# Self-Intersections of Offset Curves and Surfaces

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**Abstract**

In this paper we discuss the self-intersections of offset curves and surfaces and show how to determine the maximum offset distance such that the offset does not locally nor globally self-intersect including boundary effects. Examples illustrate the applicability of the analysis.

*Keywords:* offset curves and surfaces, self-intersections, offset strip, auto-normal chords, coincident boundary points.

## 1 Introduction

Offset *curves/surfaces*, also known as *parallel curves/surfaces*, are defined as the locus of points which are at constant distance $d$ along the normal vector from the generator (also called *progenitor*) *curve/surface*. Offsets are widely used in various applications, such as tool path generation for 2.5D pocket machining, 3D NC machining, definition of tolerance regions, rapid prototyping, brush stroke representation and in feature recognition through construction of skeletons or medial axes of geometric models. Literature surveys on these topics can be found in [4], [10].

An offset is in general more complex than its progenitor. It may self-intersect *locally* when the absolute value of the offset distance exceeds the minimum radius of curvature in a concave region. Also, an offset may intersect *globally* when the distance between two distinct points on the progenitor reaches a local minimum. These local and global self-intersections can be visualized as machining a part using a *cylindrical*/*spherical* cutter whose radius is too large for *2.5D/3D* milling.

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In a recent survey paper, Maekawa [4] raises some important outstanding issues in the area of offset surfaces. One of them is concerned with determining the maximum offset distance such that the offset surface does not locally nor globally self-intersect including boundary effects. So far the discussion in the literature [6] was based on “Given a *curve/surface* and an offset distance, does the offset curve/surface self-intersect? If so, where does it self-intersect?” However in some applications, we may encounter the problem “Given a *plane curve/surface*, what is the maximum offset distance such that its offset does not self-intersect?” In Maekawa et al. [8] it is discussed how to find the maximum possible radius of the pipe surface, which is the envelope of the smooth one-parameter family of spheres with radius $r$ centered at the spine curve, such that it will not self-intersect. The method can be easily applied to find the maximum offset distance without self-intersection by setting the spine curve as the given planar progenitor curve. In this paper we investigate a method to find the maximum offset distance such that the offset surface does not locally nor globally self-intersect including boundary effects.

The paper is organized as follows: Section 2 reviews the planar case. Part of this section serves as a motivation for Section 3 where the surface case is presented. Section 4 deals with some special cases of surfaces that arise often in applications. Section 5 describes the algorithm of the computation of self-intersections of the offset surface. Section 6 provides illustrative examples and finally, in Section 7 some concluding remarks are provided.

## 2 Planar Curves

In this section we review the case of plane curves and set up the stage for the next section. Consider a planar regular smooth curve

$$\mathbf{c} : I \to \mathbb{R}^2,\ t \mapsto \mathbf{c}(t), \qquad (d\mathbf{c}/dt \neq \mathbf{o})$$

with its unit normal vector field

$$t \mapsto \mathbf{n}(t), \qquad (\|\mathbf{n}(t)\| = 1,\ \mathbf{n}(t) \cdot d\mathbf{c}/dt(t) = 0).$$

Note that $-\mathbf{n}$ is also a normal vector field of $\mathbf{c}$. If $\mathbf{c}$ is $C^r$, then $\mathbf{n}$ is $C^{r-1}$, and thus if $\mathbf{c}$ is at least continuously differentiable, then $\mathbf{n}$ is continuous.

**Definition 2.1** *Let* $\mathbf{c}$ *be as above. Then, the offset curve* $\mathbf{c}_o(r)$ *at distance* $r$ *is defined by*

$$\mathbf{c}_o(r) : t \mapsto \mathbf{c}(t) + r\,\mathbf{n}(t).$$

*The strip between* $\mathbf{c}$ *and* $\mathbf{c}_o(r)$*, which shall be called the* offset strip *of width* $r$*, is defined as the union of all* $\mathbf{c}_o(\rho)$ *with* $0 \leq \rho \leq r$*.*

Fig. 1 shows a curve with its offset and offset strip.

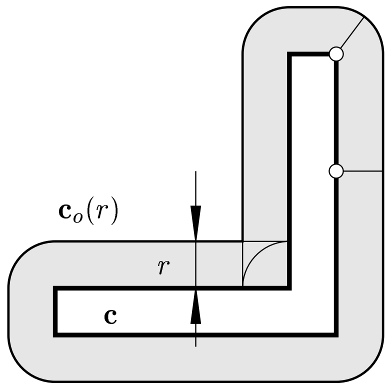


Figure 1: Curve $\mathbf{c}$ with a non-convex edge, offset curve $\mathbf{c}_o(r)$ with singularities, and offset strip (grey).

We define the term *self-intersection* to be either self-intersection of the offset strip or self-intersection of the offset curve. Eventually we shall determine

- a value $r_s$ such that for $0 \leq r < r_s$ both the offset curve and the offset strip have no self-intersections, but for $r > r_s$ the offset strip has self-intersections.
- a value $r_c$ such that for $0 \leq r < r_c$ the offset curves $\mathbf{c}_o(r)$ have no self-intersections, but at or immediately after $r = r_c$ the offset curves $\mathbf{c}_o(r)$ have self-intersections.

