ON A PROBLEM OF ELEMENTARY DIFFERENTIAL GEOMETRY AND THE NUMBER OF ITS SOLUTIONS

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ABSTRACT. If M and N are submanifolds of $\mathbb{R}^k$, and a, b are points in $\mathbb{R}^k$, we may ask for points $x \in M$ and $y \in N$ such that the vector $\overrightarrow{ax}$ is orthogonal to y's tangent space, and vice versa for $\overrightarrow{by}$ and x's tangent space. If M, N are compact, critical point theory is employed to give lower bounds for the number of such related pairs of points.

1. OVERVIEW

This short paper investigates the number of solutions of a certain problem in the elementary differential geometry of curves and surfaces:

Def. 1. *We assume that the vector space $\mathbb{R}^k$ is endowed with a positive definite scalar product $\langle\ ,\ \rangle$, and that M and N are compact C^r submanifolds of $\mathbb{R}^k$. We choose $a, b \in \mathbb{R}^k$. Points $x \in M$ and $y \in N$ are said to be related, if the tangent spaces T_xM and T_yN have the properties*

$$\overrightarrow{ax} \perp T_yN \text{ and } \overrightarrow{by} \perp T_xM. \tag{1}$$

Theorem 1. *The number of related pairs of points is ≥ 2 if not both M, N are points. It is ≥ 3 if neither of M, N has dimension zero.*

In general the number of related pairs of points is greater or equal the Lyusternik-Schnirel'man category of $M \times N$.

Theorem 2. *Generically the number of related pairs of points is greater or equal*

$$2 + |\chi(M)\chi(N) - 1 - (-1)^{\dim M + \dim N}|, \tag{2}$$

where χ denotes the Euler characteristic.

Cor. 1. *Generically there are at least four pairs of related points if*
(i) both M and N are boundaries of compact subsets of $\mathbb{R}^k$; or
(ii) at least one of M, N is of odd dimension, and the other one is not a point.

Def. 2. *Genericity as mentioned in Th. 2 and Cor. 1 means that the set of $(a, b) \in (\mathbb{R}^k)^2$ such that those statements are not true has Lebesgue measure zero.*

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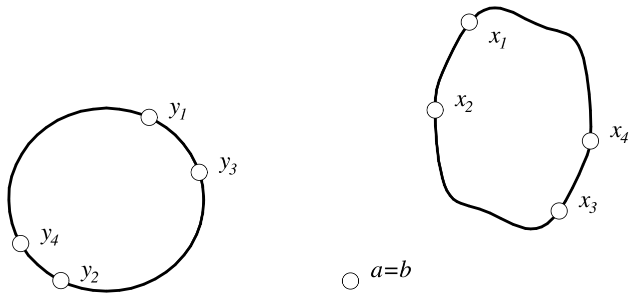

FIGURE 1. The case $\dim M = \dim N = 1$, $\chi(M \times N) = 0$.

The results above are illustrated in Fig. 1 The relation defined by Def. 1 was motivated by studying error propagation in geometric constructions [6], where it turned out to be related to computing the interval $\langle K, L \rangle$, where K and L are connected smooth or polyhedral subsets of $\mathbb{R}^k$.

After some preparations in §§2.1–4.1 we will give proofs of Th. 1, Th. 2 and Cor. 1 in §4.2.

2. FACTS

2.1. **Critical points and singular values.** We assume that M, N are C^r manifolds and $f : M \to N$ is C^r ($r \geq 2$). We use the symbol $f_*(x; v)$ for the differential of f applied to the tangent vector $(x; v) \in T_x M$. $f(x)$ is called a singular value of f if $\operatorname{rk} f_*(x; \cdot) < \dim N$. By Sard's theorem (see [10, 9]), the set of critical values is a Lebesgue zero set in N, if $r \geq \max\{1, \dim(M) - \dim(N) + 1\}$.

