On the semiaxes of touching quadrics

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Abstract: In this paper the semiaxes of quadrics which touch each other are investigated. If two quadrics Q, Q' are given, it turns out that we can rotate Q' such that it touches Q in two opposite points if and only if the squared semiaxes of Q, Q' do not separate each other. This is equivalent to the statement that for two symmetric matrices A, B there is an orthogonal matrix S with $\det(A - S^T BS) = 0$ if and only if the eigenvalues of A and B do not separate each other.
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Introduction

The aim of this paper is to investigate conditions on the semiaxes of two quadrics Q, Q', both centered at the origin, which can be rotated such that they *touch* each other. This is equivalent to the following problem: Given two symmetric matrices A, B, is there an orthogonal matrix S such that $\det(A - S^T BS) = 0$? One motivation (beside the simple characterization of touching quadrics in terms of their semiaxes) is the study of the singularities of the configuration space of surface-surface contact [6]: The configuration space is defined as the set of isometries of Euclidean n-space which transform one surface such that it touches the second. In a regular point, the configuration space locally is a $(n+2)(n-1)/2$-dimensional submanifold of the group of isometries. It turns out that a point of the configuration space is singular if and only if the indicatrices of curvature of the two surfaces touch each other in two opposite points. Thus, for a given pair of points p, q all $\alpha \in \mathrm{Isom}(E^n)$ with $\alpha(p) = q$ are regular if and only if the principal curvatures do not separate each other.

1 Eigenvalues and the Determinant of Symmetric Matrices

First we are going to repeat some well known results concerning the eigenvalues and the determinant of symmetric matrices (which actually hold for all hermitian matrices).

Consider a symmetric matrix $A \in \mathbb{R}^{n\times n}$ and its eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. We write $\langle\ ,\ \rangle$ for the scalar product. The set of k-dimensional subspaces of $\mathbb{R}^n$ is denoted by $\mathcal{M}_k$.

Theorem 1 *With the notations above, the eigenvalues of A can be expressed as follows:*

$$\lambda_k = \max_{V\in\mathcal{M}_k} \min_{\substack{x\in V\\ \|x\|=1}} \langle x, Ax\rangle = \min_{V\in\mathcal{M}_{n-k+1}} \max_{\substack{x\in V\\ \|x\|=1}} \langle x, Ax\rangle$$

This is called the Courant-Fisher minmax principle for eigenvalues (See e.g. [2]).

Theorem 2 *Let $H \in \mathcal{M}_{n-1}$ and V the matrix of the injection mapping $H \to \mathbb{R}^n$ with respect to orthonormal coordinate systems. Consider the eigenvalues $\mu_1 \geq \ldots \geq \mu_{n-1}$ of $V^T AV$. Then*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

This is a special case of the Cauchy interlacing theorem. See e.g. [1, 2, 4] for proofs and related results.

Theorem 3 *Let A, B be symmetric $n \times n$ matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively. Then*

$$\min_{\sigma \in S_n} \prod_{i=1}^{n} (\alpha_i - \beta_{\sigma(i)}) \leq \det(A - B) \leq \max_{\sigma \in S_n} \prod_{i=1}^{n} (\alpha_i - \beta_{\sigma(i)})$$

(The minimum and maximum is taken over all permutations of the indices $1, \ldots, n$.)

This theorem has been proved by M. Fiedler (see [2, 5]).

2 Semiaxes of Quadrics

A *quadric* Q is the set $\langle x, Ax \rangle = 1$, where A is a symmetric real matrix. After a suitable coordinate transformation, it has the equation

$$\left(\frac{x_1}{a_1}\right)^2 + \ldots + \left(\frac{x_r}{a_r}\right)^2 - \left(\frac{x_{r+1}}{a_{r+1}}\right)^2 - \ldots - \left(\frac{x_n}{a_n}\right)^2 = 1, \tag{1}$$

where the a_i are either positive real numbers or infinite: The matrix A has s positive eigenvalues $\lambda_1 \ldots \lambda_q$, the eigenvalues $\lambda_{q+1} = \ldots = \lambda_r$ are zero, and the eigenvalues $\lambda_{r+1}, \ldots, \lambda_n$ are negative. Then $a_i = 1/\sqrt{\lambda_i}$ for $i = 1, \ldots, s$, $a_{q+1} = \ldots = a_r = \infty$ and $a_i = 1/\sqrt{-\lambda_i}$ for $s = r+1, \ldots, n$. We say that Q has the *semiaxes*

$$0 < a_1, \ldots, a_q < \infty, \quad a_{q+1} = \ldots = a_r = \infty, \quad ia_{r+1}, \ldots, ia_n,$$

and the *squared semiaxes*

$$s_1 = 1/\lambda_1, \ldots, s_n = q/\lambda_n.$$

The quadric is an ellipsoid, if $0 < a_i < \infty$. The quadric $Q^\star$ *conjugate* to Q has squared semiaxes $s_i^\star = -s_i$. (We define $-\infty = \infty$, i.e., we consider the projective closure of the real line $\mathbb{R}$).

