CONVERGENCE AND C^1 ANALYSIS OF SUBDIVISION SCHEMES ON MANIFOLDS BY PROXIMITY

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Abstract. Curve subdivision schemes on manifolds and in Lie groups are constructed from linear subdivision schemes by first representing the rules of affinely invariant linear schemes in terms of repeated affine averages, and then replacing the operation of affine average either by a geodesic average (in the Riemannian sense or in a certain Lie group sense), or by projection of the affine averages onto a surface. The analysis of these schemes is based on their proximity to the linear schemes which they are derived from. We verify that a linear scheme S and its analogous nonlinear scheme T satisfy a proximity condition. We further show that the proximity condition implies the convergence of T and continuity of its limit curves, if S has the same property, and if the distances of consecutive points of the initial control polygon are small enough. Moreover, if S satisfies a smoothness condition which is sufficient for its limit curves to be C^1, and if T is convergent, then the curves generated by T are also C^1. Similar analysis of C^2 smoothness is postponed to a forthcoming paper.

1. Introduction

This paper defines and analyzes a wide class of curve subdivision schemes on manifolds. Curve subdivision schemes in general consist of repeated refinement of control polygons. Especially well studied are the linear schemes with rules for defining the control points at the finer level as finite linear combinations of control points in the coarser level — see e.g. [10] and [31]. Since any such convergent curve subdivision scheme is affinely invariant (cf. [10]), we prove that the rules of the scheme can be expressed in terms of repeated affine averages. Explicit representations of this type are given for the quadratic and cubic B-spline schemes, and for the 4-point interpolatory scheme of [11]. This representation of affinely invariant linear subdivision schemes, which is not unique, is used to define nonlinear schemes on manifolds in two different ways. One way is to replace affine averages by geodesic averages. The second consists of projecting affine averages onto the manifold. These constructions of nonlinear schemes from linear ones apply to surfaces, to Lie groups and in particular to matrix groups such as the Euclidean motion group, and to abstract manifolds such as the hyperbolic plane. Further applications of these two concepts can be found in [30].

The analysis of such a nonlinear scheme is performed by its proximity to the corresponding linear scheme which it was derived from. The proximity condition is proved to hold for all the above mentioned nonlinear schemes. It is shown that if the linear scheme is convergent, the proximity condition leads to the convergence of any analogous nonlinear scheme, and to the continuity of its limit curves, provided that the distances of consecutive points in the initial control polygon are small enough. Moreover, if the linear scheme satisfies a certain condition which is sufficient for C^1 limit curves, then each of its analogous convergent nonlinear schemes generate C^1 limit curves. Furthermore, the limit curves generated from the same initial control polygon by different nonlinear schemes, all derived from the same linear scheme, are close to each other.

Analysis by proximity to a linear scheme is a technique which was used before in various situations. We mention the paper by [12], which analyzes linear non-stationary schemes by proximity to linear stationary schemes. In the context of nonlinear schemes proximity is used in [5], and in the analysis of median interpolating subdivision schemes and their extensions in [33], based on the paper of [24]. In the first two papers mentioned above, the conditions of proximity required for smoothness are too restrictive. In the last two papers the nonlinearity is rather weak. Other papers related to median interpolating subdivision schemes are [8], [25], and [32].

Non-linear interpolatory schemes in Lie groups were constructed from linear schemes by [7], and used in various applications such as in smoothly interpolating a motion given at discrete instances. A similar construction of spline-like subdivision schemes on manifolds is suggested by [9]. Although these constructions are different from the constructions in this paper, we believe that the analysis tools developed here and in our next paper concerning C^2 smoothness ([29]) apply to these nonlinear schemes.

A general analysis of certain subdivision schemes on abstract Riemannian manifolds is done in [20], [19], and [21]. The geodesic analogues of the second and third degree B-spline Lane-Riesenfeld algorithms are shown to converge to smooth curves with Lipschitz derivatives.

We would like to mention a few other kinds of nonlinear schemes. In the functional setting, interpolatory schemes based on the idea of essential non-oscillation are studied in [4], a certain class of weakly nonlinear schemes are studied in [23], and shape preserving schemes are studied in [14]. In the geometric setting, examples of geometry driven schemes are presented in [16]. The analysis of the above schemes is along different lines, and applies to the particular class of schemes studied.

The outline of the paper is as follows. Section 2 discusses linear schemes and the construction of their analogous nonlinear schemes on surfaces and in matrix groups. All the proofs of the results in this section are postponed to Appendix A (Section 6). The results in Section 3 are rather general and are not confined to the schemes of Section 2. Conditions for convergence of subdivision schemes and for the C^1 smoothness of the generated limit curves are formulated. These conditions are known for linear schemes. A proximity condition between two schemes is introduced. It is shown that convergence and C^1 smoothness for a scheme T follow from the proximity condition satisfied by T and a linear scheme S, which satisfies conditions for convergence and C^1 smoothness. The proofs of the results in this section are given in Appendix B (Section 7). Section 4 returns to the schemes introduced in Section 2, and verifies that a proximity condition holds between a linear scheme and its analogous nonlinear scheme, constructed in one of the ways described by Section 2. The proofs of the results in this section are presented in Appendix C (Section 8). The results of Sections 2–4 are then combined in Section 5. Convergence and C^1 smoothness is stated and proved for the nonlinear schemes constructed in Section 2 from appropriate linear schemes, and also for schemes on abstract Riemannian manifolds and in a certain class of Lie groups.

2. Linear and Nonlinear Subdivision Rules based on Averaging

2.1. **Linear Subdivision Rules and Averaging.** We use the symbol p for a sequence of points p_i. A subdivision scheme S is a mapping which takes a point sequence p as input, and which has another point sequence Sp as output. For the sake of simplicity we consider only infinite sequences p_i, where the index i runs in the integers. Closed polygons are modeled by periodic infinite sequences. An 'ordinary' finite polygon $p_1, \ldots, p_r$ is represented by the sequence $\ldots, p_1, p_1, p_2, \ldots, p_r, p_r, \ldots$. We assume that there is an integer dilation factor $N > 1$ such that for all polygons p, q the relation $q_i = p_{i+1}$ for all i implies that $(Sq)_i = (Sp)_{i+N}$. The case of a dilation factor $N > 2$ is important to us because sometimes in the analysis it is necessary to consider several applications of a binary subdivision scheme as one round of subdivision. For the sake of a unified treatment, we allow that N assumes any value greater or equal two.

We restrict our attention to subdivision schemes whose definition uses the notion of *average* or *affine combination*. We let

$$\mathrm{av}_\alpha(x, y) := (1 - \alpha)x + \alpha y. \tag{1}$$

We write down the definition of some well-known subdivision rules in terms of the av operator: The interpolatory four-point scheme of [11] has dilation factor $N = 2$ and is defined by

$$Sp_{2i} = p_i, \quad Sp_{2i+1} = \mathrm{av}_{1/2}(\mathrm{av}_{-2w}(p_i, p_{i-1}), \mathrm{av}_{-2w}(p_{i+1}, p_{i+2})). \tag{2}$$

Degree n B-spline subdivision "$S_{(n)}$" according to [15] has $N = 2$ and is recursively defined by one splitting step and n averaging steps:

$$\begin{aligned} &(S_{(0)}p)_{2i} = (S_{(0)}p)_{2i+1} = p_i, \\ &(S_{(m)}p)_i = \mathrm{av}_{1/2}((S_{(m-1)}p)_i, (S_{(m-1)}p)_{i+1}), \quad m = 1, \ldots, n. \end{aligned} \tag{3}$$

We mention two cases explicitly: Quadratic B-Spline subdivision (Chaikin's algorithm) has the form

$$S_{(2)}p_{2i} = \mathrm{av}_{1/4}(p_i, p_{i+1}), \quad S_{(2)}p_{2i+1} = \mathrm{av}_{3/4}(p_i, p_{i+1}). \tag{4}$$

Cubic B-spline subdivision "$S_{(3)}$" reads

$$\begin{aligned} S_{(3)}p_{2i} &= \mathrm{av}_{1/2}(p_i, p_{i+1}), \\ S_{(3)}p_{2i+1} &= \mathrm{av}_{1/2}(\mathrm{av}_{1/4}(p_{i+1}, p_i), \mathrm{av}_{1/4}(p_{i+1}, p_{i+2})). \end{aligned} \tag{5}$$

For a linear scheme there exists a sequence $a = (a_i)_{i\in\mathbb{Z}}$ such that

$$Sp_j = \sum_i a_{j-Ni} p_i. \tag{6}$$

a is called the *mask* of S, and is said to be finite if only finitely many a_i's are nonzero. The subdivision scheme is affinely invariant, if

$$\sum_i a_{j-Ni} = 1, \quad j = 0, \dots, N-1. \tag{7}$$

It is trivial that a finite mask exists for subdivision schemes expressible via the av operator, and that the scheme is affinely invariant. Indeed also the converse is true:

Theorem 1. *Any affinely invariant linear subdivision rule S with finite mask is expressible via the "*av*" operator.*

The proof is given in Sec. 6.2.

The expression of a subdivision rule in terms of the averaging operator is not unique. It should be noted that convergent linear subdivision schemes are either converging towards zero or are affinely invariant, see [10].

Remark: Note that Theorem 1 guarantees only that each of the N rules of the linear scheme S is expressible by the av operator. Yet it does not imply that any affinely invariant scheme has a recursive definition by repeated averaging similar to (3). $\diamondsuit$

2.2. **Geodesic Averages in Surfaces and Geodesic Subdivision.** We would like to replace the straight lines of affine space (which are the shortest curves ending in two given points) by the geodesic lines in a surface (which again are the shortest curves, at least locally), and the average of two points by a corresponding point on the geodesic. This concept belongs to Riemannian geometry, but we study it first for surfaces. The reason for this is that our method of analyzing smoothness of nonlinear schemes requires comparison with linear schemes, and for our proofs the ambient space where a surface is immersed in is necessary. We consider abstract Riemannian manifolds only in the very end.

Geodesic lines of a surface M in $\mathbb{R}^n$ in the sense of elementary differential geometry are the solution curves $c(t)$ of the symbolic differential equation

$$\ddot{c} \perp M, \tag{8}$$

and all of them are traversed with constant velocity. It is well known that for all surface curves $c(t)$ the component of $\ddot{c}(t)$ orthogonal to M depends only on $\dot{c}(t)$:

With the tangential component "$\frac{D\dot{c}}{dt}$" of $\ddot{c}$, we have

$$\ddot{c}(t) = \frac{D\dot{c}}{dt} + \mathrm{II}_{c(t)}(\dot{c}(t), \dot{c}(t)), \tag{9}$$

where $\mathrm{II}_{c(t)}$ is the vector-valued second fundamental form of M in the point $c(t)$. II_p is a symmetric bilinear mapping which takes tangent vectors at the point p as input, and whose values are vectors orthogonal to M at p (cf. [6], §6.2). It follows that geodesic lines in surfaces are the solution curves of the differential equation

$$\ddot{c} = \mathrm{II}_c(\dot{c}, \dot{c}). \tag{10}$$

Equation (10) implies that if $c(t)$ is a geodesic, then so is any curve of the form $c(at+b)$. This property allows us to re-parametrize a given geodesic such that it is traversed with unit speed. In that case the length of the curve segment between points $c(t)$ and $c(s)$ equals $|t-s|$. The reparametrization property above means that there is never a unique geodesic ending in given points p and q.

