_{(p;n)}\gamma_{M,N}$ is orthogonal to n.*

Proof: We consider a normal vector $(p; n) \in \perp_1 f$. The subspace $V = \ker(d\pi_1)$ is tangent to the submanifold of all vectors $(p; n)$ with fixed p. It clearly is mapped in a one-to-one manner by $d\sigma_M$ onto a subspace W of $T_n S^{m-1}$.

Because $T_n S^{m-1} \perp n$, the translation which maps p to n maps $T_p f$ onto a subspace W' of $T_n S^{m-1}$. Obviously $T_n S^{m-1} = W \oplus W'$. Let $V' = d\sigma_M^{-1}(W')$. Then $T_{(p;n)}(\perp_1 M) = V \oplus V'$.[1]

Consider $v' \in V' \setminus 0$ and assume $d\gamma_{M,N}(v') = 0$. This implies

$$d\sigma_N d\pi_1(v') = d\sigma_M(v'). \tag{8}$$

Note that $u = d\pi_1(v') \neq 0$ because $v' \notin \ker(d\pi_1)$. Equ. (8) and $\mathrm{II}_n(u) = -u \cdot d\sigma(v')$ imply $\mathrm{II}_{n,N}(u) = \mathrm{II}_{n,M}(u)$ and further

$$\mathrm{II}_{n,N}(u)/\mathrm{I}(u) = \mathrm{II}_{n,M}(u)/\mathrm{I}(u). \tag{9}$$

But this means equality of normal curvatures in direction u, which contradicts our assumption. Thus also V' is mapped in a one-to-one manner and $d\gamma_{M,N}$ is nonsingular.

[1] The geometric meaning of V' is this: a curve $(p(t); n(t))$ in $\perp_1 M$ with tangent vectors contained in the respective subspaces V' is a *parallel field* of normal vectors $n(t)$ along the curve $p(t)$ in the one-sided normal bundle.

Because all tangent spaces to M, N, and S^{m-1} involved in the calculation are orthogonal to n, so is $T_{(p;n)}\gamma_{M,N}$.

N is a compact convex body and can therefore be shrunk to a point by a simple scaling λN, $0 \leq \lambda \leq 1$. As the indicatrices of λN are just the scaled versions of the indicatrices of N, the local millability criterion is fulfilled for all λN. Thus the mapping $\Gamma : (p; n, \lambda) \mapsto \gamma_{M,\lambda N}(p; n)$ is a homotopy of regular mappings between $\gamma_{M,N}$ and the identity map, which shows that $\gamma_{M,N}$ is orientation-preserving.

Suppose that $\gamma_{M,N}(p_1) = \gamma_{M,N}(p_2)$. Then there is a position of N such that N touches M in both p_1 and p_2. The local millability condition asserts that the points of contact are isolated, and $\gamma_{M,N}$ is locally injective. □

Theorem 2 *If M and N fulfill the local millability condition, and $\gamma_{M,N}$ is injective, then the general offset surface $\gamma_{M,N}(\perp_1 M)$ is a C^1 submanifold of $\mathbb{R}^m$.*

Proof: Choose $(p; n) \in \perp_1 M$ and consider all surface normal parametrizations f_i whose domains K_i contain p. All f_i can be extended to an open domain $U_i \supset K_i$. In a neighborhood of $q = \sigma_{M,N}(p)$ we can write N as a graph surface of the function $g(x_1, \ldots, x_{m-1})$ over the tangent hyperplane $T_q N$. We want to show that also $\gamma_{M,N}(\perp_1 M)$ can locally be written as a C^1 graph surface over $T_q N$. Denote the orthogonal projection onto $T_q N$ by π_1. Its complementary projection is denoted by $\pi_2 : \mathbb{R}^m \to \mathbb{R}$, where $\mathbb{R}$ is identified with the surface normal in q.