## 2.1 Self-Intersections of Offset Curves

By introducing the *endpoint map*

$$E : (u, v) \mapsto \mathbf{c}(u) + v\,\mathbf{n}(u) = \mathbf{c}_v(u) \quad (u \in I), \tag{1}$$

we see that if $E$ is one-to-one for all values $v < r$, then the offset strip of width $r$ is free of self-intersections. Moreover, for sufficiently small $r$, $E$ is always 1–1, provided that the curve has only *convex* edges. In this case we can always expect $r_c, r_s > 0$. We will restrict ourselves to curves which have only convex edges.

### 2.1.1 Auto-normal chords

We now proceed with the discussion of the different possibilities for self-intersections. For $r \geq 0$ we note that if $E(u, v)$, restricted to $0 \leq v \leq r$, is still regular (which is equivalent to the offset $\mathbf{c}_o(v)$ being regular for all $v \leq r$), then a self-intersection can occur if at some value $r_0$, the offset curve (or the offset strip) *touches* itself in an interior point of the curve.

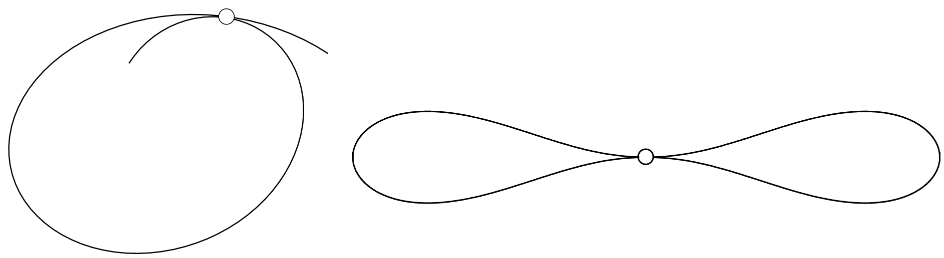

Figure 2: A curve which is in oriented contact with itself (left). A curve which touches itself, but is not in oriented contact with itself (right).

If curves are oriented (i.e., are equipped with a unit normal vector field), then we say that they are in *oriented contact*, if they touch each other in a point, and their unit normal vectors coincide in this point. The notion of oriented contact extends to contact of a curve with itself: If a curve touches itself, and the two normal vectors coincide, we say that the curve is in *oriented contact with itself* (see Fig. 2).

Since the curve $\mathbf{c}$ is oriented, so is $\mathbf{c}_o(r)$ (because the normal vectors in corresponding points are the same as for $\mathbf{c}$). If $\mathbf{c}_o(r)$ touches itself, this contact cannot be oriented contact, because then also the original curve $\mathbf{c}$ would have been in oriented contact with itself — this contradicts our assumption that $\mathbf{c}$ is one-to-one.

So we always have the following geometrical configuration (see Fig. 3): The first (in the sense of the offsetting parameter $r$) two curve points $\mathbf{p}, \mathbf{p}'$ whose offset points $\mathbf{q}, \mathbf{q}'$ coincide, have parallel tangents and the line segment joining $p, p'$ is the *shortest auto-normal chord* of the curve. An auto-normal chord of the curve is defined as the line segment $\mathbf{pp}'$ such that $\overrightarrow{\mathbf{pp}'}$ is the oriented normal at $\mathbf{p}$ and $\overrightarrow{\mathbf{p}'\mathbf{p}}$ is the oriented normal at $\mathbf{p}'$. There is also a second geometric characterization of this. Define the *cut locus curve* of a curve $\mathbf{c}$ as the set of centers of oriented circles that are in oriented contact with $\mathbf{c}$ in at least two different points, together with all limit points of this set [**?**]. It turns out that the centers of curvature in points of extremal curvature

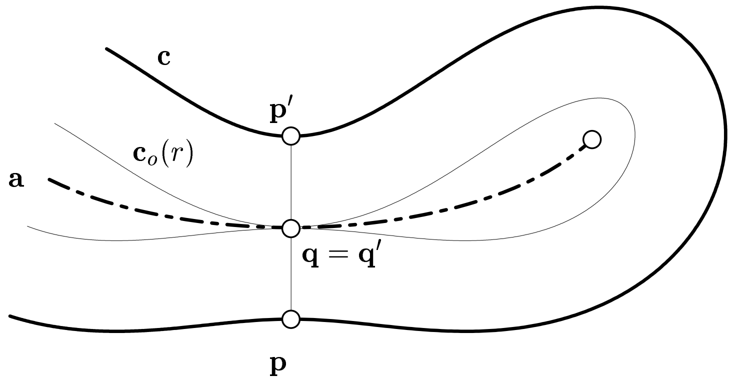


Figure 3: Curve $\mathbf{c}$, offset curve $\mathbf{c}_o(r)$, cut locus $\mathbf{a}$, and auto-normal chord $PP'$. Case 'OO'.

are exactly these limit points. When we speak of the distance between the curve and its cut locus, then this distance is always the radius of a circle which was used in the definition. It equals the distance between its center, and the points where it touches the curve. Obviously the midpoint $\mathbf{q} = \mathbf{q}'$ of the smallest auto-normal chord $\mathbf{pp}'$ of $\mathbf{c}$ is contained in the cut locus curve of $\mathbf{c}$.