A convenient way to denote the geodesic starting at p with tangent vector v is in terms of the *exponential mapping*, which is defined as follows: $\exp_p(w)$ means the point $c(t)$, if $c(t)$ is the geodesic with initial value $c(0) = p$ and initial tangent vector $v = \dot{c}(0)$, and $w = tv$. The decomposition $w = tv$ is of course not unique, but all possible ways of computing $\exp_p(w)$ yield the same result. The geodesic $c(t)$ has the property that $\exp_p(tv) = c(t)$, for all t.

If we are to replace straight lines by geodesics, we need the existence of a unique shortest geodesic which connects the two given points (unique up to reparametrization). For an introduction into this topic see e.g. [17], § 10, or Th. 3.7 and Remark 3.8 of [6].

Basically, if p and q are close enough, there is always v, smoothly dependent on q, such that $\exp_p(v) = q$, and $c(t) = \exp(tv)$ is the shortest geodesic with $c(0) = p$ and $c(1) = q$. Within a compact subset of a complete surface this is true for all points which are closer than a given small maximum distance. The properties enumerated above follow from the fact that geodesics fulfill the differential equation (10).

Our geodesic averaging (see below) requires the existence of a continuation of the geodesic $c(t)$ beyond the defining two points. This always exists locally, and for all parameters t if the surface is complete (see the references above).

In this paper we are not concerned with the problem of existence of geodesics at all, we just assume that we can carry out all necessary constructions. We define

Definition 1. *If c is the unique shortest geodesic which joins x and y, then we let*

$$\text{g-av}_\alpha(x, y) := c(\alpha t), \ \textit{if}\ c(0) = x, c(t) = y. \tag{11}$$

The g-av operator serves as a replacement of the av operator.

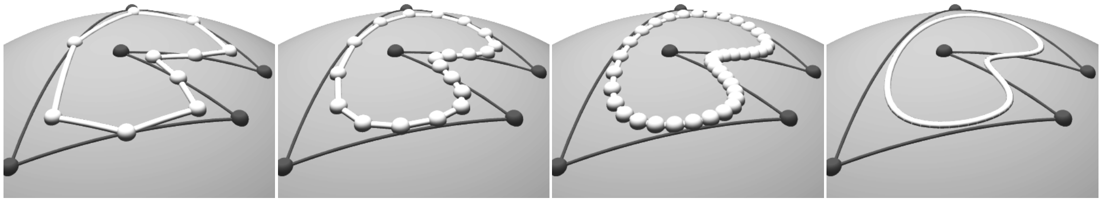

FIGURE 1. Geodesic B-Spline subdivision of degree three. From left to right: Tp, T^2p, T^3p, $T^\infty p$.

Note that both the affine average and the geodesic average fulfill the relations

$$\mathrm{av}_{1-\alpha}(y,x) = \mathrm{av}_\alpha(x,y), \quad \text{g-av}_{1-\alpha}(y,x) = \text{g-av}_\alpha(x,y). \tag{12}$$

This follows from the fact that for all geodesics $c(t)$, also $c(t_0 - t)$ is a geodesic. We should mention that even if we use the word 'average' we do not restrict the factor α to the interval $[0,1]$.

Definition 2. *The geodesic analogue T of an affinely invariant linear scheme S, which is expressed in terms of averages, is defined by replacing each occurrence of the* av *operator by the* g-av *operator.*

Fig. 1 shows the result of geodesic subdivision according to the algorithm of Lane-Riesenfeld.

2.3. **Geodesic Averages in Matrix Groups.** This section extends the concept of geodesic subdivision to the group of Euclidean motions, such that the helical motions appear as geodesic-like curves (cf. [3] or [13]). This means e.g. that the geodesic midpoint of two positions of a rigid body is found by first determining the shortest helical motion which transforms the first position (at time $t = 0$) into the other (at time $t = \tau$), and then evaluating this helical motion half way in between, i.e., at $t = \tau/2$. Fig. 2 shows the helical motions which connect given positions of a rigid body, together with the result of subdivision defined in this way.

The general concept we have to discuss here is that of a one-parameter subgroup of a matrix group, or more generally, of a Lie group. The relation between matrix groups and abstract Lie groups is in some ways similar to the relation between surfaces and abstract Riemannian manifolds. We consider the abstract case only in the end. For an introduction into Lie groups, see e.g. [22].

The curves we use for subdivision in a Lie group are called geodesics also, which will be justified when we show that they too satisfy a second order differential equation just like the geodesics in surfaces.

Let G be a linear Lie group, i.e., a smooth manifold immersed in the space of $n \times n$ matrices, which is closed with respect to matrix multiplication and matrix inversion. Prominent examples are O_n and SO_n, the groups of orthogonal matrices and of orientation-preserving orthogonal matrices. The group of Euclidean motions, here denoted by $\mathrm{SO}_n \ltimes \mathbb{R}^n$, is also a matrix group: a matrix $g \in \mathrm{SO}_n$

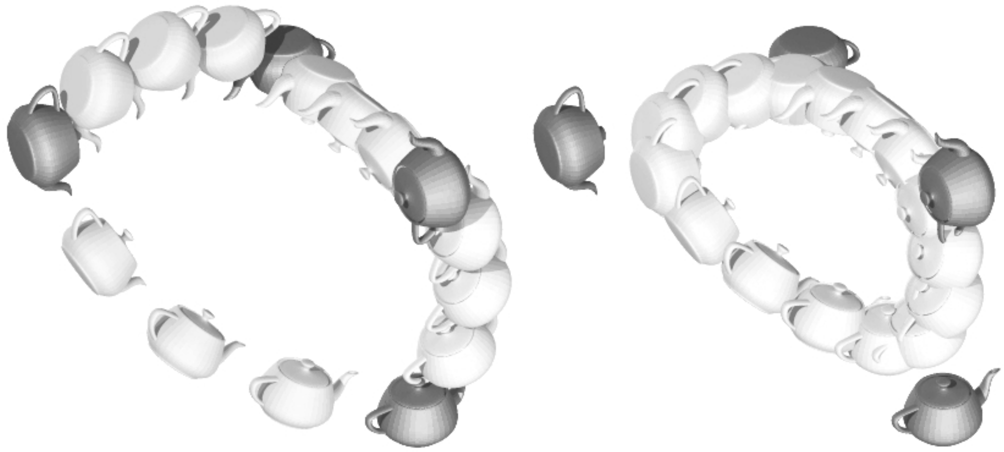

FIGURE 2. Left: Helical motions (i.e., group geodesics) which connect a sequence of positions of a rigid body. Right: Two rounds of geodesic B-spline subdivision of degree three.

and a translation vector $t \in \mathbb{R}^n$ are composed, in block matrix notation, to the $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} 1 & 0 \\ t & g \end{bmatrix} \tag{13}$$

Multiplication of two such matrices yields the result

$$\begin{bmatrix} 1 & 0 \\ t_1 & g_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ t_2 & g_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t_1 + g_1 \cdot t_2 & g_1 \cdot g_2 \end{bmatrix} \tag{14}$$

which corresponds to the composition of transformations which are represented by the matrix/vector pairs (g_1, t_1) and (g_2, t_2). Thus also the group of Euclidean motions fits the matrix group formalism.

The symbol "$\ltimes$" used in the definition of $\mathrm{SO}_n \ltimes \mathbb{R}^n$ means a certain semidirect product, and it is obvious how to define $G \ltimes \mathbb{R}^n$ for any group G of $n \times n$ matrices. For the general definition of 'semidirect product', see e.g. [22], p. 15.

One-parameter subgroups of matrix Lie groups are curves of the form

$$c(t) = \exp(tv) = \sum_{k=0}^{\infty} \frac{(tv)^k}{k!}. \tag{15}$$

The tangent vector $\dot{c}(0)$ equals v. We use those curves as the geodesics emanating from the identity element of the group. Geodesics emanating from any other element $g \in G$ are, by definition, the left translates

$$c(t) = g \cdot \exp(tv). \tag{16}$$

Why we use left translates and why the curves defined by (16) represent the helical motions, if evaluated for the Euclidean motion group, is the topic of Section 6.3.

The existence of a matrix logarithm shows that there is a neighbourhood of the identity where for all g there is a one-parameter subgroup $c(t) = \exp(tv)$ with $c(\tau) = g$, such that both v and τ depend smoothly on g. By left translation of this neighbourhood, we get an analogous local statement for any point in the group. If a concrete group like the Euclidean motion group is given, we often know how to find the shortest geodesic which ends in two given points: For $\mathrm{SO}_n \ltimes \mathbb{R}^n$, it is the shortest helical motion which connects two given positions of a rigid body.

We establish that for groups the geodesics fulfill a second order differential equation similar to the differential equation of geodesics in surfaces, and we show cases where they are traversed with constant velocity. This enables us to treat both the surface case and the Lie group case together.

Lemma 1. *Assume that G is a Lie group of $n \times n$ matrices. Then the curves of* (16) *are precisely the solution curves of the differential equation*

$$\ddot{c} = B_{c(t)}(\dot{c}(t), \dot{c}(t)), \tag{17}$$

with

$$B_g(v, w) = \frac{1}{2}(vg^{-1}w + wg^{-1}v). \tag{18}$$

If G has the property that left translations $h \mapsto gh$ are isometric with respect to a Euclidean scalar product in the linear space of $n \times n$ matrices, then the curves of (16) *are traversed with constant velocity.*

The proof is given in Sec. 6.4.

Definition 3. *A Lie group of $n \times n$ matrices is called of constant velocity, if there is a Euclidean metric in the n^2-dimensional space of matrices, such that the curves of* (16) *are traversed with constant velocity.*

We endow the n^2-dimensional vector space $\mathbb{R}^{n\times n}$ of matrices with the scalar product

$$\langle v, w\rangle = \mathrm{tr}(vw^T). \tag{19}$$

We have $\langle v, w\rangle = \langle w, v\rangle$ because of $\mathrm{tr}(vw^T) = \mathrm{tr}((vw^T)^T) = \mathrm{tr}(wv^T)$.

Lemma 2. *With the scalar product* (19)*, both G and $G \ltimes \mathbb{R}^n$ are of constant velocity, if G is a subgroup of the orthogonal group O_n. Any compact matrix group becomes a subgroup of O_n after a suitable coordinate transform.*

The proof is given in Sec. 6.4.

Lemma 2 directly applies to the (special) orthogonal groups O_n (SO_n), and also to the Euclidean motion group.

2.4. **Projecting Averages and Projection Subdivision.** The method of projection is a very general way of introducing nonlinearity.