Let $\gamma_i = \gamma_{M,N} \mid U_i$. Then $\pi_1 \circ \gamma_i$ is C^1 and regular, because $T_{(p;n)}\gamma_{M,N} = T_{(p;n)}\gamma_i$ is parallel to $T_q N$ for all i. The local mappings

$$g_i(x) = \pi_2 \circ \gamma_i \circ (\pi_1 \circ \gamma_i)^{-1} \tag{10}$$

are C^1. In a neighborhood of p the mapping $\pi_1 \circ \gamma_{M,N}$ is injective, because two $\gamma_{M,N}$-images $(x_1, \ldots, x_{m-1}, z')$ and $(x_1, \ldots, x_{m-1}, z'')$ with $z'' > z'$ in $\gamma_{M,N}(M)$ would mean that two translates N', N'' of N touch M, and we have $N'' = N' + (0, \ldots, 0, z'' - z')$. But $N'' \setminus N'$ does not have any points near p, so this assumption is a contradiction to Lemma 4.

The mapping $\pi_1 \circ \gamma_{M,N}$ is also locally surjective, because first $\gamma_{M,N}$ is an injective continuous mapping of the compact space $\perp_1 M$, and therefore a homeomorphism onto its image, and second, as $\gamma_{M,N}$ consists of finitely many γ_i all of which are C^1 with 'horizontal' tangent plane, it is Lipschitz continuous in a neighborhood of p and therefore cannot omit 'vertical' lines arbitrarily near p.

Thus it makes sense to define

$$g(x) = g_i(x) \text{ if } x \in \pi_1 \circ \gamma_i(M). \tag{11}$$

The mapping g is well defined and continuous. It is differentiable, because all g_i are differentiable with the same tangent plane. It is also continuously differentiable, because all g_i are C^1 and in an ε/δ-formulation of the continuity criterion we can choose, for a given δ, a common ε for all g_i, because there are finitely many of them.

Thus we have parametrized $\gamma_{M,N}(M)$ in the neighborhood of p as the graph of a C^1 function, and therefore it is a C^1 submanifold of $\mathbb{R}^m$. □

4 Self-Intersections of the General Offset Surface

We consider the mapping

$$\tau : \perp_1 M \times \mathbb{R} \to \mathbb{R}^m, (p; n; \lambda) \mapsto p + \lambda\sigma_{M,N}(p; n). \tag{12}$$

Its differential at $(p; 0)$ is regular, and therefore locally τ is an embedding of $\perp M$ into $\mathbb{R}^m$. In this context we will identify $\perp M$ and $\perp_1 M \times \mathbb{R}_0^+$.

Lemma 5 *For all admissible solids M with piecewise C^1 surface normal parametrization and for all C^1 strictly convex bodies N with C^1 surface normal parametrization, there is an $\varepsilon > 0$ such that $\gamma_{M,\lambda N}$ is a C^1 submanifold for all $0 < \lambda < \varepsilon$.*

The mapping τ of Equ. 12 is an embedding of $\{(p; n) \in \perp M \mid 0 < \|n\| < \varepsilon\}$ into $\mathbb{R}^m$.

Proof: We need something like the tubular neighborhood theorem [2] to prove this statement. Because ∂M ist not one of the objects which usually are proven to have tubular neighborhoods in textbooks of differential topology, we give our own proof:

In the following $B(x, r)$ denotes the set of all points whose distance to x is less or equal r. For all $x \in M$ find an $\varepsilon_0 > 0$ as follows: For all boundary d-surfaces $f : U \to \mathbb{R}^m$ with $f(u) = x$ find an $\varepsilon_0(f)$ such that the connected component of $f(U) \cap B(x, r)$ which contains x is diffeomorphic to a d-dimensional disk for all $r < \varepsilon_0(f)$. (This is accomplished by determining a δ such that $\pi_1 \circ f$ is injective in $B(u, \delta)$ and then requiring the $\varepsilon_0(f)$-neighborhood of x to lie in $f(B(u, \delta))$.) Then $\varepsilon_0 = \min_f \varepsilon_0(f) > 0$ because the number of f's involved is finite.