If all offset curves for smaller parameter values are free of self-intersections, then $r$ is the value of the *smallest distance between the curve and its cut locus.*

#### 2.1.2 Coincident points at the boundary

Upon increasing the value of $r$ and looking for self-intersections of $\mathbf{c}_o(r)$ it may happen that a boundary point $\mathbf{q}$ of $\mathbf{c}_o(r)$ coincides with another point $\mathbf{q}'$ of $\mathbf{c}_o(r)$ (see Fig. 4). (In the generic case $\mathbf{q}'$ is an interior point of $\mathbf{c}_o(r)$, but it could also be a boundary point).

The above occurs if the cut locus of $\mathbf{c}$ has its endpoint at $\mathbf{q} = \mathbf{q}'$. In that case the distance $\overline{\mathbf{pq}} = \overline{\mathbf{p}'\mathbf{q}'}$ equals the parameter $r$ of the offset curve; and if all offset curves for smaller parameter values are free of self-intersections, then $r$ is the value of the *smallest distance between the curve and its cut locus*, this minimum being attained in two points, one of which is a boundary point of the curve.

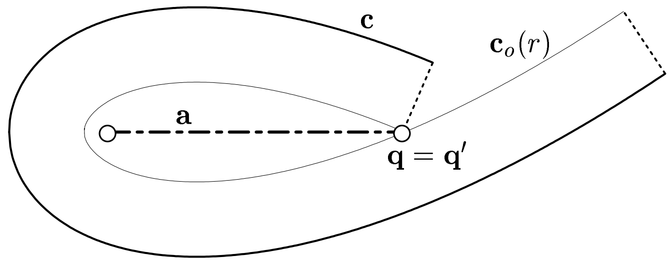


Figure 4: Boundary of offset strip touches curve.

### 2.1.3 Singular points of the offset curve

Finally, it may happen that while increasing $r$, the offset curve becomes singular for $r = r_0$, namely at a point $\mathbf{p} = \mathbf{c}(t_0)$ of extreme curvature. If the curvature is not constant in a neighborhood of $\mathbf{p}$, then a local analysis shows that there is an interval $(r, r + \epsilon)$ such that for all parameter values in this interval, the offset curve has self-intersections (see Fig. 5). If the curvature is constant, i.e., the curve is locally a circle, then the offset curve $\mathbf{c}_o(r_0)$ is itself locally constant. The curves $\mathbf{c}_o(r)$, $r > r_0$ may or may not have self-intersections.

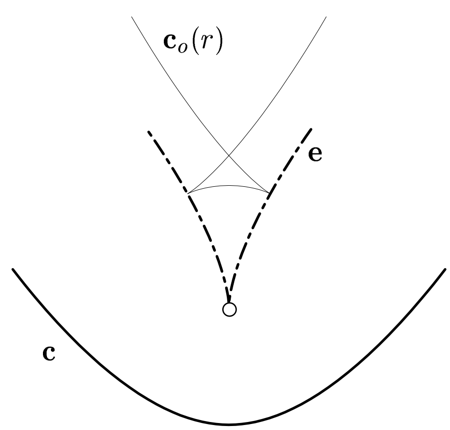


Figure 5: Behavior of involute $\mathbf{e}$ and offset curve.

Since the centers of curvature at curvature maxima are contained in the cut locus curve, the latter serves as the basic ingredient of the final statement on self-intersections of curves:

**Theorem 2.1** *The value $r_c$ equals the smallest distance between the curve* $\mathbf{c}$ *and its cut locus curve* $\mathbf{a}$*. The type of self-intersection at $r_c$ is determined by the pairs* $\mathbf{p}_i, \mathbf{q}_i$ *of points on* $\mathbf{c}$ *and* $\mathbf{a}$*, where the minimum is actually attained:*

1. *A point* $\mathbf{p}_i$ *may be a curvature maximum,*
2. *a pair* $(\mathbf{p}_i, \mathbf{q}_i)$ *may be formed from the endpoints of a smallest auto-normal chord, both being interior extremal values of the distance between the curve and its cut locus,*
3. *a pair* $(\mathbf{p}_i, \mathbf{q}_i)$ *may be such that at least one of the* $\mathbf{p}$*'s is a boundary extremum of the distance between curve and cut locus.*

## 2.2 Self-Intersection of the offset strip

In this subsection we shall show that there are several different cases of self-intersection of the offset strip, besides two cases which occur only with zero probability and which are limit cases of the previous ones.

The boundary of the offset strip consists of the curve $\mathbf{c}$ itself, of the offset curve $\mathbf{c}_o(r)$, and of line segments joining the boundary points of $\mathbf{c}$ with the respective boundary points of $\mathbf{c}_o(r)$. In the following enumeration of cases, these three components of the boundary will be denoted by the letters 'C', 'O', and 'B', respectively. The vertices where 'O' and 'B' meet, are denoted by 'V', and the endpoints of the curve $\mathbf{c}$, where 'C' and 'B' meet, by 'E'.