Definition 4. *A generalized projection P onto a submanifold M of Euclidean space is a smooth mapping onto M defined in a neighbourhood of M, such that $P(x) = x$ for all $x \in M$.*

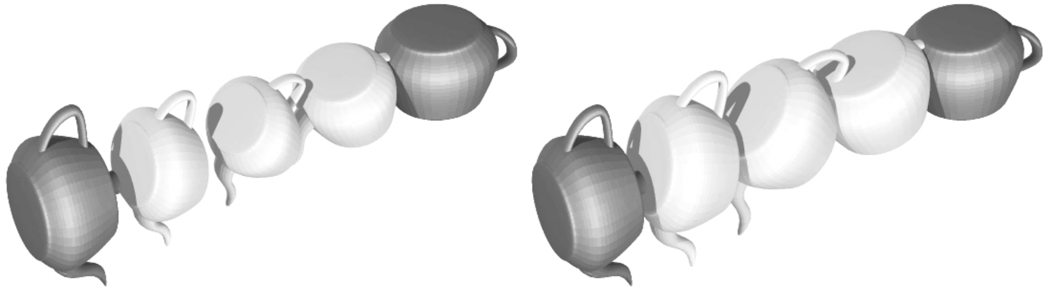

FIGURE 3. Projection onto the Euclidean motion group. (A, a) and (B, b) are two positions of the teapot, with $A, B \in \mathrm{SO}_3$ and $a, b \in \mathbb{R}^3$. Left: Positions $\mathrm{av}_\alpha((A, a), (B, b))$, Right: Positions $P\mathrm{av}_\alpha((A, a), (B, b))$. α has the values $0, 1/4, 1/2, 3/4, 1$.

How smooth exactly P must be depends on the application. We later require that the norms of first and second derivatives of P are bounded by some constants. One example of a projection is the orthogonal projection onto M.

Definition 5. *The projection analogue T of an affinely invariant linear scheme S, which is expressed in terms of averages, is defined by replacing each occurrence of the* av *operator by "Pav".*

In this way we get projection variants of the B-spline schemes and the interpolatory 4-point scheme defined by Equations (3), (4), (5), and (2), respectively.

Remark: Instead of adding a projection after each occurrence of "av", as in Def. 5, we could have defined an analogous projection scheme T by simply defining $Tp_i = PSp_i$. There is no reason why the results of this paper should not be true for this simpler definition, but the analysis is no longer analogous to the geodesic case. This is the reason why we use Def. 5 here. $\diamondsuit$

Examples of projections which are readily computable are the gradient flow towards general level set surfaces, and orthogonal projection onto selected surfaces like spheres, tori, or the Euclidean motion group. Orthogonal projections onto that group are treated by [2] and [28]. We briefly mention that if A is an $n \times n$ matrix with positive determinant, a possible projection of the affine transformation $x \mapsto A \cdot x + a$ onto the Euclidean motion group is the Euclidean motion $x \mapsto PQ \cdot x + a$, where $AJ = PDQ$ is a singular value decomposition, and J is a positive definite matrix. In applications, J is chosen as the inertia matrix of the rigid body being transformed. This projection procedure is illustrated in Fig. 3, which shows the result of both linear and projection averaging. The former in general does not yield matrices which correspond to Euclidean positions. Fig. 4 shows two rounds of interpolatory subdivision according to the projection analogue of the four-point scheme of (2).

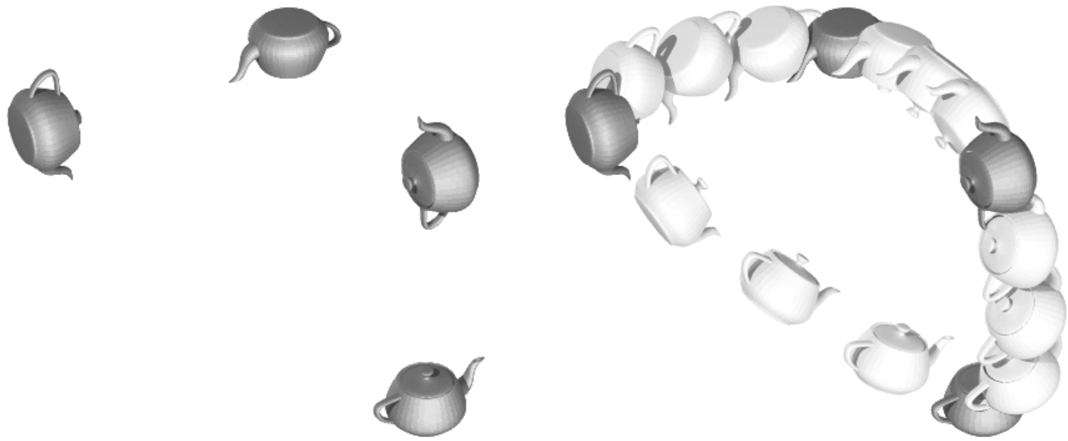

FIGURE 4. Projection subdivision in the Euclidean motion group according to the interpolatory four-point scheme of (2). Left: p. Right: $T^2 p$.

3. Convergence and Smoothness Analysis

3.1. **Convergence and Smoothness Conditions.** This section introduces conditions called 'convergence' and 'smoothness' conditions. It will be seen later that indeed they are the main ingredients in our proofs concerning the convergence of a subdivision scheme, and the continuity and smoothness of its limit curves.

If p is a sequence of points, we use the symbol Δp for the sequence of differences: $\Delta p_i = p_{i+1} - p_i$. Further we define

$$d(p) = \sup_i \|p_{i+1} - p_i\|, \quad \|p\|_\infty = \sup_i \|p_i\|. \tag{20}$$

Obviously,

$$d(p) = \|\Delta p\|_\infty. \tag{21}$$

Definition 6. *A subdivision scheme S is said to satisfy a convergence condition with factor $\mu_0 < 1$, if*

$$d(S^l p) \le \mu_0^l d(p) \text{ for all } l, p; \tag{22}$$

and it is said to satisfy a smoothness condition with factors $\mu_0, \mu_1 < 1$ and the dilation factor N, if in addition to (22) for all l, p,

$$d(N^l \Delta S^l p) \le \mu_1^l d(\Delta p). \tag{23}$$

A mixed smoothness condition is satisfied if (22) holds and there is $\mu_1 < 1$ as above such that for all l, p

$$d(N^l \Delta S^l p) \le \mu_1^l P_1(l) d(p), \tag{24}$$

where P_1 is a linear polynomial with nonnegative coefficients.

There are schemes where (22) or (23) is true only for all l greater or equal a certain number L. For example, the interpolatory 4-point scheme of (2) has $L = 2$, as is explained in more detail later. In that case we define a new subdivision rule $\overline{S} := S^L$, which then fulfills both (22) and (23). We subsequently analyze $\overline{S}$ instead of S.

Mixed conditions of the type (24) occur naturally in our smoothness analysis of nonlinear schemes. This is the reason why we consider them, instead of more familiar conditions of the form $d(N^l \Delta S^l p) \leq C_1 \mu_1^l d(p)$.

Most of our statements consider polygons whose points are contained in some subset M of $\mathbb{R}^n$, and fulfill the condition $d(p) < \varepsilon$. Such a class of polygons is denoted by $\mathcal{P}_{M,\varepsilon}$. The statements employ a scheme "S", which is linear and whose properties are known, and another scheme "T", which is to be analyzed (S is to help with the analysis). In the following we impose the additional condition that $Tp \in \mathcal{P}_{M,\varepsilon}$ if $p \in \mathcal{P}_{M,\varepsilon}$, but we don't require the same for Sp.

For instance, we will encounter the case that the smoothness conditions are true only for $p \in \mathcal{P}_{M,\delta}$ for some $\delta > 0$.

3.2. **Convergence and Smoothness of Linear Schemes.** In this section we verify that convergence and smoothness conditions actually hold for the linear schemes mentioned above. Following [10], we use the concept of k-th derived scheme S_k of a linear subdivision scheme S, which is recursively defined by

$$S_0 = S, \quad S_i(\Delta p) = N \Delta S_{i-1} p. \tag{25}$$

There may be no derived schemes. If S is affinely invariant, then S_1 exists (cf. [10]). For the convenience of the reader, we repeat some definitions here, especially because the case $N > 2$ is not so familiar. It is customary to use the terms in the sequences a (the mask of the scheme) p (the polygon), Sp (the subdivided polygon), Δp (the difference polygon) as coefficients of the formal Laurent series $a(z)$, $p(z)$ $Sp(z)$, and $\Delta p(z)$, respectively, such that e.g. $a(z) = \sum a_i z^i$. Such functions in general are called generating functions of the respective sequences, and $a(z)$ is called the *symbol* of S. By definition, and in view of (6)

$$Sp(z) = a(z) p(z^N), \quad \Delta p(z) = (1 - z) p(z) z^{-1}. \tag{26}$$

(25) implies that the symbol $a^{[1]}(z)$ of the derived scheme S_1 satisfies

$$\begin{aligned} & a^{[1]}(z) p(z^N) \frac{1 - z^N}{z^N} = N a(z) p(z^N) \frac{1 - z}{z} \\ \Longrightarrow \qquad & a^{[1]}(z) = a(z) \frac{N z^{N-1}}{1 + \cdots + z^{N-1}}. \end{aligned} \tag{27}$$

For any subdivision scheme S two rounds of subdivision yield yet another scheme, S^2. If S has dilation factor N, then the dilation factor of S^2 equals N^2. The mask c and symbol $c(z)$ of S^2 are given by

$$c_j = \textstyle\sum_i a_{j-Ni} a_i, \quad c(z) = a(z) a(z^N). \tag{28}$$

The norm $\|S\|$ of S is defined by

$$\|S\| = \sup_{\|p\|_\infty \le 1} \|Sp\|_\infty. \tag{29}$$

In terms of the mask a, we have

$$\|S\| = \max_j \textstyle\sum_i |a_{j-Ni}|. \tag{30}$$

Knowledge of the norms of derived schemes yields factors μ_0, μ_1 as required by (22):

$$d(Sp) = \|\Delta Sp\|_\infty = \frac{1}{N}\|S_1 \Delta p\|_\infty \le \frac{\|S_1\|}{N} d(p) \implies \mu_0 = \frac{1}{N}\|S_1\|; \tag{31}$$

and similarly for (23): We use $d(\Delta p) = \|\Delta^2 p\|_\infty$ and compute

$$d(N\Delta Sp) = \frac{1}{N}\|(N\Delta)^2 Sp\|_\infty \le \frac{1}{N}\|S_2\|\, d(\Delta p) \implies \mu_1 = \frac{1}{N}\|S_2\|. \tag{32}$$

B-spline subdivision of degree n according to (3) has $N = 2$ and the symbol

$$a(z) = (1+z)^{n+1}/(2z)^n \quad (n \ge 0). \tag{33}$$

Its first derived scheme is the $(n-1)$-st degree B-spline scheme. If $n \ge 2$, Equations (30), (31), and (32) show that convergence and smoothness conditions are fulfilled with factors $\mu_i = 1/2$.