We assume now that $f : M \to \mathbb{R}$ is C^2. $x \in M$ is said to be critical for f if the linear form $f_*(x; \cdot)$ is zero, in which case the Hessian f_{**} is defined by $f_{**}(x; v, w) := \partial_s \partial_t (f \circ x)(0, 0)$, where $x(t, s)$ is an M-valued C^2 surface with $x(0, 0) = x$, $\partial_t x(0, 0) = v$, and $\partial_s x(0, 0) = w$. The Hessian is a symmetric bilinear form. A critial point is called degenerate if there exists v such that $f_{**}(x; v, \cdot) = 0$. Otherwise the number of negative squares in f_{**} is called the index of f at x and is denoted by $\operatorname{ind}_x f$. f is called a Morse function if its critical points are nondegenerate.

2.2. **Topology.** We assume now that both M, N are compact and continue the discussion of 2.1. Reeb's theorem says that if f has only two critical points (degenerate or not), then M is homeomorphic to a sphere [7]. The Lyusternik-Shnirel'man category of M is the smallest number of contractible open subsets of M which cover M. It serves as a lower bound for the number of critical points of a smooth function defined in M [11].

If f is a Morse function, then $\sum_{x \text{ critical}} (-1)^{\operatorname{ind}_x f} = \chi(M)$, the Euler characteristic of M [8].

Recall that $\chi(M\times N) = \chi(M) \times \chi(N)$. If $M = \partial K$, with K compact, then $\chi(M) = 2\chi(K)$ if $\dim M$ is even; for all M of odd dimension $\chi(M) = 0$.

A sphere is not homeomorphic to $M\times N$ if $\dim M, \dim N > 0$. For these topological facts, see e.g. [1].

3. Differential geometry

Curve and surface theory in pseudo-Euclidean spaces which carry an indefinite metric is a special case of the general theory of Cayley-Klein spaces as elaborated in part in [4].

The results in this section are well known in the positive definite case, where they are often shown together [8]. As in the indefinite case principal curvatures are not generally available, we give proofs which work without regard to definiteness as far as possible.

3.1. **Distance functions.** We assume that $\mathbb{R}^l$ is endowed with a possibly indefinite scalar product $\langle\ ,\ \rangle$. Let M be a C^2 submanifold of $\mathbb{R}^l$. We use the symbols TM and $\perp M$ for tangent and normal bundle, respectively, and consider them embedded into $\mathbb{R}^{2l}$. We define endpoint map E and distance function d_p by

$$E : \perp M \to \mathbb{R}^l,\ (x; n) \mapsto x + n, \quad d_p : M \to \mathbb{R},\ x \mapsto \langle x - p, x - p\rangle.$$

Lemma 1. *$x \in M$ is critical for $d_p|M \iff p = E(x; n)$ with $n \in \perp_x M$.*

Proof. We let $n = x - p$ and consider $v \in T_xM$. Then $d_{p*}(x; v) = 2\langle n, v\rangle$. Obviously d_{p*} is the zero mapping if and only if $n \in \perp_x M$, i.e., $p = E(x, n)$. □

Lemma 2. *$x \in M$ is a degenerate critical point for $d_p|M \iff p = E(x; n)$ is a singular value of E.*

Proof. We extend x and n to C^2 functions defined in $U \subset \mathbb{R}^2$ such that

$$x : U \to M,\ n : U \to \mathbb{R}^l,\ x(0, 0) = x,\ n(0, 0) = n, \tag{3}$$

$$n(t, s) \in \perp_{x(t,s)} M,\ \partial_t x(0, 0) = v,\ \partial_s x(0, 0) = w,\ \partial_s n(0, 0) = w', \tag{4}$$

and note that $((x; n); (w, w')) \in T_{(p,n)}(\perp M)$. We compute

$$\partial_s\big\langle n, \partial_t x\big\rangle = 0 \implies \big\langle \partial_s n, \partial_t x\big\rangle + \big\langle n, \partial_t\partial_s x\big\rangle = 0. \tag{5}$$