If the curve $\mathbf{c}$ has no self-intersections, then the offset strip has neither if $r$ is small enough. We consider the smallest $r$ such that the offset strip has a self-intersection. In this situation a component of the boundary (one of 'C', 'O', 'B', 'V', 'E') meets another one. Of the 20 hypothetical combinations, the following occur in practice:

**OO** Here the offset curve touches itself, which is the case discussed above (see Fig. 3).

**CO** The offset curve $c_r$ touches the base curve $c$ (see Fig. 7a).

**BO** The offset curve touches a boundary component which is a straight line (see Fig. 8a)

**CV** A vertex of the offset strip is contained in the curve $c$ (see Fig. 6). This case contains 'BC' as a limit.

**EO** The offset meets the base curve in an endpoint (see Fig. 6 with $\mathbf{c}_o(r)$ and $\mathbf{c}$ interchanged) Cases 'EV' and 'BE' are limit cases of this.

**BV** A vertex of the offset strip is contained in a boundary component which is a straight line (see Fig. 8b).

**OV** A vertex of the offset strip is contained in a 'distant' part of the offset curve $c_r$ (see Fig. 4). Case 'VV' is a limit case of this.

The remaining cases do not occur.

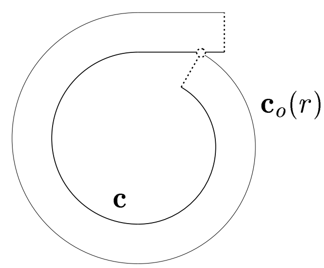


Figure 6: A vertex of the offset strip meets the original curve (case 'CV'). If $\mathbf{c}$ and $\mathbf{c}_o(r)$ are interchanged, this shows case 'EO'.

We describe an algorithm to determine the value of $r_s$.

1. Determine the number $r_c$ according to Th. 2.1, using the cut locus curve of $\mathbf{c}$. This first step looks for intersections of the types 'OO', 'OV', and 'VV'.

2. Consider the *modified cut locus curve* which is defined as the locus of centers of oriented circles which touch the curve $\mathbf{c}$ in two points, but are not in oriented contact (see e.g. the curve $\mathbf{a}$ in Fig. 7b). Let $r'_c$ equal the smallest distance of the modified cut locus to the curve $c$.

   This second step looks for self-intersections of the types 'CO' and 'EO'.

3. Consider the rays $N_i$ which emanate from the end points $E_i$ of the curve $\mathbf{c}$, and whose direction is given by the normal vector there. Intersect $\mathbf{c}$ with all rays $N_i$, which gives points $X_{i1}, \ldots, X_{ir_i}$ contained in $N_i$. Let $d_{ij} = \overline{X_{ij}E_i}$ and find the smallest value $r_n = \min_{i,j} d_{ij}$.

   This third step looks for self-intersections of type 'CV' and 'BC'.

4. Consider the rays $N_i$ as before, and compute the intersection points $X_{ij}$, if they exist. Let $d'_{ij} = \overline{X_{ij}E_i}$, and let $r'_n = \min_{i \neq j} \max(d'_{ij}, d'_{ji})$.

   This fourth step looks for self-intersections of type 'BV'.

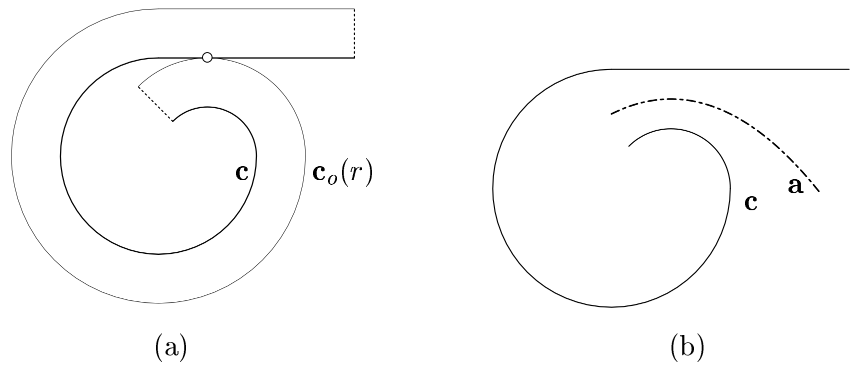


Figure 7: (a) The offset curve $\mathbf{c}_o(r)$ meets the curve $\mathbf{c}$ (case 'CO'). (b) Modified cut locus curve $\mathbf{a}$ of curve $\mathbf{c}$.