If the symbol $a(z)$ of a linear scheme S with dilation factor N has the form

$$z^l(1 + z + \cdots z^{N-1}) \prod_{j=1}^{k} ((1-\alpha_j)z + \alpha_j), \quad (l \in \mathbb{Z},\ k \ge 0), \tag{34}$$

then S is defined, apart from an index shift, by a splitting step, and k averaging steps with factors α_j. An example of such a symbol for $N = 2$ is furnished by the B-spline schemes defined by (3), whose symbol is given by (33). The symbol of the interpolatory four-point scheme of (2) has the form

$$a(z) = -w\Big(\frac{1}{z^3} + z^3\Big) + \Big(\frac{1}{2} + w\Big)\Big(\frac{1}{z} + z\Big) + 1. \tag{35}$$

For $0 < w \le 1/16$, $a(z)$ has the form (34) with $l = -3$, $N = 2$, and

$$\alpha_{1,2} = -\frac{2w + \gamma \pm \sigma_1}{2w - \gamma \mp \sigma_1},\ \ \alpha_{3,4} = -\frac{2w - \gamma \pm \sigma_2}{2w + \gamma \mp \sigma_2},\ \ \alpha_5 = \frac{1}{2},$$
$$\text{where } \gamma = \sqrt{2w(1+2w)},\ \ \sigma_{1,2} = \sqrt{2w(1 - 4w \pm 2\gamma)}.$$

It follows that the four-point scheme of (2) has a recursive definition similar to (3). It satisfies a convergence condition, but not a smoothness condition. It is known (see [10], Equations (3.22)ff) that in the case $0 < w < 1/8$, the iterated scheme S^2 has the required properties. We have $\|(S^2)_1\| = 8w + 1$, and $\|(S^2)_2\| < 4$. In view of (32), it follows that for $0 < w < 1/8$, S^2 fulfills a smoothness condition with factors $\mu_0 < 1/2$, $\mu_1 < 1$, and $N = 4$.

3.3. **Proximity Conditions.** In this section we present the inequalities which we use to quantify the differences between linear subdivision schemes of known properties and nonlinear ones, in order to conclude similar properties for the nonlinear schemes. Following [26], we define

Definition 7. *Subdivision schemes S, T satisfy a proximity condition for a class $\mathcal{P}_{M,\delta}$ of polygons p, if there is a constant C such that for all $p \in \mathcal{P}_{M,\delta}$,*

$$\|Sp - Tp\|_\infty \leq Cd(p)^2. \tag{36}$$

A higher order proximity condition which involves $d(\Delta p)$ can be used to show C^2 smoothness of limit curves. This will be the subject of [29].

As has been mentioned before, it is possible that a subdivision scheme S does not fulfill (22) or (23), and we have to consider $\overline{S} := S^L$ instead. If S and T are in proximity, then obviously this is true for S^L and T^L also. So the smoothness analysis will be applied to $\overline{S}$ and $\overline{T} := T^L$.

3.4. **Convergence from Proximity and an Approximation Result.** It is our aim to show that a convergence condition satisfied by a linear subdivision scheme S together with a proximity condition satisfied by S and T implies that T also satisfies a convergence condition, and generates continuous limit curves.

Theorem 2. *Suppose that S, T satisfy a proximity condition for all $p \in \mathcal{P}_{M,\varepsilon}$, and S satisfies a convergence condition with factor $\mu_0 < 1$. Then there is $\delta > 0$ and $\overline{\mu}_0 < 1$ such that T satisfies a convergence condition with factor $\overline{\mu}_0$ for all $p \in \mathcal{P}_{M,\delta}$. By choosing δ small enough, we can achieve that $\overline{\mu}_0 - \mu_0$ is arbitrarily small.*

The proof is given in Sec. 7.1.

It is not difficult to show that a convergence condition together with proximity ensures convergence even of a nonlinear subdivision algorithm. In order to define what that means exactly, we introduce the following auxiliary functions:

Assume that T is a subdivision scheme and that p is a polygon. For each $T^j p$ we consider the piecewise linear function $T^j f$, which is linear in the intervals $[iN^{-j}, (i+1)N^{-j}]$ ($i \in \mathbb{Z}$), and whose values at the integer multiples of N^{-j} are given by the points of $T^j p$. We use the notation

$$f = \mathcal{F}_0(p), \quad Tf = \mathcal{F}_1(Tp), \quad T^2 f = \mathcal{F}_2(T^2 p), \dots \tag{37}$$

Then

$$T^\infty f = \lim_{j \to \infty} T^j f \tag{38}$$

parametrizes the limit curve "$T^\infty p$". It is obvious by construction that

$$\|p - q\|_\infty = \|\mathcal{F}_j(p) - \mathcal{F}_j(q)\|_\infty. \tag{39}$$

Theorem 3. *We assume that S is a convergent linear subdivision scheme of finite mask, and that T is as required by Theorem 2. With the notation of Theorem 2,* *we let $f = \mathcal{F}_0(p)$ for $p \in \mathcal{P}_{M,\delta}$. Then the sequence $T^j f$ converges to a continuous limit in the maximum norm.*

The proof is given in Sec. 7.1.

Remark: It is difficult to find examples where geodesic or projection subdivision do not converge. There are nonlinear schemes which fulfill a convergence condition for all p with $d(p)$ finite: It is easy to show that the geodesic analogue T of a scheme S with symbol (34), which works by one splitting step and k rounds of averaging, has the property that $d(Tp) \le \max(|\alpha_1|, |1-\alpha_1|)d(p)$, if $0 \le \alpha_j \le 1$ for $j = 2, \ldots, k$. In the case $0 < \alpha_1 < 1$ this implies a convergence condition for T, and Theorem 3 applies. ♢

When doing subdivision in a surface, we want to ensure that the limit curve $T^\infty p$ is contained in that surface, if the surface is closed.

Lemma 3. *Suppose that T converges in the sense of Theorem 3. If there is a closed set K such that $T^j p$ is contained in K for all j, then so is the limit curve $T^\infty p$.*

The proof is given in Sec. 7.1.

Our next result concerns the distance of the limit curves of a nonlinear scheme which is in known proximity to a linear scheme, from the limit curves generated by the linear scheme. The following observation is used in the statement of the theorem: If S is affinely invariant and convergent, then the norms of the iterates of S converge to 1, implying that the norms $\|S^i\|$ are uniformly bounded.

Theorem 4. *We use the requirements and notation of Theorem 2, and we assume that S has the property that $\|S^i\| \le A$. Then for any polygon $p \in \mathcal{P}_{M,\delta}$,*

$$\|S^\infty p - T^\infty p\|_\infty \le \frac{AC}{1-\overline{\mu}^2} d(p)^2. \tag{40}$$

The proof is given in Sec. 7.1.

Remark: Theorem 4 allows to transfer stability properties of S to T. If e.g. $\|S^\infty(p+\varepsilon) - S^\infty(p)\|_\infty \le D \cdot \|\varepsilon\|_\infty$, then $\|T^\infty(p+\varepsilon) - T^\infty(p)\|_\infty \le \frac{AC}{1-\overline{\mu}^2}(d(p+\varepsilon)^2 + d(p)^2) + D\|\varepsilon\|_\infty \le \frac{AC}{1-\overline{\mu}^2}(2d(p)^2 + 4d(p)\|\varepsilon\|_\infty + 4\|\varepsilon\|_\infty^2) + D\|\varepsilon\|_\infty$ ♢

3.5. **Smoothness from Proximity.** The following theorem establishes that smoothness conditions as defined by Def. 6 follow from the proximity conditions as defined in Def. 7.

Theorem 5. *Suppose that S, T satisfy the proximity condition for $p \in \mathcal{P}_{M,\varepsilon}$, and that S satisfies a smoothness condition of type* (23) *with factors μ_0, μ_1 such that*

$$\mu_0 < \mu_0^* = \frac{1}{\sqrt{N}}, \quad \mu_1 < 1. \tag{41}$$

Then there is $\delta > 0$ such that T satisfies a mixed smoothness condition of type (24) *with factors $\overline{\mu}_i$ which also satisfy* (41), *for all $p \in \mathcal{P}_{M,\delta}$.*

The proof is given in Sec. 7.2.

With this result, it is possible to show that the curves $T^\infty p$ are C^1 if $d(p)$ is small enough.

Theorem 6. *Under the conditions of Theorem 5, with S of finite mask, the limit curves $T^\infty p$ are C^1 for all polygons p such that $T^l p$ converges.*

The proof is given in Sec. 7.2.

Remark: The completeness of the norm of the space we are working in is essential for the proofs of both Theorem 3 and Theorem 6. But we neither used the finite dimension of the space, nor the fact that the norm is induced by a scalar product. ♢

4. Verification of Proximity Conditions

4.1. Geodesic Subdivision.

We show that a linear subdivision scheme and its analogous geodesic scheme (both for a surface and for a matrix group of constant velocity) fulfill a proximity condition.

We consider a surface M contained in a Euclidean vector space, which is equipped with geodesics — either in the sense of elementary differential geometry, or in the matrix group sense. In both cases, geodesics are the solution curves of a differential equation of the form

$$\ddot{c}(t) = B_{c(t)}(\dot{c}(t), \dot{c}(t)), \tag{42}$$

where B is either the second fundamental form of (10) or the expression defined by (18). B_p is supposed to depend continuously on the point p. This is trivial for the group case, and follows from C^2 smoothness of the surface under consideration in the Riemannian case. Recall that B in both cases is symmetric and bilinear. Moreover, solution curves are traversed with constant velocity, and the reparametrization properties of Section 2.2 hold true.

We use the symbol T_xM for the tangent space of M at the point x. We consider such open subsets V of M where there exists a constant D with the property that

$$x \in V, \quad v, w \in T_xM, \quad \|v\| \leq 1, \|w\| \leq 1 \implies \|B_x(v,w)\| \leq D. \tag{43}$$

Clearly all points in M have a neighbourhood V where there exists $D > 0$ such that (43) holds true. A global D exists if M is compact. In the surface case, the fact that there exists a global D for the entire surface M means that the normal curvatures of M are bounded.

In the case of a matrix group of constant velocity, the constant D can be computed explicitly: Let $D = \max_{\|v\|,\|w\|\leq 1} \|vw\|$. Then for $\|v\|, \|w\| \leq 1$ we have $\|B_g(v,w)\| \leq \frac{D}{2}(\|v\|\|g^{-1}w\| + \|w\|\|g^{-1}v\|) = D$.

The following is easy to show:

Lemma 4. *Assume that* (43) *holds true with $D > 0$ and an open set V, and that the points x, y are joined by a unique shortest geodesic of length $\leq 1/D$. If the* *geodesic segment used in* $\text{g-av}_\alpha(x,y)$ *is contained in* V*, then*

$$\|\operatorname{av}_\alpha(x,y) - \text{g-av}_\alpha(x,y)\| \leq 2D\min(|\alpha|+\alpha^2, |\beta|+\beta^2)\|x-y\|^2, \tag{44}$$

with $\beta = 1-\alpha$*.*

The proof is given in Sec. 8.2.