There is the following version of the minmax principle for the semiaxes of quadrics, which follows immediately from Th. 1. We give a different proof which uses properties of quadrics more familiar to geometers.

Theorem 4 *Let Q be a quadric with semiaxes $a_1 \leq \ldots \leq a_r \leq \infty$ and $ia_{r+1} \geq \ldots \geq ia_n$. Then we have the following equations:*

$$a_1 = \min_{x \in Q} \|x\|, \quad a_n = \min_{x \in Q^\star} \|x\|, \quad \textit{and, if } Q \textit{ is an ellipsoid: } a_n = \max_{x \in Q} \|x\|,$$

and

$$
\begin{aligned}
a_k &= \min_{V\in\mathcal{M}_k} \max_{x\in Q\cap V} \|x\|, \quad (k=1,\dots,r)\\
a_{n-k} &= \min_{V\in\mathcal{M}_k} \max_{x\in Q^\star\cap V} \|x\|, \quad (n-k=r+1,\dots,n)\\
a_k &= \max_{V\in\mathcal{M}_{n-k}} \min_{x\in Q\cap V} \|x\|, \quad (k=1,\dots,r)\\
a_{n-k} &= \max_{V\in\mathcal{M}_{n-k}} \min_{x\in Q^\star\cap V} \|x\|, \quad (n-k=r+1,\dots,n)
\end{aligned} \tag{2}
$$

All values of maxima for indices other than the indices given are infinite.

Proof: Obviously it is sufficient to prove the statements which involve Q. We have

$$\|x\|^2 = \sum_{i\le r} x_i^2 + \sum_{i>r} x_i^2 \ge \sum_{i\le r} x_i^2 \ge \sum_{i\le r} (\frac{a_1 x_i}{a_i})^2 \ge a_1^2 \sum_{i\le r} (\frac{x_i}{a_i})^2 - a_1^2 \sum_{i>r} (\frac{x_i}{a_i})^2 = a_1^2,$$

and $(a_1,0,\dots,0)\in Q$, which shows $\min_{x\in Q}\|x\| = a_1$. If Q is an ellipsoid, $\max\|x\| = a_n$ is seen analogously. If Q is no ellipsoid, it is either void or $\|x\|$ is not bounded.

Let $k\le q$ and assume that a_k is finite. Consider the subspace $V_0 = [e_1,\dots,e_k]$. The quadric $V_0\cap Q$ is and ellipsoid and we have

$$\min_{V\in\mathcal{M}_k} \max_{x\in Q\cap V} \le \max_{x\in Q\cap V_0} \|x\| = a_k.$$

For all $V\in\mathcal{M}_k$ consider the nonzero subspace $W = V\cap[e_k,\dots,e_n]$. All $x\in W\cap Q$ are contained in the quadric $Q\cap[e_k,\dots,e_n]$, whose semiaxes are $a_k,\dots, a_r, ia_{r+1},\dots, ia_n$, which implies $\|x\| > a_k$. We set the maximum of a void set to ∞, and have

$$\min_{V\in\mathcal{M}_k} \max_{x\in Q\cap V} \|x\| \ge \min_{V\in\mathcal{M}_k} \max_{x\in Q\cap V\cap[e_k,\dots,e_n]} \|x\| \ge a_k.$$

If $a_q = a_{q+1} = \dots = a_r = \infty$, then Q is an orthogonal cylinder over a basis quadric with semiaxes $a_1,\dots,a_q, ia_{r+1},\dots,ia_n$. Let $s\le k\le r$ and $V\in\mathcal{M}_k$. If V intersects $Z := [e_{q+1},\dots,e_r]$, then $\max_{x\in V\cap Q}\|x\| = \infty$, because $Q\cap V$ either is void or contains straight lines.

If V does not intersect Z, let p equal the projection with kernel Z onto $[e_1\dots, e_q, e_{r+1},\dots, e_n]$. In this case $p|V$ is 1–1, $p(V)\cap Q = p(V\cap Q)$, and $\max_{x\in p(V)\cap Q}\|x\| = \infty$, which implies $\max_{x\in V\cap Q}\|x\| = \infty$.

If $k>r$, then $V\cap Q$ s a non-void quadric which is no ellipsoid, and so $\max_{x\in Q\cap V}\|x\| = \infty$.

This implies the first of the four statements in (2). The proof of the others is similar. □

We use Th. 4 to prove the following statement about the changes in the semiaxes if a quadric undergoes a linear transformation.

