INFLECTION POINTS AND SINGULARITIES ON C-CURVES**

QINMIN YANG* AND GUOZHAO WANG†

Abstract. We show that all so-called C-curves are affine images of trochoids or sine curves and use this relation to investigate the occurrence of inflection points, cusps, and loops. The results are summarized in a shape diagram of C-Bézier curves, which is useful when using C-Bézier curves for curve and surface modeling.

Key words: Trochoid; C-curve; C-Bézier curve; Affine invariant; Inflection point; Singularity

1. Introduction

Bézier, B-spline, and other curves are piecewise polynomial, and due to their simplicity and beautiful geometric properties they are powerful tools in Computer-Aided Geometric Design. *Hybrid polynomial curves* on the one hand may be more complicated, but on the other hand they may have some properties which are convenient in special cases.

This paper deals with C-curves, which are the linear combinations of the basis functions $\sin t, \cos t, t, 1$. Ever since the advent of analytic geometry such curves have been considered, their most famous representative being the cycloid. The first analysis of their shapes in general was probably given by [3]. As the above mentioned basis is a Tchebycheff system if restricted to an interval of length less than π, corresponding T-Bézier and TB-spline curves may be defined by blossoming as described by [8]. Shape analysis then makes use of the general theory of Tchebycheff systems. Our basis has been studied by [10] under the name of *helix splines*. [16] constructed C-Ferguson-, C-Bézier-, C-B-spline- and other curves, which incidentally are the same as defined by the blossoming procedure. Previous work on the topic of characterizing the existence of inflection points and singularities can be found in [1, 2, 4, 5, 6, 7, 9, 11, 12, 14].

In this paper C-curves and C-Bézier curves are discussed. Some of their properties are similar to those of cubic curves. Moreover, trochoids, circular arcs,

⋆ Institute of Geometry, Vienna University of Technology, A-1040 Wien, Austria.

† Department of mathematics, Zhejiang University, Hangzhou 310027, P.R. China, and Institue of Computer Graphics and Image Processing, Zhejiang University, Hangzhou 310027, P.R. China.

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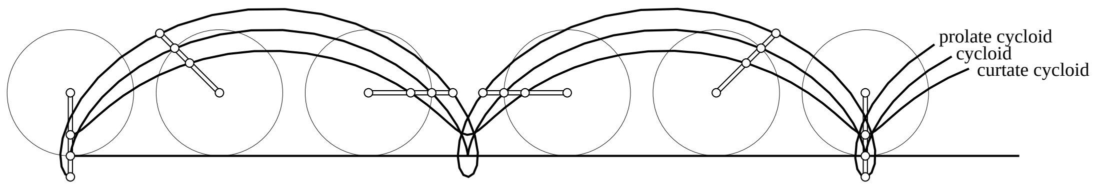

Figure 1. Trochoids

elliptical arcs, and sine curves may be represented as C-Bézier curves by control polygons. We give conditions on the existence of inflection points, cusps, and loops, which are undesirable features in many cases where one wants to design 'fair' curves. The results are summarized in a *shape diagram* like the one in [13].

This paper is organized as follows: First we show that nondegenerate C-curves are affine images of trochoids or sine curves. This correspondence is used to describe the occurrences of inflection points, cusps, and loops. At last we apply those conditions to C-Bézier curves.

2. The shape of trochoids and C-curves

A trochoid is the path of a point at some distance $b > 0$ from the center of a circle of radius a, while this circle is rolling on a fixed line. If we let $a = 1$ it has the parametric equation $c(t) = (x(t), y(t))$, with $x(t) = t - b\sin t$ and $y(t) = 1 - b\cos t$.

For $b < 1$, the curve is a *curtate cycloid* and has inflection points at $t = 2k\pi \pm \arccos b$, $k \in \mathbb{Z}$. For $b = 1$, the curve is a *proper cycloid*, or simply *cycloid*. It has cusps at $t = 2k\pi$, $k \in \mathbb{Z}$. For $b > 1$, the curve is a *prolate cycloid* and it has loops which are located at $c(t_1) = c(t_2)$, where $t_{1,2} = 2k\pi \pm \tau$, $k \in \mathbb{Z}$, $\tau > 0$ is the smallest zero of the function $t/\sin t - b$. Examples are shown by Fig. 1. The interested reader is also referred to [15].