By using the elementary estimate of Lemma 4 several times, we are able to prove the following general result:

Lemma 5. *Let* V *and* D *be as in* (43)*. Consider an affinely invariant subdivision scheme* S *and its analogous geodesic scheme* T*. Let the class* $\mathcal{P}'_{V,\delta}$ *consist of all polygons* p *in* V *with* $d(p) < \delta$ *and which have the property that all geodesic segments used in subdividing according to* T *are contained in* V*.*

Then S *and* T *fulfill a proximity condition for all polygons* $p \in \mathcal{P}'_{V,\delta}$*. The constant* C *in the proximity condition depends on* T*,* D*, and* δ*.*

The proof is given in Sec. 8.2.

Remark: As a consequence of the proof of Lemma 5 we see that it holds also for nonstationary schemes if the factors used in averaging are bounded: The upper bound on $\|Sp - Tp\|_\infty$ as required by the proximity condition then is of the form $Cd(p)^2$, where C depends on an upper bound of these factors. ♢

4.2. **Taylor's Formula.** In proving the proximity condition for projection subdivision schemes, we represent the projection operator by its Taylor expansion. For the convenience of the reader, we write down Taylor's formula in the form we use it. If P is a mapping of sufficient smoothness from $\mathbb{R}^n$ to $\mathbb{R}^m$, then for all x, h such that the line segment with endpoints x and $x+h$ is contained in P's domain,

$$P(x+h) = P(x) + \frac{d_x P(h)}{1!} + \cdots + \frac{d_x^k P(h,\ldots,h)}{k!} + \frac{d_{x+\vartheta h}^{k+1} P(h,\ldots,h)}{(k+1)!}, \tag{45}$$

for some $\vartheta \in [0,1]$. The k-th derivative of P in the point x, $d_x^k P$, is a k-linear mapping $(\mathbb{R}^n)^k \to \mathbb{R}^m$ of the form

$$d_x^k P(u_1,\ldots,u_k) = \sum_{i_j \in \{1,\ldots,n\}} u_1^{i_1} \cdots u_k^{i_k} \frac{\partial^k P(x)}{\partial x^{i_1} \cdots \partial x^{i_k}}, \tag{46}$$

where the vectors u_i have coordinates u_i^j. Its norm is defined by

$$\|d_x^k P\| := \max\{\|d_x^k P(u_1,\ldots,u_k)\| : \|u_i\| \leq 1\}. \tag{47}$$

If the domain of P is an interval, then $d_x^k P(u,\ldots,u) = u^k P^{(k)}(x)$, and $\|d_x^k P\| = |P^{(k)}(x)|$.

4.3. **Projection Subdivision.** In order to show proximity results for projection subdivision, we require the existence of upper bounds for the norms of the projection's derivatives. In compact subsets, upper bounds always exist in analogy to the constant D of (43).

We consider an open subset U of $\mathbb{R}^n$ (the space where the surface under consideration is contained in), where there are constants $D, D' \geq 0$ such that

$$x \in U \implies \|d_x P\| \leq D, \ \|d_x^2 P\| \leq D'. \tag{48}$$

If P is the orthogonal projection onto M, then D measures a certain curvature of M, and D' has an interpretation as a change of curvature.

The following is a simple application of Taylor's formula. It is similar to Lemma 4.

Lemma 6. *Assume that U, D, D' are as in* (48)*, and that the straight line segment which contains the points $x, y, (1-\alpha)x + \alpha y$ is contained in U. Then*

$$\| \operatorname{av}_\alpha(x,y) - P\operatorname{av}_\alpha(x,y)\| \leq \frac{D'}{2} \min(|\alpha| + \alpha^2, |\beta| + \beta^2)\|x-y\|^2, \tag{49}$$

where $\beta = 1 - \alpha$.

The proof is given in Sec. 8.3.

Similar to the geodesic case, we have

Lemma 7. *(the projection analogue of Lemma 5) Let U, D, and D' be as in* (48)*. Consider an affinely invariant subdivision scheme S and its analogous projection scheme T. Let the class $\mathcal{P}'_{U,\delta}$ consist of all surface polygons p with $d(p) < \delta$, and such that the line segments used in averaging in the application of T are inside U.*

Then S and T fulfill a proximity condition for all polygons $p \in \mathcal{P}'_{U,\delta}$. The constant C in the proximity condition depends on T, D, D', and δ.

The proof is given in Sec. 8.3.

5. Results

We give a definition, which collects the requirements we impose on a linear subdivision scheme S.

Definition 8. *We call a linear subdivision scheme S 0-admissible, if it is affinely invariant and fulfills the convergence condition* (22) *with a factor $\mu_0 < 1$. S is called 1-admissible if in addition a smoothness condition* (23) *holds true, such that the factors μ_0, μ_1 are bounded according to* (41).

By the analysis of linear schemes (cf. [10]) a k-admissible subdivision scheme S produces C^k limit curves ($k = 0, 1$).

Theorem 7. *If S is a k-admissible scheme, $k = 0, 1$, and T is its analogous geodesic scheme in a surface in the sense of Section 2.2, then T converges and $T^\infty p$ is a C^k curve for all p with $d(p)$ small enough.*

Proof. Lemma 5 says that S and T meet a proximity condition. In the case $k = 0$, Theorem 2 together with Theorem 3 shows the convergence of T and the continuity of its limit curves. In the case $k = 1$, Theorem 5 shows a mixed smoothness condition for T, and Theorem 6 shows that the limit curves of T are C^1. □

We say a few words concerning the sentence "all p with $d(p)$ small enough" in the statement of Theorem 7. It does not mean that for a given surface there is a global constant δ such that for all p with $d(p) < \delta$ the theorem holds. Such a δ in general exists only in a compact subset of M, but can exist globally if there exists a global constant D such that (43) holds.

One inference however can safely be made: If a nonlinear subdivision scheme happens to converge, if applied to a given finite polygon, then the limit curve is smooth, if the appropriate conditions as set down in Theorem 7 are met. This is because p itself is contained in a compact set, for which there exists $\delta > 0$ such that the theorem applies; and in the process of subdivision, $d(p)$ converges towards zero.

In Section 2.2 we defined geodesic averaging and the geodesic analogue of an affinely invariant linear scheme by expressing it in terms of averages, and by replacing the affine average by the geodesic average. The same definition applies to Riemannian manifolds, if geodesics and the exponential mapping are understood in the Riemannian sense, see [6]. We give an example below.

Corollary 1. *Theorem 7 applies to geodesic subdivision in Riemannian manifolds.*

Proof. By the global embedding theorem of [18], any Riemannian manifold can be embedded as a surface of the same smoothness into a Euclidean space of sufficiently high dimension. Theorem 7 applies to this surface. □

Example Figure 5 shows four points connected by geodesics in the conformal disk model of the hyperbolic plane H^2. It consists of the points of the open unit disk in Euclidean $\mathbb{R}^2$. For an introduction into this topic, see e.g. [1]. The vector model of H^2 consists of the points of the upper sheet of the two-sheeted hyperboloid with equation $z^2 = x^2 + y^2 + 1$ in $\mathbb{R}^3$, which is one half of the unit sphere with respect to the pseudo-euclidean scalar product $\langle (x_1, y_1, z_1), (x_2, y_2, z_2)\rangle = x_1x_2 + y_1y_2 - z_1z_2$. Mapping a point (x, y) from the disk model to the vector model is defined by projecting the point $(x, y, 0)$ from the point $(0, 0, -1)$ onto the hyperboloid. The scalar product and norm of tangent vectors in the vector model is defined via the scalar product above. H^2 is thus equipped with a Riemannian metric. So far it is not a surface in a Euclidean space, because the scalar product we use is not Euclidean. Small pieces of H^2 are realizable as a surface of Gaussian curvature -1 in $\mathbb{R}^3$, but it has been shown that there is no C^2 immersion of the entire hyperbolic plane into $\mathbb{R}^3$. For that, we have to resort

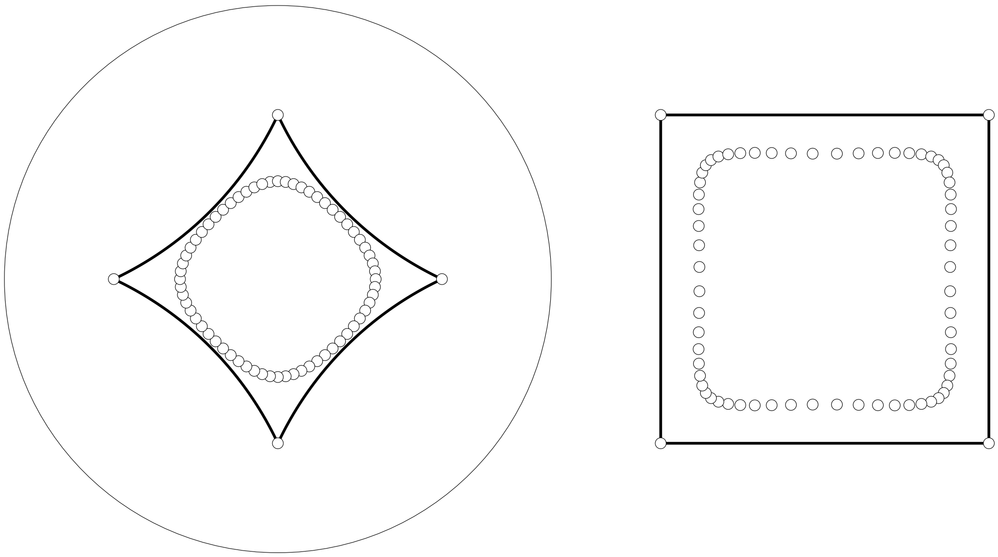

FIGURE 5. Geodesic B-Spline subdivision of degree three in the hyperbolic plane. Left: Polygons p and T^4p. Right: Polygons Δp and $2^4\Delta T^4 p$.

to higher dimensions. We never actually use this embedding except by referring to its existence in the proof of Cor. 1.

Recall that in the Euclidean unit sphere of $\mathbb{R}^3$, geodesics are defined via $\exp_p(tv) = \cos t \cdot p + \sin t \cdot v$, if v is a unit vector; and the geodesic distance $\delta(p,q)$ of points p,q fulfills $\cos\delta(p,q) = \langle p, q\rangle$. In the vector model of H^2, this is similar: $\exp_p(tv) = \cosh t \cdot p + \sinh t \cdot v$, if $\langle v, v\rangle = 1$; and $\cosh\delta(p,q) = |\langle p,q\rangle|$. In the disk model, geodesics appear as circles which intersect the unit circle orthogonally.