Theorem 5 *Let Q be a quadric with semiaxes $a_1,\dots,a_r$ and $ia_{r+1},\dots,ia_n$. Let α be a linear automorphism of $\mathbb{R}^n$ with singular values $w_1\le\dots\le w_n$. Then $\alpha(Q)$ has semiaxes $a'_1,\dots,a'_r, ia'_{r+1},\dots,ia'_n$ which fulfill the inequalities*

$$w_1 a_j \le a'_j \le w_n a_j \quad (j=1,\dots,n)$$

Here $w_j\cdot\infty = \infty$. When treating each a'_j separately, this inequality cannot be further improved.

Proof: The singular values of α are defined as the semiaxes of the α-images of the unit sphere of $\mathbb{R}^n$. Thus $w_1 = \min_{\|x\|=1} \|\alpha(x)\|$ and $w_n = \max_{\|x\|=1} \|\alpha(x)\|$, which implies

$$w_1\|x\| \leq \|\alpha(x)\| \leq w_n\|x\|.$$

Then the statement follows from $\max_{x\in\alpha(V)\cap\alpha(Q)} \|x\| = \max_{x\in V\cap Q} \|\alpha(x)\|$ and Theorem 4.

Further it is easy to find for each j a linear transformation such that either $w_1 a_j = a'_j$ or $w_n a_j = a'_j$. □

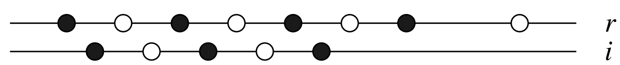
There is the following version of the interlacing theorem for quadrics, which follows from Th. 2, or from Th. 4:

Theorem 6 *Let Q be a quadric with semiaxes $a_1 \leq \ldots \leq a_r \leq \infty$ and $\infty > ia_{r+1} \geq \ldots \geq ia_n$. Let H be a hyperplane which contains the origin. Then the quadric $Q' = H \cap Q$ has semiaxes $a'_1, \ldots, a'_{r-1}, ia_r, \ldots, ia'_{r-1}$ or $a'_1, \ldots, a'_r$, $ia'_{r+1}, \ldots, ia'_{r-1}$ which fulfill the inequalities:*

$$a_1 \leq a'_1 \leq a_2 \leq a'_2 \leq \ldots \leq a_{r-1} \leq a'_{r-1} \tag{3}$$

$$a_{r+1} \geq a'_{r+1} \geq a_{r+2} \geq a'_{r+2} \geq \ldots \geq a'_{n-1} \geq a_n. \tag{4}$$

If Q' has r real semiaxes, then $a'_r \geq a_r$, otherwise we have $a_r \geq a_{r+1}$. The semiaxes on the real and imaginary axis can by symbolized by the following diagram:



Proof: We have

$$\begin{aligned}
\min_{V\in\mathcal{M}_k} \max_{x\in Q\cap V} \|x\| &\geq \min_{V\in\mathcal{M}_k} \max_{x\in Q\cap V\cap H} \|x\| \\
\max_{V\in\mathcal{M}_{n-k}} \min_{x\in Q\cap V} \|x\| &\leq \max_{V\in\mathcal{M}_{n-k}} \min_{x\in Q\cap V\cap H} \|x\|.
\end{aligned}$$

This implies $a_i \leq a'_i \leq a_{i+1}$ for $i = 1, \ldots, r-1$, and $a'_r \geq a_r$ if Q' has r real semiaxes. The other inequality follows if we use $Q^\star$ instead of Q. □

The well-known theorem on the semiaxes of ellipsoids one of which is contained in the other, also can be generalized without much effort:

Theorem 7 *Let Q, Q' be two quadrics centered in the origin with semiaxes $a_1 \leq \ldots \leq a_r \leq \infty$, $\infty > ia_{r+1} \geq \ldots \geq ia_n$ and $a'_1 \leq \ldots \leq a'_n \leq \infty$, i.e., Q' is an ellipsoid or cylinder with ellipsoidal base quadric. Let Q' be contained in the connected component of $\mathbb{R}^n \setminus Q$ which contains the origin. Then*

$$a'_i \leq a_i \quad i = 1, \ldots, r \tag{5}$$

Proof: Let $a'_1, \ldots, a'_q$ be finite and $a'_{q+1} = \ldots = a'_n = \infty$. Consider a quadric Q'' with semiaxes $a''_1 \leq \ldots \leq a''_j$, $a''_j = a'_j$ for $j \leq q$ and $a''_{q+1}, \ldots, a''_r > a'_q$. Then Q'' is contained in the interior of Q'. Every ray emanating from the origin which intersects Q in a point p, also intersects Q'' in a point $p'' = \lambda p$ with $\lambda \leq 1$. Thus for all subspaces $V \in \mathcal{M}_k$ we have

$$\max_{x\in Q\cap V} \|x\| \geq \max_{x\in Q''\cap V} \|x\|,$$

which implies $a_j \geq a''_j$ for all $j \leq r$. As a''_j was arbitrary for $q < j \leq r$, this shows $a_j = \infty$ in this case. □