Now we can express d_{p**} in terms of E_*: $d_{p**}(x; v, w) = \partial_t\partial_s\langle x - p, x - p\rangle\big|_{s,t=0} = 2\big(\big\langle \partial_t x, \partial_s x\big\rangle - \big\langle x - p, \partial_t\partial_s x\big\rangle\big)\big|_{s,t=0} = 2\big\langle v, w\big\rangle + 2\big\langle \partial_s n, \partial_t x\big\rangle\big|_{s,t=0} = 2\big\langle \partial_s(x + n), v\big\rangle\big|_{s=0} = 2\big\langle E_*\big((x; n); (w, w')\big), v\big\rangle$. We see that x is degenerate $\iff$ there exists v such that $E_*(T_{(x;n)}\perp M) \in v^\perp$, i.e., E_* does not have full rank at $(x; n)$. □

3.2. **Curvatures.** If $T_xM \cap \perp_x M = 0$, both orthogonal projections π and π' onto T_xM and $\perp_x M$, respectively, are well defined, and the restriction of $\langle\ ,\ \rangle$ to T_xM is nonsingular. The second fundamental form II_x at x is defined by $\mathrm{II}_x(v,w) = \pi'(\partial_s\partial_t x)$, if $x(t,s)$ and $n(t,s)$ are as in (3) and (4). It is a vector-valued symmetric bilinear form. (5) implies that $\langle \mathrm{II}_x(v,w), n\rangle = \langle -w', v\rangle$. The Weingarten mapping $\sigma_{x,n} : w \mapsto -\pi(w')$ is well defined by the previous formula. It is selfadjoint and its eigenvalues $\kappa_i^{(n)}$ (if any) are called principal curvatures with respect to n. Obviously $\sigma_{x,\lambda n} = \lambda\sigma_{x,n}$, and $\kappa_i^{(\lambda n)} = \lambda\kappa_i^{(n)}$. In that way the principal curvatures are linear forms in the one-dimensional subspace $[n] \in \perp_x M$ (For the existence of eigenvalues of selfadjoint mappings, see [5], Th. 5.3.)

Lemma 3. *Suppose that $T_xM \cap \perp_x M = 0$ and $p = E(x,n)$. Then x is degenerate $\iff$ there is a tangent vector w with $w = \sigma_{x,n}(w)$ $\iff$ a curvature $\kappa_i^{(n)} = 1$.*

Proof. d_{p**} is symmetric. So x is degenerate $\iff \exists w \forall v : d_{p**}(w,v) = \langle E_*((x;n); (w,w')),\ v\rangle = \langle w + w', v\rangle = 0 \iff \pi(w+w') = 0 \iff w = \sigma_{x,n}(w)$. ☐

Remark: The singular values of the endpoint map depend only on the subspaces $\perp_x M$. As "$\perp$" is actually a C^r mapping of Grassmann manifolds, the points where $T_xM \cap \perp_x M \neq 0$ are not as special as Lemma 3 suggests. ♢

4. Critical points of the scalar product

4.1. **The metric in product space.**

Lemma 4. *Related pairs $(x,y) \in M\times N$ are precisely the critical points of the function $f : M\times N \to \mathbb{R}$, $f(x,y) = \langle x-a, y-b\rangle$.*

Proof. We compute $f_*((x,y);(v,w)) = \langle x-a, w\rangle + \langle v, y-b\rangle$. This linear mapping of (v,w) is zero if and only if $\langle v, y-b\rangle = \langle x-a, w\rangle = 0$ for all v, w. ☐

In order to apply the previous lemmas concerning distance functions, we introduce the following indefinite scalar product on $(\mathbb{R}^k)^2$:

$$(6)\qquad \langle\ ,\ \rangle_{\mathrm{pe}} : (\mathbb{R}^{2k})^2 \to \mathbb{R} \quad \langle (v_1,v_2),(w_1,w_2)\rangle_{\mathrm{pe}} := \frac{1}{2}\big(\langle v_1, w_2\rangle + \langle v_2, w_1\rangle\big).$$