We demonstrate the geodesic analogue of the cubic B-spline scheme in the hyperbolic plane in Fig. 5. $\diamondsuit$

Theorem 7 not only applies to geodesic subdivision in surfaces as defined in Sec. 2.2 but also for the case of matrix groups treated in 2.3. We will however be able to show a stronger result (Theorem 8 below). The difference between these two theorems is that now there is a global constant $\delta > 0$, depending only on the scheme and the group, which ensures convergence of $T^l p$ if $d(p) < \delta$. The reason for this is that in the matrix group case, there is a global constant D for (43).

Theorem 8. *Assume that G is a matrix group of constant velocity. If S is a k-admissible scheme with $k = 0$ or $k = 1$, and T the analogous geodesic scheme in G, then there exists $\delta > 0$ such that T converges and $T^\infty p$ is a C^k curve for all p with $d(p) < \delta$.*

Proof. The result is based on Lemma 5 (which establishes proximity of S and T), Theorem 2 and Theorem 3 (for convergence and continuity), and Theorem 5 and Theorem 6 (for C^1 smoothness). $\square$

In order to extend this result to abstract Lie groups, we give the following definition, which extends the definition of matrix groups of constant velocity.

Definition 9. *A Lie group is called of constant velocity, if it is locally isomorphic to a matrix Lie group of constant velocity.*

Remark: The condition regarding constant velocity refers to the Lie algebra of a Lie group, because this is the object shared by all Lie groups which are locally isomorphic. In particular, all real Lie groups whose Lie algebra is compact (cf. [22]), are locally isomorphic to a compact Lie group (loc. cit., p. 228), which in turn is realizable as a matrix Lie group (loc. cit., p. 241). So in view of Lemma 2, all real Lie groups with compact Lie algebra are of constant velocity. $\diamondsuit$

Corollary 2. *Theorem 8 holds with G replaced by a Lie group of constant velocity, with geodesics defined by* (16)*, and with geodesic averages as in Def. 1.*

Proof. Geodesics are invariant with respect to left translation in the group, and therefore so is geodesic subdivision. Both continuity and smoothness are local properties. It is therefore sufficient to consider a neighbourhood of the identity in the group. By the constant velocity assumption, we may assume, without loss of generality, that such a neighbourhood is realized in a matrix group. Geodesics are invariant with respect to this local embedding of the group, so Theorem 8 regarding matrix groups applies. $\square$

There is a result very similar to Theorem 7, which concerns projection subdivision:

Theorem 9. *If S is a k-admissible scheme, $k = 0, 1$, and T is its analogous projection scheme, then T converges and $T^\infty p$ is a C^k curve for all p with $d(p)$ small enough.*

Proof. Lemma 7 establishes proximity of S and T. Then we invoke Theorem 2 and Theorem 3 to show convergence of T and the continuity of its limit curves (in the case $k = 0$), and Theorem 5 and Theorem 6 to show C^1 smoothness (in the case $k = 1$). $\square$

Remark: In Theorem 7, Theorem 8, and Theorem 9, proximity was essential for the convergence of T. Yet T depends on a particular representation of S in terms of averages, and proximity holds for all possible choices of T. By Theorem 4, any two schemes T and $\overline{T}$, which are analogues of S, satisfy

$$\|T^\infty p - \overline{T}^\infty p\|_\infty \le \frac{2AC}{1-\mu_0^2} d(p)^2, \tag{50}$$

where p is such that proximity holds. This shows that the actual choice of the representation in terms of averages has a very small influence on the limit curves. $\diamondsuit$

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6. Appendix A: Proofs of Results in Section 2

6.1. **Preliminary Results.** To begin with, we enumerate some simple properties of norms of polygons.

$$\|p-q\|_\infty \leq \varepsilon \implies d(p) \leq d(q) + 2\varepsilon. \tag{51}$$

This follows from the triangle inequality, because $\max \|p_i - q_i\| \leq \varepsilon$ implies that

$$\|p_{i+1} - p_i\| \leq \|p_{i+1} - q_{i+1}\| + \|q_{i+1} - q_i\| + \|q_i - p_i\| \leq \varepsilon + \|p-q\|_\infty + \varepsilon.$$

Likewise it is obvious that

$$\|\Delta p - \Delta q\|_\infty \leq 2\|p-q\|_\infty, \quad d(\Delta p) \leq 2d(p). \tag{52}$$

Note that because of (52), Equ. (24) is also a weaker form of (23) with a constant polynomial.

6.2. **Proof of Theorem 1.** The proof of Theorem 1 is elementary linear algebra: The affine combination of (6) defines, for each j, a rule $F_j(p)$ for computing the point Sp_j. There are essentially only N different F_j's, because $F_{j+N}(p) = F_j(\sigma p)$, where σ is the right shift operator $(\sigma p)_i = p_{i+1}$. We would like to express the rules F_j $(j = 1, \dots, N)$ in terms of averages. As the mask is finite, only finitely many p_i's contribute to $F_j(p)$, so we have $F_j(p) = F_j(p_s, \dots, p_r)$.

It is well known from linear algebra that for points p_i in some $\mathbb{R}^d$, repeatedly computing affine combinations by (1) yields all the points of the smallest affine subspace U which contains the points p_i; and it is also well known that U equals the set of all affine combinations $\sum b_i p_i$ with $\sum b_i = 1$.

This means that for all j and p there is a rule $G_{j,p}$, whose definition employs only averages, and with the property that $G_{j,p}(p) = F_j(p)$. The statement of this lemma is that we can choose $G_{j,p}$ independent of p, i.e., G_j is a rule for computing $F_j(p)$ whose definition employs only averages.

Both F_j and $G_{j,p}$ are affinely invariant in the sense that for all affine mappings α we have

$$\alpha(F_j(p_r, \dots, p_s)) = F_j(\alpha(p_r), \dots, \alpha(p_s)) = G_{j,p}(\alpha(p_r), \dots, \alpha(p_s)). \tag{53}$$

If we choose $s-r$ points $\overline{p}_r, \dots, \overline{p}_s$ as a basis of $\mathbb{R}^{s-r}$, then for any $p_r, \dots, p_s$ there is an affine mapping α with $\alpha(\overline{p}_i) = p_i$. It follows that for all p, we have

$$F_j(p) = F_j(\alpha(\overline{p}_r), \dots) = G_{j,\overline{p}}(\alpha(\overline{p}_r), \dots) = G_{j,\overline{p}}(p). \tag{54}$$

Note that the rule $G_{j,\overline{p}}$ is independent of p. □

6.3. **Lie groups: Why left translates?** We want to convince ourselves that left translates of one-parameter subgroups are indeed what we want for applications. In particular we want to see the connection with the helical motions.

Let G be a group of $n \times n$ matrices. For reasons which will become clear later, G is to act on $\mathbb{R}^n$ by

$$g^{-1} \cdot x, \tag{55}$$

if $g \in G$ and $x \in \mathbb{R}^n$. The fact that we do not use the more canonical action $g \cdot x$ does not make any difference for applications. We could say that instead of representing a linear mapping by its matrix, we represent it by the inverse matrix.

Group multiplication is defined as matrix multiplication. Because of

$$(gh)^{-1} \cdot x = h^{-1} \cdot g^{-1} \cdot x \tag{56}$$

the meaning of the product gh in terms of the group's action is to apply g first and h afterwards.

The action of a matrix/vector pair (g, t) with $g \in \mathrm{SO}_n$ and $t \in \mathbb{R}^n$ is given by

$$\begin{bmatrix} 1 \\ x \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ t & g \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ x \end{bmatrix}. \tag{57}$$

The product of the matrix/vector pairs (g_1, t_1) and (g_2, t_2) acts by applying (g_1, t_1) first and then (g_2, t_2):

$$\begin{bmatrix}1\\x\end{bmatrix} \mapsto \begin{bmatrix}1&0\\t_2&g_2\end{bmatrix}^{-1} \cdot \begin{bmatrix}1&0\\t_1&g_1\end{bmatrix}^{-1} \cdot \begin{bmatrix}1\\x\end{bmatrix} = \begin{bmatrix}1&0\\t_1+g_1\cdot t_2&g_1\cdot g_2\end{bmatrix}^{-1}\begin{bmatrix}1\\x\end{bmatrix} \tag{58}$$

Wee see that the multiplication of matrices in (14) is consistent with the action (57).

One-parameter subgroups of the motion group of Euclidean $\mathbb{R}^3$ are the helical motions $\alpha(t)$, which in a suitable Cartesian coordinate system are represented by the matrices

$$c_0(t) = \begin{bmatrix}1&0&0&0\\0&\cos(\omega t)&-\sin(\omega t)&0\\0&\sin(\omega t)&\cos(\omega t)&0\\pt&0&0&1\end{bmatrix} = \exp\left(t\begin{bmatrix}0&0&0&0\\0&0&-\omega&0\\0&\omega&0&0\\p&0&0&0\end{bmatrix}\right). \tag{59}$$

$c_0(0)$ is the identify transformation, and it is clearly seen that $c_0(-t)$ is the transformation inverse to $c_0(t)$. The general form of a helical motion $c(t)$ emanating from the identity at $t = 0$ then is of the form

$$c(t) = \beta^{-1}c_0(t)\beta = \exp(t\cdot\beta^{-1}v\beta), \tag{60}$$

where β represents a change of coordinate system. Obviously, $c(-t)$ is the transformation inverse to $c(t)$. This means that inverting all matrices changes the sense in which the one-parameter subgroups are traversed. For a general introduction to the kinematics of Euclidean space, see [13].

Now suppose that we are given points x_i, and two "positions" g and h of these points. We write simple g and h here, even if g and h are block matrices. The points at "position g" and "position h" are the points

$$g^{-1}x_i, \quad h^{-1}x_i. \tag{61}$$

If $c(t)$ is a helical motion which transforms position g (for $t = 0$) into position h (for $t = \tau$), then necessarily

$$c(\tau)^{-1}g^{-1}x_i = h^{-1}x_i, \quad \text{or} \quad (gc(\tau))^{-1}x_i = h^{-1}x_i. \tag{62}$$

This means that left translates $g\cdot c(t)$ of one-parameter subgroups are the curves which have a meaning in applications, if we adopt the inverted action (55).

6.4. **Lie groups: The differential equation of geodesics.** This section is devoted to the proof of the statements of 2.3.

Proof of Lemma 1: We first verify that (17) holds. Assume that $c(t) = g\exp(tv)$. By differentiating Equation (15) we get $\dot{c}(t) = g\exp(tv)v$, $\ddot{c}(t) = g\exp(tv)v^2$, which implies that $\ddot{c} = cv^2 = cvc^{-1}cv = B_c(\dot{c},\dot{c})$.