3 Semiaxes of conics which touch each other

Given are two conics Q, Q' (quadrics of $\mathbb{R}^2$) centered in the origin. We look for a rotated version $\alpha(Q')$ of Q' such that $Q, \alpha(Q')$ *touch* each other. This touching relation will be slightly different from the usual notion of touching of conics in the sense of projective geometry, i.e., there is a pair (point, tangent) which belongs to both conics.

Definition: Two affine conics Q, Q' of $\mathbb{R}^2$ centered at the origin touch each other, if they are equal or if the linear pencil of conics spanned by Q and Q' contains a double line through the origin. We say that Q and Q' are related, if they can be rotated such that they touch each other, and we write $Q \sim Q'$.

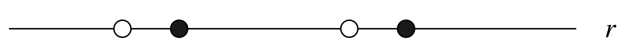
Obviously $\sim$ is a reflexive and symmetric relation.

If the pencil of conics spanned by Q and Q' contains a double line l through the origin, there are the following possibilities:

- If l intersects Q in two points $\pm x$, then $\pm x = l \cap Q = l \cap Q'$ and Q, Q' touch each other in $\pm x$. If Q or Q' is an ellipse, this is always the case.
- If both Q and Q' are pairs of straight lines, then l is parallel to both of them.
- If both Q, Q' are hyperbolas and l is an asymptote of Q, then l is also an asymptote of Q' and Q, Q' hyperosculate at infinity (see Fig. 2).
- In the remaining cases l intersects neither Q nor Q', but intersects their conjugate conics in two points $\pm y = l \cap Q^\star = l \cap Q'^\star$, which then touch each other in $\pm y$.

We say that two families a_i and b_i of points in $\mathbb{R} \cup \{\infty\}$ separate each other if there are i, j, k, l with $i \neq j, k \neq l$ such that $a_i \leq b_k \leq a_j \leq b_l$ or $b_k \leq a_i \leq b_l \leq a_j$. Note that if there is an a_i which equals a b_k, then for all a_j, b_l this condition is fulfilled.

Lemma 8 *Given are two ellipses Q, Q' with semiaxes a_1, a_2 and a'_1, a'_2. We have $Q \sim Q'$ if and only if the pairs a_1, a_2 and a'_1, a'_2 separate each other:*



This means that either $a_1 \leq a'_1 \leq a_2 \leq a'_2$ or $a'_1 \leq a_1 \leq a'_2 \leq a_2$.

Other ellipses are not related, and an ellipse is not related to a conic with imaginary semiaxes ia'_1, ia'_2.

Proof: If $a_i = a'_j$ for some pair i, j, then clearly $Q \sim Q'$, so we restrict ourselves to the case of strict inequality: If $a_1 < a'_1$ and $a_2 < a'_2$ then the ellipses $Q : (x_1/a_1)^2 + (x_2/a_2)^2 = 1$ and $Q' : (x_1/a'_1)^2 + (x_2/a'_2)^2 = 1$ have no points in common.

If in addition $a'_2 > a_1$, and α_0 is a rotation about 90 degrees, then Q and $\alpha_0(Q')$ have four points in common, and for continuity reasons, there is an $\alpha \in \mathrm{SO}_2$ such that Q and $\alpha(Q')$ touch each other in two points. (see Fig. 1).

If, on the other hand, $a_1, a_2 < a'_1, a'_2$ then $\max_{x \in Q} \|x\| < \min_{x \in Q'} \|x\|$, and we have $Q \not\sim Q'$. If $a_1 < a'_1 \leq a'_2 < a_2$, we assume that Q has equation $(x_1/a_1)^2 + (x_2/a_2)^2 = 1$, and have a look at

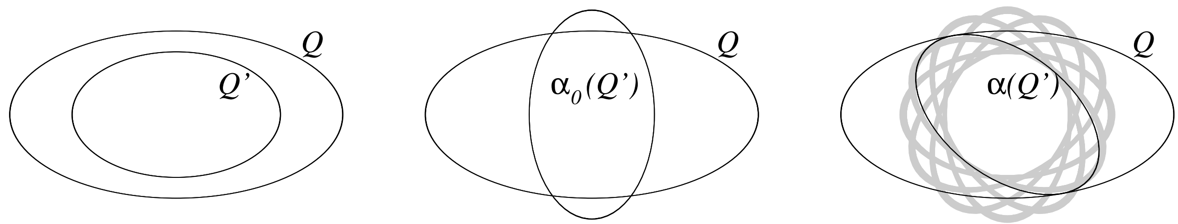


Figure 1: Ellipses touching each other

the intersection points $\pm P_1$ and $\pm P_2$ of $\alpha(Q')$ with the x_1- and x_2-axis, respectively. Theorem 4 implies that $\pm P_1$ is outside Q and $\pm P_2$ is inside Q. By continuity, $\#Q\cap\alpha(Q')=4$ and therefore $Q\not\sim Q'$.