Assume that $v_i = (a_i, b_i) \in \mathbb{R}^2$ $(i = 0, 1, 2)$, and $p_3 = (a_3, b_3) \in \mathbb{R}^2$ is a point. A C-curve in general has the parametrization

$$c(t) = v_0 \sin t + v_1 \cos t + v_2 t + p_3, \quad t \in \mathbb{R}. \tag{1}$$

Define the determinants $d_0 = \det(v_1, v_2)$, $d_1 = \det(v_2, v_0)$, and $d_2 = \det(v_0, v_1)$. If $d_0 = d_1 = 0$, $c(t)$ is a line or an ellipse, and we will disregard these cases in the following. We let $b = |d_2|/\sqrt{d_0^2 + d_1^2}$ and choose ω such that $\tan\omega = d_1/d_0$, and such that $\omega \in [-\pi/2, \pi/2]$ or $\omega \in (\pi/2, 3\pi/2)$ depending on whether $d_0 d_2 \geq 0$ or $d_0 d_2 < 0$.

Lemma 1. *If $d_2 \neq 0$, (1) is the affine image of a trochoid.*

Proof. We notice that

$$\begin{bmatrix} -d_0 & -d_1 \\ d_1 & -d_0 \end{bmatrix} \begin{bmatrix} b_1 & -a_1 \\ -b_0 & a_0 \end{bmatrix} \frac{c(t) - p_3}{d_0^2 + d_1^2} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t - b\sin(t + \omega) \\ 1 - b\cos(t + \omega) \end{bmatrix}.$$

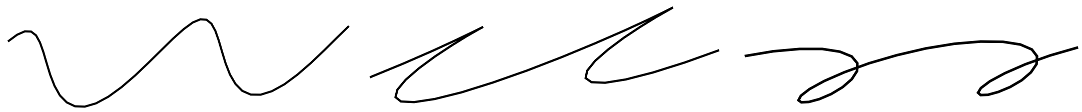

FIGURE 2. C-curves with inflection points, cusps, and loops (from left to right).

□

Lemma 2. *If $d_2 = 0$, then* (1) *is the affine image of a sine curve.*

Proof. We assume that $d_1 \neq 0$, the case $d_0 \neq 0$ being analogous. It is easily verified that $\begin{bmatrix} b_0 & -a_0 \\ b_2 \sin\omega & -a_2 \sin\omega \end{bmatrix} \frac{c(t) - p_3}{d_1} + \begin{bmatrix} \omega \\ 0 \end{bmatrix} = \begin{bmatrix} t+\omega \\ \cos(t+\omega) \end{bmatrix}$. □

Lemma 3. *If* (1) *is not a straight line or an ellipse (i.e., $d_0^2 + d_1^2 > 0$), then:*

(i) (1) *has inflection points at $c(t)$ if and only if $b < 1$; then $t = 2k\pi - \omega \pm \arccos b$ ($k \in \mathbb{Z}$).*
(ii) (1) *has cusps if and only if $b = 1$; they are located at $t = 2k\pi - \omega$ ($k \in \mathbb{Z}$).*
(iii) (1) *has loops at $c(t_1) = c(t_2)$ if and only if $b > 1$; then $t_{1,2} = 2k\pi - \omega \pm \tau$ ($k \in \mathbb{Z}$).*

Proof. Being an inflection point, a cusp, or a loop is invariant under affine transformations ([13]). The result follows immediately from Lemmas 1 and 2. □

Lemma 3 shows that inflection points, cusps, and loops are mutually exclusive. Examples of C-curves which exhibit these features can be seen in Fig. 2. The following theorem again summarizes the previous results, this time with a view towards C-Bézier curves:

Theorem 1. *We consider the C-curve* (1) *to be defined in the interval $(0, \alpha)$ with $\alpha \leq \pi$. Then $c(t)$ has two inflection points if and only if $b < 1$ and $\omega \in (\beta - \alpha, -\beta)$, where $\beta = \arccos b$. There is one inflection point if and only if $b < 1$ and*

$$\omega > 2\pi - \beta - \alpha \textit{ or } \begin{cases} -\beta - \alpha < \omega < -\beta \\ \omega \leq \beta - \alpha \end{cases} \textit{ or } \begin{cases} \beta - \alpha < \omega < \beta \\ \omega \geq -\beta. \end{cases}$$

There is a cusp if and only if $b = 1$ and $\omega \in (-\alpha, 0) \cup (2\pi - \alpha, 3\pi/2)$. There is a loop if and only if $b > 1$ and $\omega \in (\tau - \alpha, -\tau) \cup (2\pi + \tau - \alpha, 2\pi - \tau)$.