We verify that $\|\dot{c}\| = \text{const}$: Left multiplication in the group was supposed to be isometric, so the relation $\dot{c} = cv$ implies that $\|\dot{c}\| = \|v\|$, i.e., is constant. □

Proof of Lemma 2: We show that left translations $h \mapsto g \cdot h$ in the group are isometric if G is a subgroup of O_n. We have to show that a tangent vector v attached to the point g does not change its norm if it undergoes the linear mapping $v \mapsto g \cdot v$. We use the relation $g^T = g^{-1}$.

$$\|gv\|^2 = \mathrm{tr}(gvv^T g^T) = \mathrm{tr}(g^T gvv^T) = \mathrm{tr}(vv^T) = \|v\|^2. \tag{63}$$

A similar computation is performed for the group $G \ltimes \mathbb{R}^n$. We apply left translation by the element $\begin{bmatrix} 1 & 0 \\ t & g \end{bmatrix}$ to the tangent vector $\begin{bmatrix} 0 & 0 \\ u & v \end{bmatrix}$:

$$\left\| \begin{bmatrix} 1 & 0 \\ t & g \end{bmatrix} \begin{bmatrix} 0 & 0 \\ u & v \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} 0 & 0 \\ gu & gv \end{bmatrix} \right\|^2 = \mathrm{tr} \begin{bmatrix} 0 & 0 \\ 0 & (gu)(gu)^T + (gv)(gv)^T \end{bmatrix}$$
$$= \mathrm{tr}(gg^T uu^T) + \mathrm{tr}(gg^T vv^T) = \mathrm{tr}(uu^T) + \mathrm{tr}(vv^T) = \left\| \begin{bmatrix} 0 & 0 \\ u & v \end{bmatrix} \right\|^2.$$

As to compact matrix groups acting on a Euclidean vector space, it is well known that the definition of $\|x\|^2_{\mathrm{new}}$ as the average (with respect to a left invariant measure on G) over $\|gx\|^2$ yields a positive definite quadratic form, such that G is a subgroup of the orthogonal group defined by $\|\cdot\|_{\mathrm{new}}$. So also in this case we may assume (by changing the coordinate system to a basis which is orthonormal with respect to $\|\cdot\|_{\mathrm{new}}$) that G is a subgroup of O_n. This concludes the proof. $\square$

Remark: In groups $G \ltimes \mathbb{R}^n$, right multiplication usually is not isometric with respect to the scalar product (19). It is therefore necessary to consider left translates of one-parameter subgroups, and not right translates. This is achieved by letting the group act by inversion. $\diamondsuit$

7. Appendix B: Proofs of Results in Section 3

7.1. **Convergence and Continuity.** The proof of Theorem 2 concerning the convergence condition which follows from proximity works by induction:

Proof of Theorem 2: We start from (22) and want to show that there is $\overline{\mu}_0 < 1$ such that

$$d(T^l p) \leq \overline{\mu}_0^l d(p) \text{ for all } l, p \text{ with } d(p) < \delta. \tag{64}$$

We let $d_l := d(T^l p)$. By (51) and (36),

$$d_l \leq d(ST^{l-1}p) + 2Cd(T^{l-1}p)^2. \tag{65}$$

We use (22) to get the recursion formula

$$d_l \leq \mu_0 d_{l-1} + 2Cd_{l-1}^2. \tag{66}$$

We choose $\delta > 0$ such that

$$\overline{\mu}_0 := \mu_0 + 2C\delta < 1. \tag{67}$$

We show that if $d(p) = d_0 < \delta$, then

$$d_{l+1} \leq \overline{\mu}_0 d_l. \tag{68}$$

Clearly (68) implies (64). (68) is clear for $l = 0$ because of the choice of δ:

$$d_1 \leq d_0(\mu_0 + 2Cd_0) \leq (\mu_0 + 2C\delta)d_0 = \overline{\mu}_0 d_0. \tag{69}$$

If we assume that (68) holds true for d_{l-1}, then

$$\begin{aligned} d_l &\leq \mu_0 d_{l-1} + 2Cd_{l-1}^2 \leq \overline{\mu}_0^{l-1} d(p)(\mu_0 + 2C\overline{\mu}_0^{l-1} d(p)) \\ &\leq \overline{\mu}_0^{l-1} d(p)(\mu_0 + 2C\delta) \leq \overline{\mu}_0^{l} d(p). \end{aligned} \tag{70}$$

This shows that (68) must also be true for d_l. Thus, by induction we have shown (64). The statement about the difference $\overline{\mu}_0 - \mu_0$ is clear from (67). □

The fact that convergence conditions ensure convergence, as they do for linear schemes, is stated by Theorem 3. Its proof depends on the fact that for a linear convergent scheme S of finite mask the rate of convergence towards its limit is well known: There is a constant C, depending only on S and neither on j nor on p, such that

$$\|\mathcal{F}_{j+1}(Sp) - \mathcal{F}_j(p)\|_\infty \leq Cd(p). \tag{71}$$

This follows e.g. from Equations (3.8)–(3.10) of [10].

Proof of Theorem 3: We use (71) and (36) together with (39) to compute

$$\begin{aligned} &\|T^{j+1}f - T^j f\|_\infty \leq \|\mathcal{F}_{j+1}(T^{j+1}p) - \mathcal{F}_{j+1}(ST^j p)\|_\infty \\ &+\|\mathcal{F}_{j+1}(ST^j p) - \mathcal{F}_j(T^j p)\|_\infty \leq C_0 d(T^j p)^2 + Cd(T^j p). \end{aligned}$$

By Theorem 2, this expression is bounded by a factor times μ^j, with $\mu < 1$. It follows that $T^j f$ is a Cauchy sequence with respect to the maximum norm, i.e., the limit curve exists and is continuous. □

Proof of Lemma 3: We use the functions $T^k f$ with $f = \mathcal{F}_0(p)$ and want to show that $T^\infty f(t) \in K$ for all t in the parameter domain.

Choose a sequence t_k with $\lim t_k = t$ such that $|t_k - t| < N^{-k}$ and $T^k f(t_k)$ is one of the points of $T^k p$. Our construction is such that $T^k f(t_k) \in K$. We have

$$\begin{aligned} &\|T^\infty f(t) - T^k f(t_k)\| \leq \|T^\infty f(t) - T^k f(t)\| + \|T^k f(t) - T^k f(t_k)\| \\ \leq\ &\|T^\infty f(t) - T^k f(t)\| + d(T^k p). \end{aligned} \tag{72}$$

Because both $\|T^\infty f(t) - T^k f(t)\|$ and $d(T^k p)$ converge to zero, we have

$$T^\infty f(t) = \lim T^k f(t_k). \tag{73}$$

It follows that $T^\infty f(t) \in K$, as K is closed. □

We come to the proof of the approximation result of Theorem 4:

Proof of Theorem 4: We assume that S meets a convergence condition of the form (22) with factor μ_0. By Theorem 2, T does likewise, with factor $\overline{\mu}$. We show that

$$\|S^l p - T^l p\|_\infty \le C d(p)^2 \sum_{i=0}^{l-1} \|S^i\| \overline{\mu}^{2(l-1-i)}. \tag{74}$$

This is obvious for $l = 0$, if we define an empty sum to be zero. For $l > 0$, we assume that (74) holds for $l-1$ and perform an induction step:

$$\begin{aligned} \|S^l p - T^l p\|_\infty &\le \|S^l p - S^{l-1}Tp\|_\infty + \|S^{l-1}Tp - T^{l-1}Tp\|_\infty \\ &\le \|S^{l-1}\| \, \|Sp - Tp\|_\infty + C d(Tp)^2 \sum_{i=0}^{l-2} \|S^i\| \overline{\mu}^{2(l-2-i)} \\ &\le \|S^{l-1}\| C d(p)^2 + C\overline{\mu}^2 d(p)^2 \sum_{i=0}^{l-2} \|S^i\| \overline{\mu}^{2(l-2-i)}, \end{aligned} \tag{75}$$

which equals the right hand side of (74). Thus we have

$$\sum_{i=0}^{l-1} \|S^i\| \overline{\mu}^{2(l-1-i)} \le A \sum_{i=0}^{l-1} \overline{\mu}^{2(l-1-i)} \le A \sum_{i=0}^{\infty} \overline{\mu}^{2(l-1-i)} = \frac{A}{1-\overline{\mu}^2}$$

$$\implies \quad \|S^l p - T^l p\|_\infty \le \frac{CA}{1-\overline{\mu}^2} d(p)^2 \tag{76}$$

for all l. Now (40) follows and the proof is complete. □

7.2. **Smoothness Properties. Proof of Theorem 5:** We assume that (23) holds. By Theorem 2, we know that there is $\delta > 0$, such that

$$d(T^l p) \le \overline{\mu}_0^l d(p), \tag{77}$$

if $d(p) < \delta$. We want to show that there is $\overline{\mu}_1 < 1$, and a linear polynomial $\overline{P}_1$ with nonnegative coefficients such that

$$d(N^l \Delta T^l p) \le \overline{\mu}_1^l \overline{P}_1(l) d(p) \tag{78}$$

for all p with $d(p) < \delta$. Analogous to the proof of Theorem 2, we let

$$d_l = d(N^l \Delta T^l p), \quad q = T^{l-1}p. \tag{79}$$

Equations (51) and (52) together with the proximity condition show that

$$\begin{aligned} d_l = d(N^l \Delta Tq) &\le d(N^l \Delta Sq) + 2 \cdot N^l \|\Delta Sq - \Delta Tq\|_\infty \\ &\le d(N^l \Delta Sq) + 2 \cdot 2 \cdot N^l \|Sq - Tq\|_\infty \\ &\le \mu_1 \, d(N^{l-1} \Delta q) + 4N^l C d(q)^2. \end{aligned} \tag{80}$$

In view of Theorem 2, we can now replace $d(q)$ by an upper bound, and obtain

$$4N^l C d(q)^2 \leq 4N^l C(\overline{\mu}_0^{l-1} d(p))^2 = 4NC(N\overline{\mu}_0^2)^{l-1} d(p)^2 \tag{81}$$

Convergence of the subdivision process was an assumption, so we can without loss of generality assume that $d(p)$ is arbitrarily small. In view of Theorem 2, we may also assume that $|\mu_0 - \overline{\mu}_0|$ is so small that with (41) we have $\overline{\mu}_0 < 1/\sqrt{N}$. This implies that

$$N\overline{\mu}_0^2 =: \widetilde{\mu}_1 < N(\frac{1}{\sqrt{N}})^2 = 1. \tag{82}$$

Thus

$$4CN^l d(q)^2 \leq \widetilde{\mu}_1^{l-1} d(p)^2 P_0, \tag{83}$$

with a positive constant P_0. From (80), we get

$$d_l \leq d_{l-1}\mu_1 + P_0 d(p)^2 \widetilde{\mu}_1^{l-1}. \tag{84}$$

Repeated application of (84), starting with $l = 1$, implies that

$$d_l \leq \mu_1^l d_0 + P_0 d(p)^2 \sum\nolimits_{j=0}^{l-1} \mu_1^{l-j-1} \widetilde{\mu}_1^j. \tag{85}$$

Defining

$$\overline{\mu}_1 := \max\{\mu_1, \widetilde{\mu}_1\} < 1, \tag{86}$$

we get

$$d_l \leq \overline{\mu}_1^l d_0 + d(p)^2 \overline{\mu}_1^{l-1} l P_0. \tag{87}$$

There is $C' > 0$ such that $\delta C' < \overline{\mu}_1$, so that we have $d(p)^2 \overline{\mu}_1^{l-1} \leq d(p) C' \overline{\mu}_1^l$. Now (87) implies the inequality

$$d_l \leq \overline{\mu}_1^l \Big[d(\Delta p) + C' d(p) l P_0 \Big] \leq \overline{\mu}_1^l (2 + C' l P_0) d(p), \tag{88}$$

for $d(p) < \delta$. We let $\overline{P}_1(x) = 2 + P_0 C' x$, and the proof is complete. □

The smoothness condition (23) does not express the existence of a first derivative as such, but rather its continuity. For linear schemes, however, it is known that these conditions ensures C^1 smoothness of limit curves. We show now that this is true also for nonlinear schemes, provided they are in proximity to linear ones.