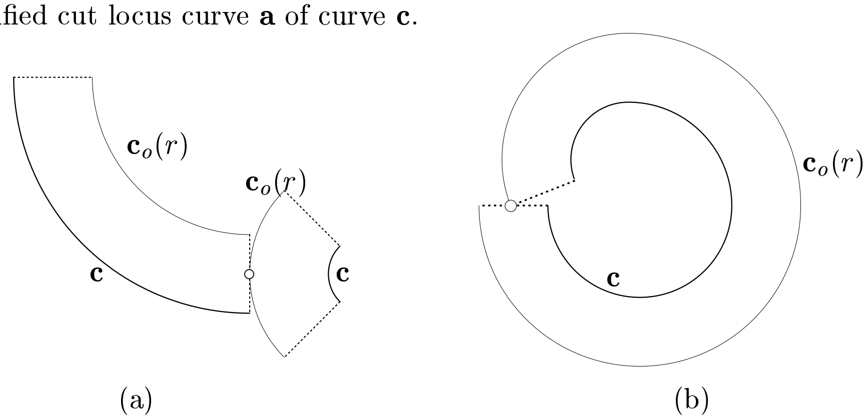


Figure 8: Self-intersection of the offset strip. (a): Case 'BO', (b); Case 'BV'.

5. Consider the rays $N_i$ as before. For all rays $N_i$, find the rays $\widetilde{N}_{ij}$, which emanate from a point $\mathbf{c}(t_{ij})$ of the curve $\mathbf{c}$, whose direction is given by the unit normal there, and which are *orthogonal* to the ray $N_i$. If it exists, the intersection of $N_i$ and $\widetilde{N}_{ij}$ is denoted by $X_{ij}$.

   Let
$$d''_{ij} = \overline{X_{ij}E_i}, \quad \widetilde{d}''_{ij} = \overline{X_{ij}\mathbf{c}(t_{ij})}, \quad r''_n = \min_{d''_{ij} \le \widetilde{d}''_{ij}} \widetilde{d}''_{ij}.$$

   This fifth step looks for self-intersections of type 'BO'.

**Theorem 2.2** *The value $r_s$ equals $\min(r_c, r'_c, r_n, r'_n, r''_n)$, where the various values are computed by the algorithm above.*

# 3 Surfaces

Let $M \subset \mathbb{R}^3$ be a smooth orientable surface, which is piecewise curvature continuous and has a piecewise curvature continuous boundary. To each point $\mathbf{p} \in M$ we assign the unit normal vector $\mathbf{n}(\mathbf{p})$.

We restrict ourselves to surfaces having only convex edges, because the offset surface will always have self-intersections, if $M$ possesses a concave edge. In a boundary point $\mathbf{p}$ of $\partial M$ there is a certain set $N(\mathbf{p})$ of unit normal vectors, depending on the smoothness on the boundary. If we do not want to round off the offset at the boundary, we assign to each boundary point the unit normal vector orthogonal to the limit tangent plane there.

We want to give a parametrization to the set of normal vectors: If the surface, in its smooth parts, is parametrized by a regular smooth function $\mathbf{g}(u,v) = (g_1(u,v),\, g_2(u,v),\, g_3(u,v))$, then the normal vector at $\mathbf{p} = \mathbf{g}(u,v)$ is given by $\mathbf{n}(p) = \pm\partial\mathbf{g}/\partial u \times \partial\mathbf{g}/\partial v \;/\; \|\partial\mathbf{g}/\partial u \times \partial\mathbf{g}/\partial v\|$.

We can also define a new object, the *unit normal bundle* $\perp_1 M$ of $M$. It consists of all pairs $(\mathbf{p};\mathbf{n})$ where $\mathbf{n}$ is a unit normal vector at the point $\mathbf{p}$. It has a smooth parametrization and makes it possible to define the *endpoint map* $E$ and the *offset surface* $M_o(r)$ of $M$ at distance $r$:

**Definition 3.1** *For $M$ as above and $r \in \mathbb{R}$, we define*

$$E : \perp_1 M \times \mathbb{R} \to \mathbb{R}^3, \qquad E((\mathbf{p};\mathbf{n}), r) = \mathbf{p} + r\,\mathbf{n}$$

*and $M_o(r)$ to be the image of $E$.*

We also define the offset strip of width $r$ as the union of all $M_o(d)$ for $0 \le d \le r$.

## 3.1 Self-intersections of Offset Surfaces

As in the planar case, we want to find values $r_M$ and $r_s$ of the offset parameter such that the offset surfaces $M_o(r)$ have no self-intersections for $r < r_M$, and have self-intersections at or immediately after $r = r_M$; and that the offset strip of width $r$ has no self-intersections for $r < r_s$, but has self-intersections for some $r \ge r_s$.

Again, it can be shown that there exists $r_0$ such that the endpoint map is one-to-one when restricted to positive values $r < r_0$. Like in the planar case we let $r$ grow from $r = 0$ until a singularity occurs.

We define the *cut locus surface* as the set of all centers of spheres which touch $M$ in at least two points, together with the limit points of this set.

The discussion is similar to the planar case. We discuss compact $C^2$ surfaces without boundary (closed surfaces) and find values of $r$ such that $M_o(r)$ is nonsingular and free of self-intersections. Surfaces with boundary are studied in [13].

### 3.1.1 Smooth Surfaces

If the surface is closed, we can identify its unit normal bundle $\perp_1 M$ with $M$. For $\mathbf{q} \in M$ consider the normal line $l$ to $M$ that goes through $\mathbf{q}$. It consists of all points $\mathbf{q} + t\,\mathbf{n}(\mathbf{q})$, where $\mathbf{n}(\mathbf{q})$ is the unit normal vector of $M$ at $\mathbf{q}$.