Proof. This is a direct consequence of Lemma 3. □

3. The shape of C-Bézier curves

A C-Bézier curve as defined in [16] is the C-curve $c(t)$ defined in the interval $(0, \alpha)$ with $\alpha \leq \pi$, and having the form

$$c(t) = \frac{1}{\alpha - s} \left[\begin{array}{cccc} \sin t & \cos t & t & 1 \end{array} \right] \left[\begin{array}{cccc} c & 1-c-M & M & -1 \\ -s & (\alpha-K)M & -KM & 0 \\ -1 & M & -M & 1 \\ \alpha & -(\alpha-K)M & KM & 0 \end{array} \right] \left[\begin{array}{c} q_0 \\ q_1 \\ q_2 \\ q_3 \end{array} \right], \tag{2}$$

where

$$s = \sin\alpha,\ c = \cos\alpha,\ K = (\alpha - s)/(1-c),\ M = \left\{ \begin{array}{ll} 1 & \text{if} \quad \alpha = \pi, \\ \frac{s}{\alpha - 2K} & \text{if} \quad \alpha < \pi \end{array} \right. ,\ q_i \in \mathbb{R}^2.$$

The points $q_0, \ldots, q_3$ are called control points. We want to apply the shape classification of C-curves to C-Bézier curves and therefore have to express (2) in terms of (1).

Let q be the intersection of the lines q_0q_1 and q_2q_3, and define λ, μ by

$$e_1 := q - q_0,\ e_2 := q_3 - q,\ \lambda e_1 = q_1 - q_0,\ \mu e_2 = q_3 - q_2.$$

(see Fig. 3). If q does not exist, we let $\lambda = \mu = 0$.

If we rewrite (2) as a C-curve (cf. Equ. (1)), we get

$$\begin{aligned} v_0 &= \frac{1}{\alpha - s}[(M + \lambda - M\lambda - c\lambda - 1)e_1 + (M - M\mu - 1)e_2], \\ v_1 &= \frac{1}{\alpha - s}[(s\lambda + KM\lambda - KM)e_1 + KM(\mu - 1)e_2], \\ v_2 &= \frac{1}{\alpha - s}[(1 - M + M\lambda)e_1 + (1 - M + M\mu)e_2]. \end{aligned}$$

Now we can express the shape parameters b and ω in terms of α, λ and μ. Theorem 2 describes the shape of (2) in terms of regions of $\mathbb{R}^2$ where the point (λ, μ) lies in. These regions depend on α and constitute a *shape diagram* for C-Bézier curves (see Fig. 4)

We define the matrices L_{ij}, I_{ij}, N_{ij}, where i is the row index and j is the column index, both starting with 0.

$$\begin{aligned} &L_{11} = s(c-1), L_{10} = c(\alpha - s), L_{01} = \alpha - s, L_{00} = I_{10} = I_{01} = I_{00} = 0, \\ &I_{22} = -2(c-1)^3, I_{21} = I_{12} = 2s(c-1)(\alpha - s), I_{20} = I_{02} = (\alpha - s)^2, \\ &I_{11} = 2c(\alpha - s)^2, N_{12} = \alpha(c-1)^3, N_{11} = (1-c)(\alpha - s)(\alpha s - c + 1), \\ &N_{10} = N_{01} = -s(\alpha - s)^2, N_{02} = (c-1)^2(\alpha - s), N_{00} = 0, \\ &I = \sum\nolimits_{i,j=0}^{2} I_{ij}\lambda^i\mu^j, \\ &f(\lambda, \mu) = I \arccos\left(\frac{1}{\sqrt{I}} \sum\nolimits_{i,j=0}^{1} L_{ij}\lambda^i\mu^j\right) + \lambda \sum\nolimits_{i=0}^{1} \sum\nolimits_{j=0}^{2} N_{ij}\lambda^i\mu^j. \end{aligned}$$

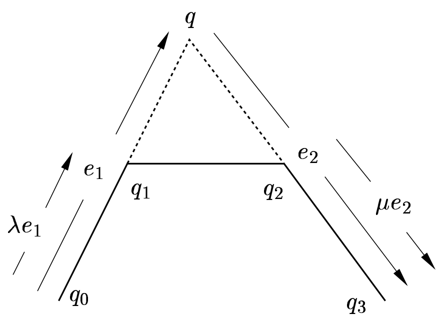

FIGURE 3. Definition of λ and μ.