If Q has no real points and Q' is an ellipse, then $Q\cap Q'=\emptyset$ implies that $Q\not\sim Q'$. □

Lemma 9 *Given is an ellipse Q with semiaxes $a_1\le a_2$ and a hyperbola Q' with semiaxes a'_1, ia'_2. Then $Q\sim Q'$ if and only if $a_1\le a'_1\le a_2$: If Q is a conic with semiaxes $ia_1\le ia_2$, then $Q\sim Q'$ if and only if $a_1\le a'_2\le a_2$.*

Proof: The proof is similar to that of Lemma 8. If $a_i=a'_1$, then obviously $Q\sim Q'$. If $a_1<a'_1<a_2$, there is a position of Q such that $\#Q\cap Q'=0$ and after a rotation α_0 about 90 degrees we have $\#\alpha_0(Q)\cap Q'=4$, so there is an $\alpha\in SO_2$ such that $\alpha(Q)$ and Q' touch each other.

If $a_1,a_2<a'_1$, then $\alpha(Q)\cap Q'$ is void for all α, and if $a_1,a_2>a'_1$, then $\alpha(Q), Q'$ have four intersection points for all α.

The statement on the conic with two imaginary semiaxes follows from the fact that $Q\sim Q'$ if and only if $Q^\star\sim Q'^\star$. □

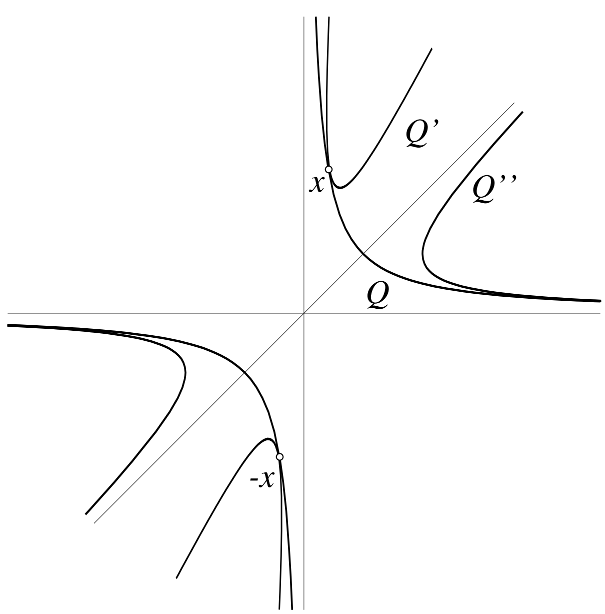


Figure 2: Hyperbolas: Q, Q'' osculate at infinity, whereas Q, Q' touch in the points $\pm x$.

Lemma 10 *Given are two hyperbolas Q, Q' with semiaxes a_1, ia_2, a'_1, ia'_2, $a'_1 \geq a_1$. We can rotate Q' such that it then touches Q in two real points if and only if*

$$a_1 a_2 > a'_1 a'_2$$

If $a_1 a_2 = a'_1 a'_2$, then Q' can be rotated such that it then hyperosculates Q at infinity.

Proof: It is easily seen that the only way how Q and a rotated version of Q' can touch each other is like in Fig. 2, and this happens if and only if a'_2 is smaller than a''_2, where the hyperbola with semiaxes a'_1, ia''_2 hyperosculates Q at infinity.

Suppose that one of the asymptotes of Q is the x-axis. Then the hyperbolas Q' hyperosculating Q in the horizontal point at infinity are precisely the images of Q under the area-preserving shear transformations $x' = x + ky$, $y' = y$. To prove $a_1 a_2 = a'_1 a'_2$ we note that all parallelograms whose vertices lie on the asymptotes of Q and two of whose edges touch Q have equal area. Now this area remains invariant under shear transformations, which implies the result. □

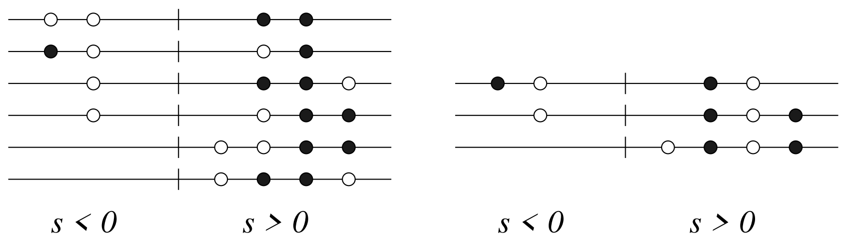


Figure 3: Squared semiaxes of conics Q, Q'. Left: $Q \not\sim Q'$ Right: $Q \sim Q'$