**Lemma 3.1** *The endpoint map $E : (\mathbf{p}, t) \mapsto \mathbf{p} + t\,\mathbf{n}(t)$ has nullity $> 0$ (that is, its Jacobian matrix is noninvertible) if and only if $t = 1/K_i(\mathbf{p})$, where $K_i(\mathbf{p})$ are the nonzero principal curvatures of $M$ at $\mathbf{p}$.*

Intuitively, the point $\mathbf{p} + (1/K_i(\mathbf{p}))\,\mathbf{n}(\mathbf{p})$ is a point in $\mathbb{R}^3$ where nearby normals intersect (cf. ([9], p. 34, Lemma 6.3).

The proof of this is an exercise in differential geometry: consider the spherical map $n : \mathbf{p} \mapsto \mathbf{n}(\mathbf{p})$ from $M$ to the unit sphere $S^2$, and its differential $d\mathbf{n}$. The principal curvatures are defined as the eigenvalues of $-d\mathbf{n}$. There are tangent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $d\mathbf{n}(\mathbf{v}_i) = -K_i \mathbf{v}_i$. Then the endpoint map $(\mathbf{p}, t) \mapsto \mathbf{p} + t\,\mathbf{n}(\mathbf{p})$ has nullity $> 0$ at $(\mathbf{p}, t) = (\mathbf{p}, 1/K_i)$: We evaluate its differential $dE = d\mathbf{p} + t\,d\mathbf{n}(\mathbf{p}) + dt\,\mathbf{n}(\mathbf{p})$ at the tangent vector $(d\mathbf{p}, dt) = (\mathbf{v}_i, 0)$, which gives $dEu(\mathbf{v}_i, 0) = \mathbf{v}_i + (1/K_i)(-K_i\mathbf{v}_i) = 0$.

Because the $E$-images of the tangent vectors $(d\mathbf{p}, dt)$ with $dt = 0$ span the tangent space of the offset surface $M_o(r)$, we have also shown the following:

**Lemma 3.2** *The parametrization $\mathbf{p} \mapsto \mathbf{p} + r\mathbf{n}(\mathbf{p})$ of the offset surface $M_o(r)$ is singular at $\mathbf{p}$ if and only if $r$ equals one of the two principal curvature radii $1/K_i$ of $\mathbf{p}$.*

We will consider how to find the maximum $r$ so that $M_o(r)$ will not self-intersect in a global manner, i.e., $M_o(r)$ is an embedded two-dimensional submanifold of $\mathbb{R}^3$. For this, we define the map

$$G : M \times M \to \mathbb{R}, \quad G(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|^2$$

Obviously, $G(\mathbf{x}, \mathbf{y}) > 0$, for $\mathbf{x} \neq \mathbf{y}$. Second, note that $G$ has a critical value $r > 0$ since $M \times M$ is compact. To see that, consider $\mathbf{x}, \mathbf{y} \in M$ with $G(\mathbf{x}, \mathbf{y}) = r$. Then $\mathbf{n}(\mathbf{x}) = \pm\mathbf{n}(\mathbf{y})$ and the vector $\mathbf{x} - \mathbf{y}$ is parallel to both

$\mathbf{n}(\mathbf{x})$ and $\mathbf{n}(\mathbf{y})$. (Otherwise there would be curves on $M$ passing through $\mathbf{x}$ and $\mathbf{y}$ such that the squared distance is not stationary at the pair $(\mathbf{x}, \mathbf{y})$). We claim that if

$$\gamma = \inf\{r > 0 \mid r \text{ is a critical value of } G\} \tag{2}$$

then $\gamma$ is positive. For if $\gamma$ were to be zero, we would then have that, for each arbitrarily small positive $\delta$ there exists a pair of points $(\mathbf{x}_\delta, \mathbf{y}_\delta)$ so that

1. $\|\mathbf{x}_\delta - \mathbf{y}_\delta\| < \delta$,

2. The vector $\mathbf{x}_\delta - \mathbf{y}_\delta$ is parallel to both $\mathbf{n}(\mathbf{x}_\delta), \mathbf{n}(\mathbf{y}_\delta)$.

Choose a sequence of $\delta$'s converging to zero. We may assume (since $M$ is compact) that the sequence $(\mathbf{x}_\delta, \mathbf{y}_\delta)$ converges to $(\mathbf{a}, \mathbf{a})$ with $\mathbf{a} \in M$. If $\delta$ is small enough, $\mathbf{x}_\delta, \mathbf{y}_\delta$ have to belong to the same component of $M$, because unions of components are closed. There is a neighborhood $U$ of $\mathbf{a}$, a local coordinate system around $\mathbf{a}$ and a $C^2$ function $h$ defined in a neighborhood $V$ of 0 in $\mathbb{R}^2$ such that that $M \cap U$ is the graph surface $\{(u_1, u_2, h(u_1, u_2) \mid (u_1, u_2) \in V\}$ and $(0, 0, h(0, 0)) = \mathbf{a}$. Now the normal vector field is continuous, and

$$\|(u_1, u_2, h(u_1, u_2)) - (u_1', u_2', h(u_1', u_2'))\| \geq \|(u_1, u_2, 0) - (u_1', u_2', 0)\|,$$

which contradicts the existence of $\mathbf{x}_\delta, \mathbf{y}_\delta$. We now consider a $r > 0$ such that