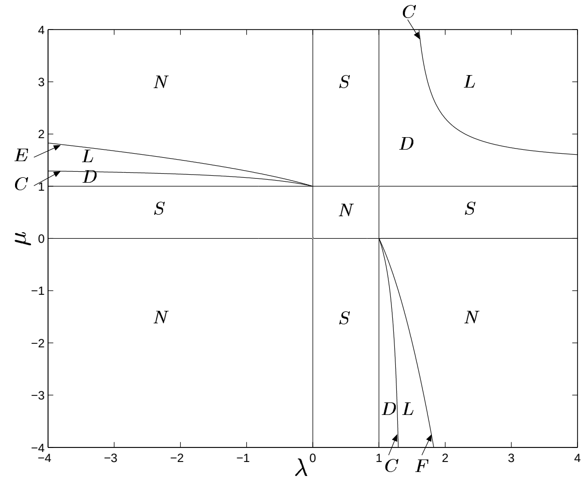

FIGURE 4. Shape diagram of C-Bézier curves.

We define curves C, E, F by

$$C : (c-1)(\alpha^2 + 2c - 2)\lambda\mu + 2(\alpha - s)^2(\lambda + \mu - 1) = 0, \text{ where } \lambda, \mu \notin [0, 1];$$
$$E : f(\lambda, \mu) = 0 \text{ where } \lambda < 0, \mu > 1; \quad F : f(\mu, \lambda) = 0 \text{ where } \lambda > 1, \mu < 0.$$

The curves C, E, F together with the lines $\lambda, \mu = 0, 1$ are partitioning the (λ, μ)-plane into the regions D, L, S, N according to Fig. 4.

Theorem 2. *The curve* (2) *has an inflection point, if* $(\lambda, \mu) \in S$*, it has two of them if* $(\lambda, \mu) \in D$*, it has a cusp if* $(\lambda, \mu) \in C$*, it has a loop if* $(\lambda, \mu) \in L$*, and is has none of them if* $(\lambda, \mu) \in N$*.*

Proof. Multipling d_0, d_1, and d_2 with the factor d, which is defined as

$$d := (\alpha - s)^2(\alpha + \alpha c - 2s)/s$$

will not affect the result. So we have

$$\begin{aligned} d_0 &= s(c-1)\lambda\mu + (\alpha - s)(c\lambda + \mu), \\ d_1 &= \lambda[(c-1)^2\mu - s(\alpha - s)], \\ d_2 &= \alpha(c-1)\lambda\mu + (\alpha - s)(\lambda + \mu). \end{aligned}$$

It follows directly from the defintion of the angles ω, τ, and β that

$$\cos\omega = bd_0/d_2, \ \sin\omega = bd_1/d_2, \ \tau/\sin\tau = b, \ 0 < \tau < \pi, \ \sin\beta = \sqrt{1 - b^2}.$$

We are going to consider the cases of (i) two inflection points, (ii) one inflection point, (iii) a cusp, and finally (iv) a loop separately. The conditions that we have case (i), as given by Theorem 1, are equivalent to

$$\begin{aligned} & b < 1,\ \sin(-\omega+\beta) > 0,\ \cos(-\omega+\beta) > c,\ \sin(-\omega-\beta) > 0,\ \cos(-\omega-\beta) > c, \\ \iff\ & t := d_0^2 + d_1^2 - d_2^2 > 0,\ d_1 d_2 + |d_0|\sqrt{t} < 0,\ (d_0^2+d_1^2)c + |d_1|\sqrt{t} < d_0 d_2. \end{aligned}$$

Substitute d_0, d_1, and d_2 into the above inequalities yields the definition of the region D shown by Fig. 4.