Theorem 11 *Consider two conics Q, Q' with squared semiaxes s_1, s_2, s'_1, s'_2. Then $Q \sim Q'$ if and only if the pairs (s_1, s_2) and (s'_1, s'_2) separate each other. (see Fig. 3).*

Proof: We will show two proofs of this result. The first uses the continuity arguments of the previous lemmas, whereas the second is purely algebraic.

1. If $s_i = s'_j$, then obviously $Q \sim Q'$, so assume that $s_i \neq s'_j$ for all $i, j = 1, 2$.

   If both Q and Q' are ellipses, the result follows from Lemma 8. If one of Q, Q' is an ellipse and the other is a hyperbola, it follows from Lemma 9. If one of Q, Q' is without real points, the results follows from the fact that $Q \sim Q' \iff Q^\star \sim Q'^\star$.

   If both Q and Q' are hyperbolas, then without loss of generality assume $s_1, s'_1 > 0$, $s_2, s'_2 < 0$. Then we have the four cases (i): $s_1 < s'_1, s_2 > s'_2$. (ii): $s_1 > s'_1, s_2 < s'_2$, (iii): $s_1 < s'_1, s_2 < s'_2$, (iv): $s_1 > s'_1, s_2 > s'_2$, In cases (iii) and (iv) the pairs separate each other, in cases (i) and (ii) they don't. In case (i) we have $s'_1 > s_1$ and $|s_1 s_2| < |s'_1 s'_2|$, so Lemma 10 implies that Q' cannot be rotated such that it touches Q in two real points. We have also $|s'_2| > |s_2|$, so Lemma 10 again shows that also $Q'^\star$ cannot be rotated such that it then touches $Q^\star$ in two real points. Thus $Q \not\sim Q'$. Case (ii) is the same with the roles of Q, Q' interchanged.

   In Case (iii) we have both $|s_1|' > |s_1|$ and $|s_2| > |s'_2|$, so Lemma 10 shows that either we can rotate Q' such that it then touches Q in two real points (if $|s_1 s_2| > |s'_1 s'_2|$) or the same with the conjugate hyperbolas (if $|s_1 s_2| < |s'_1 s'_2|$). If $|s_1 s_2| = |s'_1 s'_2|$, then we can rotate Q' such that it then hyperosculates Q at infinity, and we have shown that (iii) implies $Q \sim Q'$. Case (iv) is similar.

If Q is a pair of lines then it behaves similar to an ellipse and some simple arguments show the result.

2. Denote the reciprocals of s_1, s_2 and s'_1, s'_2 by λ_1, λ_2 and λ'_1, λ'_2, respectively. The matrix of a rotation about an angle ϕ is given by $S = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$. We can rotate Q such that it touches Q' if and only if there is a ϕ such that

$$\det\left(S^T \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} S + \begin{pmatrix} \lambda'_1 & \\ & \lambda'_2 \end{pmatrix} \right) = 0.$$

The left hand side of this equation equals $(\lambda'_1 - \lambda_1)(\lambda'_2 - \lambda_2) + \sin^2\phi(\lambda_2 - \lambda_1)(\lambda'_2 - \lambda'_1)$. A detailed discussion shows easily that there is a solution if and only if the pairs λ_1, λ_2 and λ'_1, λ'_2 separate each other. This equivalent to the statement of the theorem.

□

4 Semiaxes of touching quadrics

As in the case of conics, we define when two quadrics touch each other:

Definition: Two affine quadrics Q, Q' of $\mathbb{R}^n$ touch each other, if the unique quadric contained in the pencil spanned by Q, Q' which contains the origin has a vertex space of dimension greater or equal one. If there is an $\alpha \in \mathrm{SO}_n$ such that Q touches $\alpha(Q')$, then we say that Q, Q' are related to each other and write $Q \sim Q'$.

Obviously $\sim$ is again a symmetric and reflexive relation.

The quadrics with such a vertex space are easily enumerated in low dimensions: In $\mathbb{R}^2$, only the double line has this property. In $\mathbb{R}^3$, there is the union of two planes, and the double plane. In $\mathbb{R}^4$, there is the double hyperplane, a union of two hyperplanes, the cone with vertex line and regular conic as base curve.