It is well known that C^1 smoothness of the limit curve $T^\infty f$ as defined by (38) follows from existence of the limit

$$\lim_{l \to \infty} \mathcal{F}_l (N^l \Delta T^l p), \tag{89}$$

with respect to the maximum norm, provided this limit is continuous. It then equals the derivative of the curve $T^\infty f$ (cf. [31], §3.1.4). For the convenience of the reader, we give this result in a form which directly applies to our setting.

Lemma 8. *Assume that the sequence p^l of polygons has the property that $\lim_{l\to\infty} \mathcal{F}_l(p^l) = f$ with respect to the maximum norm. We let*

$$g_l := \mathcal{F}_l(N^l \Delta p^l). \tag{90}$$

If g_l is a Cauchy sequence, and $\lim d(N^l \Delta p^l) = 0$, then f is C^1 with $f' = \lim_{l\to\infty} g_l$ (with respect to the maximum norm).

Proof. The derivatives f_l' are piecewise constant and in general not continuous. The functions g_l linearly interpolate the values $(f_l)'_+$ at each point of discontinuity of $(f_l)'$, and

$$\|f_l' - g_l\|_\infty \leq d(N^l \Delta p^l). \tag{91}$$

The sequence g_l is a Cauchy sequence by our assumption, and the previous equation shows that f_l' is also. Especially for all t the pointwise limit $\lim f_l'(t)$ exists. Thus, for any finite a, b, the dominated convergence theorem yields (see [27]):

$$f(b) - f(a) = \lim(f_l(b) - f_l(a)) = \lim \textstyle\int_a^b f_l' = \int_a^b \lim f_l'. \tag{92}$$

Equ. (92) expresses the fact that $f' = \lim f_l'$. Equation (91) implies that $f' = \lim g_l$, so f' is continuous. □

Proof of Theorem 6: The case $k = 0$ is Theorem 3. For $k = 1$ we consider the derived scheme S_1 and proceed analogously. The defining equation (25) implies that the smoothness condition of (23) which is supposed to hold for S, is nothing but a convergence condition of the form (22) for the derived scheme S_1. Analogous to the proof of Theorem 3 we rewrite (71) for S_1:

$$\|\mathcal{F}_{j+1}(S_1 \Delta p) - \mathcal{F}_j(\Delta p)\|_\infty \leq C d(\Delta p). \tag{93}$$

By Theorem 5, a mixed smoothness condition of the form (24) holds for T. We verify that Lemma 8 applies to the polygons $T^l p$. We define β_l by

$$\begin{aligned} & \|\mathcal{F}_{l+1}(N^{l+1}\Delta T^{l+1} p) - \mathcal{F}_l(N^l \Delta T^l p)\|_\infty \\ \leq\ & \|\mathcal{F}_{l+1}(N\Delta T - S_1\Delta) N^l T^l p\|_\infty + \|(\mathcal{F}_{l+1} S_1 - \mathcal{F}_l)\Delta N^l T^l p\|_\infty \\ \leq\ & (2N)\|(T - S)(N^l \Delta T^l p)\|_\infty + C d(\Delta N^l T^l p) \\ \leq\ & (2N) C' N^l (\mu_0 d(p))^{2l} + C \mu_1^l P_1(l) d(p) \ =: \beta_l. \end{aligned} \tag{94}$$

For (94), we have used (36), (22), and (24). By (41), $\mu_0^2 N < 1$, and $\sum_l \beta_l < \infty$. This shows that the sequence $g_l := \mathcal{F}_l(N^l \Delta T^l p)$ is a Cauchy sequence. T satisfies a smoothness condition by Theorem 5, which implies that $d(N^l \Delta T^l p) \to 0$. Thus Lemma 8 applies, and the limit curve $T^\infty f$ is C^1. □

8. Appendix C: Proofs of Results in Section 4

8.1. Preliminary Results. We first show some simple lemmas, which are needed later. The first one is concerned with the distance of endpoints of a curve which is traversed with unit velocity:

Lemma 9. *Assume that c is a curve with $\|\dot c\| = 1$ and $\|\ddot c\| \le C$. Then*

$$\|c(0) + t\dot c(0) - c(t)\| \le \frac{Ct^2}{2}, \quad |t| - Ct^2/2 \le \|c(t) - c(0)\|, \tag{95}$$

$$t < 1/C \implies |t| \le 2\|c(t) - c(0)\|. \tag{96}$$

Proof. Taylor's formula $c(t) = c(0) + t\dot c(0) + \frac{t^2}{2}\ddot c(\vartheta t)$ with $\vartheta \in [0,1]$ implies that

$$\|c(t) - c(0) - t\dot c(0)\| = \|\frac{t^2}{2}\ddot c(\vartheta t)\|, \tag{97}$$

$$\|c(t) - c(0)\| = \|t\dot c(0) - \ddot c(\vartheta t)\| \ge \|t \cdot c\| - \|\ddot c(\vartheta t)\| \tag{98}$$

Equation (97) and (98) immediately imply (95).

The function $\phi(t) := t - Ct^2/2$ is monotonically increasing for $t \in [0, 1/C]$ with $\phi(1/C) = 1/2C =: L_{\max}$. ϕ is also concave in this interval (and its inverse function ϕ^{-1} is convex), so $\phi(t) \ge t/2$ if $t \in [0, 1/C]$, and $\phi^{-1}(L) \le 2L$, if it exists in $[0, 1/C]$.

As $\psi(t) := \|c(t) - c(0)\|$ has the property that $\psi(t) > \phi(t)$, it follows that $\psi(t) \le L$ implies $\phi(t) \le L$, and (by monotonicity and concavity)

$$t \le \min(\phi^{-1}(L), 1/C) \le \min(2L, 1/C).$$

This implies (96). □

The next lemma already points towards comparing linear and nonlinear averages.

Lemma 10. *Assume that c is a curve with $\|\ddot c\| < C$. Then*

$$\| \operatorname{av}_\alpha(c(0), c(t)) - c(\alpha t)\| \le \frac{|\alpha| + \alpha^2}{2} Ct^2. \tag{99}$$

Proof. We use Taylor's formula and find that the left hand side of (99) expands to $\frac12\|\alpha t^2 \ddot c(\vartheta t) - \alpha^2 t^2 \ddot c(\vartheta' \alpha t)\|$ with $\vartheta, \vartheta' \in [0,1]$, which implies the upper bound given by (99). □

8.2. Geodesic subdivision. The next lemma uses the norm of the bilinear mapping B used in the differential equation of geodesics to give a simple upper bound of their second derivatives. It follows directly from from (42) and (43).

Lemma 11. *If $c(t) = \exp_p(tv)$ with $\|\dot c\| = 1$, then*

$$\|\ddot c\| \le D, \tag{100}$$

with D from (43).

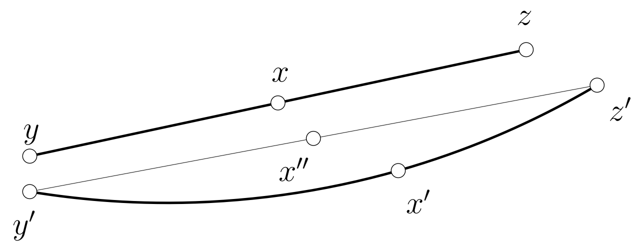

FIGURE 6. Proof of Lemma 5

We the above lemmas, the proof of Lemma 4 is easy.

Proof of Lemma 4: We assume that c is the minimal geodesic with $c(0) = x$, $c(t) = y$. By (99) and (100), an upper bound is given by $D\frac{|\alpha|+\alpha^2}{2}t^2$. Because of the symmetry of the geodesic average expressed by (12), this relation remains true if we replace α by $1-\alpha$. Thanks to Lemma 9, $t \le 2\|x-y\|$, which completes the proof. $\square$

The proof of Lemma 5 proceeds by induction.

Proof of Lemma 5: By Theorem 1, the scheme S is expressible in terms of averages. If S is defined by N different rules, each of which involves the average operator at most once in the form

$$Sp_{iN+j} = \mathrm{av}_{\alpha_j}(p_{i+r_j}, p_{i+s_j}), \tag{101}$$

then Lemma 4 implies immediately that there is a constant C such that

$$\|Tp_{iN+j} - Sp_{iN+j}\| \le Cd(p)^2.$$

As to two or more steps of averaging, we perform an induction step. We assume that points x and x' are defined in a linear and a nonlinear way, respectively, by

$$x = \mathrm{av}_\alpha(y, z), x' = \text{g-av}_\alpha(y', z'), \tag{102}$$

as illustrated by Fig. 6. We also assume that

$$\|y - z\| \le Cd(p),\ \ \|y - y'\|, \|z - z'\| \le C'd(p)^2. \tag{103}$$

Our aim is to show that also x and x' meet a proximity condition. By induction, this would show that S and T are in proximity.

We introduce $x'' = \mathrm{av}_\alpha(y', z')$ (see Fig. 6) and use Lemma 4 again:

$$\begin{aligned}\|x - x'\| &\le \|x' - x''\| + \|x - x''\| \le C''\|y' - z'\|^2 + \|\,\mathrm{av}_\alpha(y - y', z - z')\| \\ &\le C''(\|y - y'\| + \|y - z\| + \|z - z'\|)^2 + C'''\max(\|y - y'\|, \|z - z'\|).\end{aligned}$$

Thus, by (103) and $d(p) < \delta$,

$$\|x - x'\| \le C''''d(p)^2.$$

This is what we wanted to show. $\square$

8.3. **Projection subdivision.** The following is an immediate consequence of the definition of D' by (48):

Lemma 12. *If $c(t) = P(x + tv)$ with $\|v\| = 1$, then $\|\ddot{c}\| < D'$.*

We now turn to the proof of the lemmas already stated above.

Proof of Lemma 6: We consider the curve $c(t) = P(x + vt)$ with $x + v\tau = y$ and $\tau = \|y - x\|$ and apply Lemma 9. It follows that the left hand expression of (49) is bounded by $(|\alpha| + \alpha^2)D'\frac{\tau^2}{2}$. Exchanging x and y yields an analogous estimate with $1 - \alpha$ instead of α. □

Proof of Lemma 7: This is very similar to the proof of Lemma 5. We replace the reference to Lemma 4 by a reference to Lemma 6. □

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