1. For each point $\mathbf{x} \in M$ neither principal curvature of $M$ at $\mathbf{x}$ is equal to $\pm 1/r$, and

2. $2r < \sqrt{\gamma}$, where $\gamma$ is as in (2).

and show that in this case $M_o(r)$ is nonsingular and has no self-intersections:

Suppose that $r$ satisfies condition 1 above and it is the *first* positive number for which the obvious mapping $M \to M_r$ is not one-to-one. Then there exist $\mathbf{x} \neq \mathbf{y}$ for which $\mathbf{x} + r\mathbf{n}(\mathbf{x}) = \mathbf{y} + r\mathbf{n}(\mathbf{y})$. This implies $\mathbf{x} - \mathbf{y} = r(\mathbf{n}(\mathbf{y}) - \mathbf{n}(\mathbf{x}))$. We claim that in this case $(\mathbf{x}, \mathbf{y})$ is a critical point of $G$: For if not, the locally 2-dimensional manifold $M_o(r)$ has a transverse self-intersection at $E(\mathbf{x})$, and that contradicts the minimality of $r$. Thus, $\|\mathbf{x} - \mathbf{y}\| = |2r|$, and $(2r)^2$ is a critical value of $G$, so condition 2 is violated. This shows the following:

**Theorem 3.3** *Let $M$ be a compact orientable $C^2$ surface without boundary. Then $M_o(r)$ is nonsingular and without self-intersections, if $r$ satisfies conditions 1 and 2 as above.*

We will see that in this case $r_s$ equals either the minimum of $1/K_i$ or $\sqrt{\gamma}/2$.

### 3.1.2 Singular Points in Offset Surfaces

Notice that if $E$ becomes singular for some value of $r$, then the offset strip of width $r$ has a self-intersection because a mapping which is singular and maps all lines $((\mathbf{p};\mathbf{n}),r)$, $(\mathbf{p};\mathbf{n})$ fixed, to a straight line in space, cannot be 1–1. We also have

1. If $E$ becomes singular in a smooth point $\mathbf{p}$ of $M$, this is caused by the fact that $r_0$ equals the smallest principal curvature radius $1/K_i(\mathbf{p})$.
2. If $E$ becomes singular in a point $(\mathbf{p};\mathbf{n})$ of the unit normal bundle, such that $\mathbf{p}$ is located in an edge $e$ of $M$, then this is caused by the fact that $r_0$ equals the normal curvature radius $r(e,\mathbf{n})$ of this edge with respect to the surface normal $\mathbf{n}$ (see [13]).

In both cases it is possible to show that the center of the 'offending' curvature is contained in the cut locus surface.

Unlike the curve case, it is not as easy to determine when $M_o(r)$ actually has self-intersections, i.e., where the endpoint map $\perp_1 M \to M_o(r)$ is not injective. Even if $M_o(r)$ has a singularity, the endpoint map may still injective. For values $r > d$, the offset surface $M_o(r)$ could again be regular and free of self-intersections (consider a sphere, a cylinder, or any tubular surface and their interior offsets). All we can say is that if $M_o(r)$ is singular, the offset strip has self-intersections for values $> d$.

### 3.1.3 Auto-normal chords

Like the planar case, the offset surface $M_o(r)$ is oriented (per definition) by the normal vectors inherited from $M$. Again it is not possible that $M_o(r)$ touches itself, but is not in oriented contact. So for all coincident *interior* points $\mathbf{q} = \mathbf{q}'$ of $M_o(r)$, the corresponding points $\mathbf{p}, \mathbf{p}'$ in $M$ define the *shortest auto-normal chord* of $M$. For smooth surfaces without boundary, this has been shown above. For other surfaces, see [13].

It is also clear that $\mathbf{q} = \mathbf{q}'$ is located in the cut locus surface, and when disregarding the boundary values of the distance function, the pairs $\mathbf{pq}$ and $\mathbf{p}'\mathbf{q}$ are local interior minima of the distance function between $M$ and its cut locus surface, and among those local minimal they are the ones where the smallest value of the distance is attained..

If $M$ has no boundary, we have therefore the following:

**Theorem 3.4** *Let $M$ be a closed surface without boundary. Then the value $r_s$ equals the minimum distance between $M$ and its cut locus surface. The value $r_M$ is greater or equal to $r_M$.*

### 3.1.4 Coincident Boundary points

The situation is exactly the same as in the planar case. We state the corresponding result:

**Theorem 3.5** *The value $r_M$ equals the smallest distance between the surface $M$ and its cut locus surface. The type of self-intersection at $r_M$ is determined by the pairs $\mathbf{p}_i, \mathbf{q}_i$ of points like in the curve case.*

### 3.1.5 Self-intersections of the offset strip

The offset strip has the following boundary: The offset surface $M_o(r)$, the surface $M$ itself (if we did not consider a two-sided offset); and a strip $S$ of the ruled surface which consists of the surface normals in the points of $\partial M$ (see Fig. 9). Depending on the smoothness of $M$'s boundary, the ruled surface strip is smooth or has edges. There are, again depending on the smoothness of $M$'s boundary, many different cases of intersection or touching of various components of the offset strip's boundary. We will not attempt to enumerate them completely. Most of them are straightforward generalizations of two-dimensional cases.