The connected component of $B(x, \varepsilon_0)$ which contains x is denoted by V. The set $M \setminus V$ is compact, and there is an $\varepsilon_1 < \varepsilon_0$ such that $B(x, \varepsilon_1) \cap M \setminus V$ is empty.

There is further an ε_2 such that in an ε_2-neighborhood of x, the mapping τ is an embedding of $\perp M$ into $\mathbb{R}^m$. At last choose $\varepsilon(x) < \min\{\varepsilon_1/2, \varepsilon_2\}$.

Then the sets $U(x) = B(x, \varepsilon(x)) \cap M$ are connected, and the one-sided normal bundle defines a fibration in the union of the $B(x, \varepsilon(x))$. As M is compact, finitely many $U(x_i)$, $i = 1, \ldots, r$, cover M, and there is an $\varepsilon > 0$ such that $\gamma_{M,\lambda N}(M)$ is contained in $U(x_1) \cup \ldots \cup U(x_r)$ for all $\lambda < \varepsilon$. This means that τ restricted to normal vectors $(p; n)$ with $\|n\| < \varepsilon$ is an embedding. □

Lemma 6 *If $\gamma_{M,\lambda N}$ is injective and regular for all $\lambda \in [0, 1]$, then $\gamma_{M,N}$ is the boundary of an admissible solid, denoted by $M + N$. The surface normals of M give a piecewise C^1 surface normal parametrization of $\gamma_{M,\lambda N}(\perp_1 M)$. Furthermore, $(M + \lambda N) + \mu N = M + (\lambda + \mu)N$, if all sums are defined.*

Proof: The solid in question consists of the original M and the set $\{p+n \mid (p; n) \in \perp M, \|n\| \leq 1\}$. Its boundary is a C^1 submanifold of $\mathbb{R}^m$ which has the C^1 surface normal parametrization inherited from M, which also implies the equation $(M + \lambda N) + \mu N = M + (\lambda + \mu)N$. □

The previous lemma moreover asserts that there is an $\varepsilon > 0$ such that $\gamma_{M,\lambda N}$ is injective and regular even for $\lambda \leq 1+\varepsilon$:

Corollary 1 *The set of all $\lambda > 0$ with the property that $\gamma_{M,\lambda N}$ is a C^1 submanifold for all $\mu \leq \lambda$ is open.*

Definition: The *injectivity size* $s(M,N)$ of N with respect to M is the minimal positive λ such that $\gamma_{M,\lambda N}$ is not a regular C^1 submanifold. If such a λ does not exist, then we set $s(M,N)=\infty$.

Lemma 6 shows that $s(M,N) > 0$ for all admissible M, N. Regarding whether or not the image of $\gamma_{M,\lambda N}$ is a C^1 submanifold, there are three possibilities:

1. M is convex. This means that all indicatrices of M's boundary surfaces are void and all $\gamma_{M,\lambda N}$ are regular, because always $\mathrm{II}_{f,n}(v) \leq 0$.

   Moreover, the convexity of M easily shows that injectivity of $\gamma_{M,\lambda N}$ for all $\lambda > 0$. This shows that $s(M,N)=\infty$.

2. If M is not convex, then there are f, p, n, v, such that the second fundamental form $\mathrm{II}_{f,n}(v)$ of f at p is positive. There is a smallest $\lambda > 0$ such that $\beta = \mathrm{II}_{N,n} - \mathrm{II}_{f,n}$ is degenerate but still positive semidefinite. Consider a tangent vector w in the radical of β, i.e., $\beta(v,w)=0$ for all v. We note that $\mathrm{II}_{\lambda N,n} = \frac{1}{\lambda}\mathrm{II}_{N,n}$, and calculate $0 = \frac{1}{\lambda}(\mathrm{II}_{N,n} - \lambda_{f,n})(v,w) = \mathrm{II}_{\lambda N,n}(v,w) - \mathrm{II}_{f,n}(v,w) = v \cdot (d\sigma_{\lambda N}(w) - d\sigma_f(\bar{w}))$, where $\bar{w} \in T_{p;n}\perp_1 f$ is such that $d\pi_1(\bar{w}) = w$. The scalar product in $T_p f$ is nondegenerate, so we have $d\sigma_{\lambda N}(w) - d\sigma_f(\bar{w}) = d\sigma_{\lambda N}(d\gamma_{M,N}(\bar{v})) = 0$. Because $d\sigma_{\lambda N}$ is an isomorphism, $d\gamma_{M,N}(\bar{v}) = 0$, and $s(M,N) \leq \lambda$, i.e., $s(M,N)$ is finite.