Case (ii) is similar: The conditions given by Theorem 1 are equivalent to either one

$$\begin{aligned} & b < 1,\ \sin(-\omega+\beta) > 0,\ \cos(-\omega+\beta) > c,\ \sin(-\omega-\beta) \le 0 \\ \text{or}\quad & b < 1,\ \sin(-\omega-\beta) > 0,\ \sin(-\omega-\beta-\alpha) < 0,\ \sin(-\omega+\beta-\alpha) \ge 0 \end{aligned}$$

If we use the abbreviation $t := d_0^2 + d_1^2 - d_2^2$, this is equivalent to

$$\begin{aligned} & t > 0, \quad d_1 d_2 - d_0 \operatorname{sgn}(d_2)\sqrt{t} < 0, \\ & d_1 d_2 + d_0 \operatorname{sgn}(d_2)\sqrt{t} \ge 0, \quad (d_0^2+d_1^2)c - d_0 d_2 < d_1 \operatorname{sgn}(d_2)\sqrt{t} \\ \text{or}\quad & t > 0, \quad d_1 d_2 + d_0 \operatorname{sgn}(d_2)\sqrt{t} < 0, \quad d_2(cd_1 + sd_0) \le (cd_0 - sd_1)\operatorname{sgn}(d_2)\sqrt{t}, \\ & d_2(cd_1 + sd_0) > (sd_1 - cd_0)\operatorname{sgn}(d_2)\sqrt{t} \end{aligned}$$

By substituting d_0, d_1, and d_2 into above inequalities we get the definition of the region S shown by Fig. 4.

Again according to Theorem 1, case (iii) (a cusp) is equivalent to

$$\begin{aligned} & b = 1, \quad \sin(-\omega) > 0, \quad \sin(-\omega-\alpha) < 0, \\ \iff\ & d_0^2 + d_1^2 - d_2^2 = 0, \quad d_1 d_2 < 0 \quad (sd_0 + cd_1)d_2 > 0. \end{aligned}$$

These inequalities define the curve C of Fig. 4.

As to the last case (iv) of a loop, Theorem 1 shows that it is equivalent to:

$$\begin{aligned} & b > 1,\ \sin(-\omega) > 0,\ \sin(-\omega-\alpha) < 0, \\ & \tau < \arccos(\cos(-\omega)),\ \tau < \alpha - \arccos(\cos(-\omega)), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & d_0^2 + d_1^2 - d_2^2 < 0, \quad d_1 d_2 < 0, \quad (sd_0 + cd_1)d_2 > 0, \\ & (d_0^2+d_1^2)\arccos\frac{d_0\operatorname{sgn}(d_2)}{\sqrt{d_0^2+d_1^2}} + d_1 d_2 > 0, \\ & (d_0^2+d_1^2)(\arccos\frac{d_0\operatorname{sgn}(d_2)}{\sqrt{d_0^2+d_1^2}} - \alpha) + (cd_1 + sd_0)d_2 < 0. \end{aligned}$$

By substituting d_0, d_1, and d_2 into the above inequalities we get the definition of the region L of Fig. 4. $\square$

Examples of these five different shapes of C-Bézier curves are shown in Fig. 5.

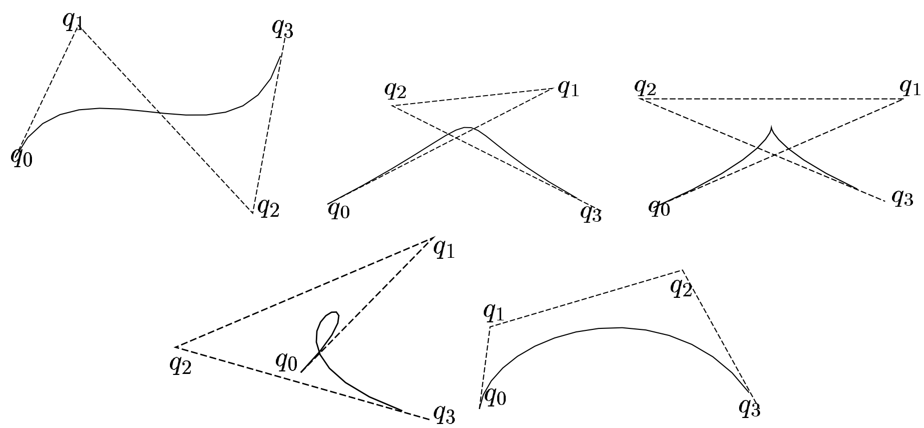

FIGURE 5. Different shapes of C-Bézier curves

CONCLUSION

We investigated the existence of inflection points, cusps, and loops in C-curves and C-Bézier curves by showing that these curves are affine images of trochoids, and finally computed a *shape diagram*. The conditions are useful for classifying the shapes of other special forms of C-curves as well, such as C-Ferguson curves or C-B-spline curves.

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