There are the following possibilities for touching quadrics Q, Q': Let l be a 1-dimensional subspace which is contained in the vertex subspace of the quadric mentioned above. If $l \cap Q = \pm x$, then Q and Q' touch each other in the usual sense in $l \cap Q = l \cap Q'$. If $l \cap Q^\star = \pm y$, then $Q^\star$ and $Q'^\star$ touch each other in the usual sense in $\pm y$. If both $l \cap Q$ and $l \cap Q'$ are void, then consider the trace pencil in any 2-dimensional subspace $\varepsilon \supset l$. It contains the conics $k = \varepsilon \cap Q$, $k' = \varepsilon \cap Q'$ and the double line l. l intersects neither k nor its conjugate $k^\star$, which means that either k_1, k_2 hyperosculate at infinity, or both are degenerate.

Lemma 12 *Let Q, Q' be two quadrics with squared semiaxes $0 < s_1 \leq \ldots \leq s_n < \infty$ and $s'_1 \leq \ldots \leq s'_n \leq \infty$, i.e., Q is an ellipsoid. If the s_i and the s'_i do not separate each other, then $Q \not\sim Q'$.*

Proof: We let $I = [s_1, s_n]$ and $I' = [s'_1, s'_n]$. Because the s_i don't separate the s'_j, there is no s_i in I' or no s'_j in I or neither. If there is an s'_j in I but no s_i in I', then all s'_j are finite and positive and Q' is also an ellipsoid. Thus without loss of generality we may assume that no s'_j is contained in I, which means that either $s'_j < s_1$ or $s'_j > s_n$ for all $j = 1, \dots, n$.

We note that the touching relation is invariant with respect to linear automorphisms of $\mathbb{R}^n$. Consider the semiaxes $a_j = \sqrt{s}_j$ of Q and define the linear mapping ϕ by

$$e_j \mapsto \frac{a_1}{a_j} e_j.$$

Its singular values are a_1/a_j, their minimum equals a_1/a_n, and their maximum equals 1. All semiaxes of $\phi(Q)$ equal a_1, and the semiaxes a''_i of $\phi(Q')$ fulfill the inequalities

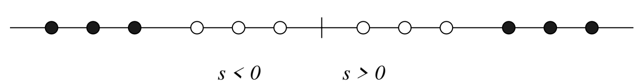
$$\frac{a_1}{a_n} a'_j \leq a''_j \leq a'_j,$$

where $a'_j = \sqrt{s'_j}$ are the semiaxes of Q'.

If a'_j is real and greater than a_n, then $a''_j \geq (a_1/a_n)a'_j > (a_1/a_n)a_n = a_1$. If a'_j is real and less than a_1, then $a''_j \leq a'_j < a_1$. This implies that no semiaxis a''_j equals a_1. Because $\phi(Q)$ is a sphere, we conclude that $\phi(Q)$ and $\phi(Q')$ do not touch, and therefore neither do Q, Q', which means $Q \not\sim Q'$. □

Lemma 13 *Let Q, Q' be two quadrics with squared semiaxes $-\infty < s_1 \leq \ldots \leq s_r < 0$, $0 < s_{r+1} \leq \ldots \leq, s_n < \infty$, $1 \leq r \leq n-1$. and $s'_1 \leq \ldots \leq s'_n$, i.e., Q is a non-void quadric with finite semiaxes which is no ellipsoid. If the s_i do not separate the s'_j, then $Q \not\sim Q'$.*

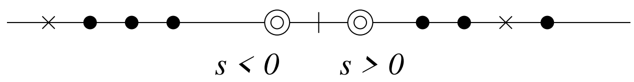
Proof: Let $I = [s_1, s_n]$ and $I' = [s'_1, s'_n]$ and have a closer look at the case that there is an s'_j in I but no s_i is in I': Here also Q' has only finite semiaxes. If Q' is an ellipsoid, then Lemma 12 implies the result. If Q' has no real points, then $Q \sim Q' \iff Q^\star \sim Q'^\star$ and Lemma 12 show that also in this case we have $Q \not\sim Q'$. If Q' is a non-void quadric which is no ellipsoid, then we interchange Q and Q', so without loss of generality we can assume that there is no s'_i in I.



(Squared semiaxes of Q are marked by empty circles, those of Q' by filled ones.) We assume a coordinate system such that Q has normal form. The semiaxes of Q, Q' are denoted by a_i, ia_k and a'_j, ia'_l, resp. (black circles). Let $a = \min(a_r/a_1, a_{r+1}/a_n)$, and define a linear transformation ϕ by $e_j \mapsto (aa_r/a_j)e_j$ if $1 \leq j \leq r$ and $e_j \mapsto (aa_{r+1}/a_j)e_j$ if $r < j \leq n$. The minimum and maximum singular values of ϕ are a and 1.