The smallest λ such that $\gamma_{M,N}$ is no more regular may nor may not equal $s(M,N)$. If it does, we say that *local* properties of M and N restrict the injectivity size of N.

If for all $\lambda \leq s(M,N)$ the general offset $\gamma_{M,N}$ is regular, we say that at $\lambda = s(M,N)$, the solid M is *not globally millable* by λN, although locally it is.

Theorem 3 *If M is not globally millable by λN at $\lambda = s(M,N)$, but locally it is, $\gamma_{M,N}$ has a self-intersection which is not transverse, but the surface touches itself.*

Proof: Because of $\gamma_{M,N} = \gamma_{M+\varepsilon N,(1-\varepsilon)N}$ we may assume that M is C^1 and identify $M = \perp_1 M$.

We have to show that if at $\gamma_{M,\lambda N}(x_1) = \gamma_{M,\lambda N}(x_2)$ the tangent planes are not parallel then there are $\lambda' < \lambda$, and x_1', x_2', such that $\gamma_{M,\lambda' N}(x_1') = \gamma_{M,\lambda' N}(x_2')$. We study the behavior of solutions of this equation in a neighborhood of the original solution by solving the corresponding equation of differentials $(d\lambda, v, w)$.

$$v - \lambda d\sigma_{M,N}(v) - d\lambda\sigma_{M,N}(x_1) = w - \lambda d\sigma_{M,N}(w) - d\lambda\sigma_{M,N}(x_2), \qquad (13)$$

which gives $d\lambda(\sigma(x_1)-\sigma(x_1) = v - \lambda d\sigma_{M,N}(v) - w + \lambda d\sigma_{M,N}(w)$. The span of all $v - \lambda d\sigma_{M,N}(v)$ equals the whole $(m-1)$-dimensional tangent plane of $\gamma_{M,\lambda N}$, because the differential $d\gamma_{M,N}$

was assumed to be regular. So the above equation of differentials is solvable for $d\lambda \neq 0$ if the span of the right hand side equals $\mathbb{R}^m$, which is the case if the normal vectors at x_1, x_2 are not parallel.

Thus there is a local submanifold of solutions of $\gamma_{M,\lambda' N}(x_1') = \gamma_{M,\lambda' N}(x_2')$ with $\lambda' < \lambda$. □

Theorem 4 *If M is globally millable by λN (cf. the discussion preceding Th. 3), then λN can be moved such that for all $(x; n) \in \perp_1 M$, the translate of λN touches M at x with common normal vector n. This is not possible for $\lambda > s(M, N)$.*

Proof: Consider a point $(x; n) \in \perp_1 M$. There is a $\lambda > 0$ and a vector $r(\lambda, x) \in \mathbb{R}^m$ such that the translate $\lambda N + r(\lambda, x)$ of λN touches M at x having normal vector $-n$ there, and such that the intersection $Z := M \cap (\lambda N + r(\lambda, x))$ equals the singleton $\{x\}$.

We let λ increase until, at a smallest $\lambda = \lambda_1$, the intersection set Z is greater than $\{x\}$. If x is not an isolated point of Z, then the various indicatrices of M at $(x; n)$ can no longer be contained in the exterior of $i_{N,n}$. In this case $\gamma_{M,\lambda_1 N}$ is no longer regular for all $\lambda \leq \lambda_1$.