Lemma 5. *We have $f = d_{(a,b)}|(M\times N)$, where $d_{(a,b)}(x,y) = \langle (a,b)-(x,y), (a,b)-(x,y)\rangle_{\mathrm{pe}}$ to $M\times N$ is a distance function with respect to $\langle\ ,\ \rangle_{\mathrm{pe}}$.*

The tangent and normal spaces of $M\times N$ are given by $T_{(x,y)}(M\times N) = T_xM \times T_yN$, $\perp_{(x,y)}(M\times N) = \perp_y N \times \perp_x M$.

Proof. Expand the definitions. ☐

4.2. **Proofs.**

Proof. (of Th. 1) The function f of Lemma 4 is C^2, has a maximum (x_1,y_1) and a minimum (x_2,y_2). By Lemma 4, criticality of (x,y) is equivalent to x and y being related, so the first statement of Th. 2 follows.

If $\dim M, \dim N \geq 1$, then $M\times N$ is not homeomorphic to a sphere and Reeb's theorem shows that there are at least three pairs of related points.

The last statement follows directly from the result on the Lyusternik-Shnirel'man category quoted in §2.2. □

Proof. (of Th. 2) By Sard's theorem, almost all (a, b) (in the sense of Lebesgue measure) are not singular values of the endpoint map with respect to $\langle\ ,\ \rangle_{\mathrm{pe}}$ and $f = d_{(a,b)}|(M\times N)$ is a Morse function (by Lemma 5 and Lemma 2). With C as its set of critical points, we have

$$\chi(M\times N) = \sum\nolimits_{(x,y)\in C} (-1)^{\mathrm{ind}_{(x,y)} f}.$$

The indices of the maximum and minimum are known: $\mathrm{ind}_{(x_1,y_1)} f = 0$, and $\mathrm{ind}_{(x_2,y_2)} f = \dim(M\times N)$. As the number of remaining critical points must fulfil

$$\#C - 2 \quad \geq \quad \left|\chi(M)\chi(N) - \sum_{i=1}^{2} (-1)^{\mathrm{ind}_{(x_i,y_i)} f}\right|,$$

the statement follows. □

Proof. (of Cor. 1) We assume the generic case, i.e., f is a Morse function.

(i) If M and N are boundaries, then $\chi(M)\chi(N) \in 4\mathbb{Z}$ As $\dim M = \dim N = k-1$, we have a lower bound of $2 + |\chi(M)\chi(N) - 1 - (-1)^{2(k-1)}| \geq 4$.

(ii) We assume without loss of generality that $\dim(M)$ is odd, so $\chi(M) = 0$. With the notations of the previous proof, we let $C' = C \setminus \{(x_1, y_1), (x_2, y_2)\}$. The case that N is of dimension zero is trivial, and in all other cases we already know that $M\times N$ is not homeomorphic to a sphere, so $\#C \geq 3$ and $\#C' \geq 1$. Regardless of $\dim N \times M$, $1 + (-1)^{\dim N\times M}$ is even, so the equation

$$\sum_{(x,y)\in C} (-1)^{\mathrm{ind}_{(x,y)} f} = 1 + (-1)^{\dim N\times M} + \sum_{(x,y)\in C'} (\pm 1) = \chi(M)\chi(N) = 0$$

implies that $\#C'$ is even, i.e., $\#C' \geq 2$ and $\#C \geq 4$. □

Remark: There are many relations between critical points and the topology of manifolds, which could be used to improve Cor. 1. However, this discussion would lead us too far. See e.g. [8] for computing the homotopy type of a compact manifold from a Morse function, and [3, 11, 2] for results on the Lyusternik-Shnirel'man category and its relation to the minimum number of critical points. ♢

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