Then $\phi(Q)$ has semiaxes $i\bar{a}_1 = \ldots = i\bar{a}_r = iaa_r$ and $\bar{a}_{r+1} = \ldots = \bar{a}_n = aa_{r+1}$ (their squares are marked by empty circles below). The quadric $\phi(Q') = \bar{Q}'$ has semiaxes $\bar{a}'_j$ which fulfill the inequalities $aa'_j \leq \bar{a}'_j \leq a'_j$.

Thus if ia''_j is an imaginary semiaxis of $\bar{Q}'$, then $a''_j \geq aa'_j > aa_1 \geq aa_r = \bar{a}_1$. If a''_j is a real semiaxis of $\bar{Q}'$,t then $a''_j \geq aa'_j > aa_n \geq aa_{r+1} = \bar{a}_n$.



Now suppose that Q touches Q' and consider a 1-dimensional subspace contained in a vertex subspace of a quadric of the pencil spanned by $\bar{Q}, \bar{Q}'$. Let $l_1 = [e_1, \ldots, e_r] \cap (l + [e_{r+1}, \ldots, e_n])$ and $l_2 = [e_{r+1}, \ldots, e_n] \cap (l + [e_1, \ldots, e_r])$. The conic $[l_1, l_2] \cap \bar{Q}$ is a hyperbola with axes $\bar{a}_1, \bar{a}_n$. The conic $[l_1, l_2] \cap \bar{Q}'$ has semiaxes (their squares are marked by crosses in the figure above), which, according to the iterated interlacing theorem (Th. 6) do not separate $\bar{a}_1, \bar{a}_n$. Thus these two conics do not touch, which implies in turn that neither $\bar{Q}, \bar{Q}'$ nor Q, Q' touch each other, and therefore $Q \not\sim Q'$. □

Now we are able to prove our main theorem on touching quadrics:

Theorem 14 *Let Q, Q' be two quadrics with squared of semiaxes $s_1, \ldots, s_n$ and $s'_1, \ldots, s'_n$. The n-tuples (s_i) and (s'_i) separate each other if and only if $Q \sim Q'$.*

Proof: We denote the unit vectors in direction of the jth axis of the quadric Q (Q', resp.) by e_j (e'_j, resp.). If we can find two pairs (s_j, s_k), (s'_l, s'_m) which separate each other then there is an $\alpha_1 \in \mathrm{SO}_n$ which maps the $[e'_l, e'_m]$ onto $[e_j, e_k]$. Then the conic $\alpha_1(Q' \cap [e'_l, e'_m])$ can be rotated in $[e_j, e_k]$ such that it touches $Q \cap [e_j, e_k]$. There is a unique α_2 in SO_n which extends this rotation and leaves the span of the other e_i pointwise fixed. Then Q and $\alpha_2\alpha_1(Q')$ touch each other and we have $Q \sim Q'$.

We now show the converse. Assume that the semiaxes of Q, Q' do not separate each other. There are at least two ways to show that $Q \not\sim Q'$, the second of which is more elementary.

1. Consider the equations $\langle x, Ax \rangle$ and $\langle x, Bx \rangle = 1$ of Q and Q'. A rotated version of Q' then has equation $\langle x, S^T BSx \rangle = 1$ with an orthogonal matrix S. The unique quadric contained in the pencil spanned by Q, Q' which contains the origin has equation $\langle x, (A - S^T BS)x \rangle = 0$. It has a nonzero vertex space if and only if $\det(A - S^T BS) = 0$.

   The eigenvalues of A and $S^T BS$ are $1/s_i$ and $1/s'_i$, respectively. Without loss of generality assume that no $1/s_i$ is between two $1/s'_j$. Then there are k eigenvalues of $S^T BS$ which are greater than all eigenvalues of A, and $n-k$ eigenvalues of $S^T BS$ which are less than all eigenvalues of A. Th. 3 applied to A and $S^T BS$ implies that the sign of $\det(A - S^T BS)$ equals $(-1)^k$, and so $\det(A - S^T BS) \neq 0$, which implies $Q \not\sim Q'$.

2. If both Q, Q' have infinite semiaxes, then $Q \sim Q'$, so assume that Q has only finite semiaxes. Q is an ellipsoid, the result follows from Lemma 12. If Q' is without real points, the results follows from the fact that $Q \sim Q' \iff Q^\star \sim Q'^\star$ and Lemma 12. If Q is non-void and no ellipsoid, the result follows form Lemma 13. □

Corollary 15 *Let A, B be symmetric $n \times n$ matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively. There are $\alpha_i, \alpha_j, \beta_k, \beta_l$ with $\alpha_i \leq \beta_j \leq \alpha_k \leq \beta_l$ or vice versa (i.e., $\geq$ instead of $\leq$), if and only if there is an orthogonal $n \times n$ matrix S with $\det(A - S^T BS) = 0$.*

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