If, however, x is an isolated point of Z, then $\lambda_1 N + r(\lambda_1, x)$ actually *touches* M in all points of Z. This can be seen by an argument similar to the proof of Th. 3. It is easily seen that if the tangent planes at an intersection point are not parallel, there is a $\lambda_2 < \lambda_1$ with $Z \supsetneq \{x\}$.

But if $\lambda_1 N$ touches M in two different points x and y, then $\gamma_{M,\lambda_1 N}(x) = \gamma_{M,\lambda_1 N}(y)$, and so $\gamma_{M,\lambda_1 N}$ is no longer injective. □

Thus the notion 'globally millable' is justified, if we think of N as of a milling tool shaping the solid M.

This leads to an algorithm for determining the value of $s(M, N)$. First we define a general normal of a surface element:

Definition: For every point p of N there is a unique second point q of N with $T_p N \parallel T_q N$. We call the line spanned by p and q the *general normal* (in the points p, q).

For any surface element $(x; n) \in \perp_1 M$ we define the *general normal* $\nu(x; n)$ as the line which passes through x and is parallel to the line which joins $\sigma_N^{-1}(n)$ and $\sigma_N^{-1}(-n)$.

The *generalized distance* of two points x, y of a general normal parallel to the points $p, q \in N$ is the factor d in the equation $\|x - y\| = d\|p - q\|$.

Now Theorems 3 and 4 imply:

Theorem 5 *Search for the smalled generalized length s_0 of a general normal chord which joins two points $(x_1; n)$ and $(x_2; -n)$ of $\perp_1 M$. Further determine the smallest real number s_1 such that the indicatrices of M, scaled by the factor s_1, are not contained in the interior of the respective indicatrix of N. Then*

$$s(M, N) = \min\{s_0, s_1\}. \tag{14}$$

5 Non-smooth N

In applications we cannot expect N to be smooth — cutting tools in three-axis milling will often have edges. It is however easy to show that the general offset surface is smooth again under certain circumstances.

It is obvious how to modify the definition of σ_N and $\sigma_{M,N}$ if N is not smooth. As we did for M, we consider the unit normal bundle $\perp_1 N$ and the spherical mapping $\sigma_N : \perp_1 N \to S^{m-1}$. Then $\sigma_{M,N} : \perp_1 M \to \perp_1 N$, $\sigma_{M,N} = \sigma_N^{-1} \circ \sigma_M$. Because of our definition of strict convexity, σ_N again is one-to-one.

The local millability criterion is to be modified in the following way: Let $(p;n) \in \perp_1 M$ and $(q;-n) \in \perp_1 N$ be corresponding pair of normal vectors, and let f, g be surface patches of M and N, respectively, containing p and q. Then there are the indicatrices $i_{f,n}$ and $i_{g,-n}$. Assume that N is translated such that it touches M at $(p;n)$. Then for all pairs of boundary surfaces f, g of M, N which contain the point of contact and have the property that $T_p f \subseteq T_p g$ the indicatrix $i_{g,n}$ is contained in the *interior* of $i_{f,n}$, and for all pairs f, g with $T_p g \subseteq T_p f$ the indicatrix $i_{f,n}$ is contained in the *exterior* of $i_{g,n}$. Note that a pair f, g such that neither of $T_p f$, $T_p g$ is contained in the other, does not contribute to the criterion.

Theorem 6 *If M, N are piecewise C^1 surfaces with piecewise C^1 surface normals, the local millability criterion is fulfilled for all possible pairs of surface patches for all pairs $(p;n) \in \perp_1 M$ and $(q;-n) = \sigma_{M,N}(p;n) \in \perp_1 N$, and either*

- *(i) one of the two surfaces is smooth, or*
- *(ii) the tangent spaces of the boundary surfaces containing p and q, respectively, are complementary subspaces of $\mathbb{R}^m$,*

then $\gamma_{M,N}(\perp_1 M)$ is an immersed C^1 submanifold of $\mathbb{R}^m$, if $\gamma_{M,N}$ is injective, then its image is a C^1 submanifold of $\mathbb{R}^n$ with piecewise C^1 surface normals.

Proof: First suppose that we have a corresponding pair of normal vectors $(p;n) \in \perp_1 M$ and $(q;-n) \in \perp_1 N$ with M smooth at p. Then there is an $\varepsilon > 0$ such that the *inner* parallel surface M' at distance ε with respect to the unit ball is smooth in a neighborhood of p. Analogously the *outer* parallel surface N' of N at distance ε is smooth. But the general offset of M with respect to N' equals the general offset of M' with respect to N', which is smooth.

Now suppose that the pair $(p;n)$ and $(q;-n)$ belongs to edges of M and N, respectively, whose tangent spaces are complementary subspaces of $\mathbb{R}^m$. Then $\gamma_{M,N}$ is the surface of translation defined by the two edges, which is smooth if both edges are. $\square$

6 Special Cases

There are situations where there are no general normal chords between points of M. One example is the case of a star-shaped surface: Recall that a solid is called *star-shaped* with respect to the point o, if for all $p \in M$ the line segment $[op] = tp + (1-t)o$, $0 \leq t \leq 1$, is contained in M.

Corollary 2 *If M is both star-shaped and locally millable by N, then it is also globally millable by N.*

Proof: It is easily seen that for all $(p;n) \in \perp_1 M$ the line segment $[op]$ is completely contained in the half-space $(x-p)n \leq 0$. Otherwise n could not be in $N(p)$.

Suppose that there are points $(p;n)$ and $(q;-n)$ in $\perp_1 M$. The general normal chord $[pq]$ leaves M in p, so we have $(q-p)n > 0$. We also have $(o-p)n \leq 0$ and $(o-q)(-n) \leq 0$. These three inequalities contradict each other, as is clearly seen when we write them like this: $qn > pn$, $pn \geq on$, $on \geq qn$. □

The *convex core* $cc(M)$ of a star-shaped solid M is the set of all o with respect to which M is star-shaped. It is easily seen to be convex.

A set which is a 'hole' $\mathbb{R}^m \setminus M$ in $\mathbb{R}^m$ is said to be admissible, if $D(0,r) \setminus M$ is admissible for some large r.

Corollary 3 *A convex body N is able to globally mill the interior of a star-shaped solid M, i.e., is able to mill $\mathbb{R}^m \setminus M$, if $\mathbb{R}^m \setminus M$ and N fulfill the local millability condition and N is contained in the convex core of M.*

Proof: It is easily seen that (see also the proof of Cor. 2) that a point $o \in cc(M)$ fulfills the equation $(o-p)n \geq 0$ for all $(p;n) \in \perp_1(\mathbb{R}^m \setminus M)$. Thus a general normal chord defined by $(p;n)$ and $(q;-n)$ forces $cc(M)$ to lie in the strip

$$\{x \in \mathbb{R}^m \mid pn \leq xn \leq qn\}. \tag{15}$$

The corresponding chord of N has therefore general length ≤ 1 and thus we have $s(M,N) \leq 1$. □

If M is the inside of a convex body, then there is the following generalization of the theorem of Blaschke [1] which says that sphere can roll freely inside a convex surface if its radius is smaller than the smallest principal curvature of that surface. It follows directly from the previous corollary, because for convex M we have $cc(M) = N$.

Corollary 4 *A strictly convex body N contained in the convex body M is able to globally mill $\mathbb{R}^m \setminus M$, if the local millability condition is fulfilled for $\mathbb{R}^m \setminus M$ and N.*

A similar result has been proved by J. Rauch [9], who showed a stronger result if both surfaces have strictly positive curvature: in that case it is not necessary to assume that N is contained in M, it is automatically true (see also [5]). A detailed study of generalizations of Blaschke's rolling theorem in the smooth and non-smooth case can be found in [3].

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