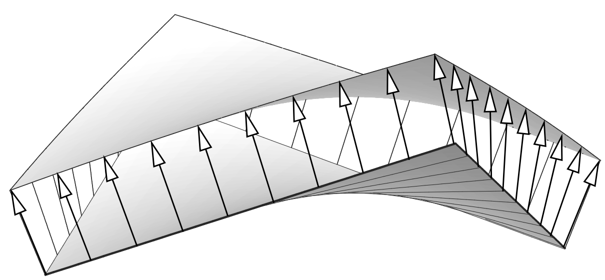

Figure 9: Offset strip of a surface with its boundary.

Touching of $M$ and $M_o(r)$, can, like in the curve case, be tested by considering the modified cut locus surface. This also includes the boundaries of $M$ and $M_o(r)$. The only case which did not occur in the two-dimensional problem is the case of self-intersection of $S$. We are going to describe how to compute this self-intersection.

Suppose that $S$ meets itself in the point

$$\mathbf{q} = \mathbf{p} + r_0\,\mathbf{n}(\mathbf{p}) = \mathbf{p}' + r_0'\,\mathbf{n}(\mathbf{p}')$$

with $\mathbf{p}, \mathbf{p}' \in \partial M$ and $r_0' < r_0$: When growing the offset strip, at $r = r_0'$ it begins to intersect $S$, while not intersecting itself, but at $r = r_0$ it finally intersects itself. Thus for finding the maximum $r$ such that this type of self-intersection does not happen, we have to compute the complete intersection curve of $S$ with itself.

For each $\mathbf{p} \in \partial M$ there is a value $\lambda(\mathbf{p})$ such that $\mathbf{p}+\lambda(\mathbf{p})\mathbf{n}(\mathbf{p})$ is the first (in the sense of $\lambda$) intersection point of the ray $\mathbf{p} + t\mathbf{n}(\mathbf{p})$, $(t > 0)$ with $S$. If there is no such point, we let $\lambda(p) = \infty$. If $\lambda(p) < \infty$, i.e., there actually is an intersection, there is another value $\lambda'(\mathbf{p})$ and another point $\mathbf{p}' \in \partial M$ such that

$$\mathbf{p} + \lambda(\mathbf{p})\,\mathbf{n}(\mathbf{p}) = \mathbf{p}' + \lambda'(\mathbf{p})\,\mathbf{n}(\mathbf{p}').$$

and the maximum $r_S$ is found by

$$r_S = \min_{\mathbf{p}\in\partial M} \max(\lambda(\mathbf{p}), \lambda'(\mathbf{p})).$$

In order to understand the algorithm better, we show how to intersect *two* ruled surfaces $S_1(u, v) = \mathbf{c}_1(u) + v\mathbf{n}_1(u)$ and $S_2(u, v) = \mathbf{c}_2(u) + v\mathbf{n}_2(u)$. To intersect $S$ with itself, we then let $S_1 = S_2 = S$.

We start at a parameter value $u$ and intersect the line $l_1(u) : \mathbf{c}_1(u) + v\mathbf{n}_1(u)$ with $S_2$, by looking for a parameter $u'_0$ such that the line $l_2(u') : \mathbf{c}_2(u') + v\mathbf{n}_2(u')$ intersects $l_1$. This is done by solving

$$\det(\mathbf{c}_2(u') - \mathbf{c}_1(u), \mathbf{n}_1(u), \mathbf{n}_2(u')) = 0. \tag{3}$$

(This absolute value of this equals the distance of the lines $l_1(u)$ and $l_2(u')$ multiplied by $\|\mathbf{n}_1(u) \times \mathbf{n}_2(u')\|$.) Having found the solution $u'$, we let $\mathbf{p} = \mathbf{c}_1(u)$, $\mathbf{p}' = \mathbf{c}_2(u')$ and easily calculate $\lambda$ and $\lambda'$ such that $S_1(u, \lambda) = S_2(u', \lambda')$.

# 4 Special cases

For convex and star-shaped[1] surfaces we have some information concerning the auto-normal chords [11]:

- If a *convex surface* is oriented such that its normals point to the *outside*, then the offset surface is regular and the endpoint map is always one-to-one for all $r > 0$, because there are no auto-normal chords and all principal curvatures are negative.

[1]A surface is star-shaped if it is the boundary of a star-shaped body. A star-shaped body $S$ is a subset of Euclidean space which has a point $o$ in its interior such that the straight line segment $op$ is in $S$ for all $p \in S$.

- If a *star-shaped surface* $M$ is oriented such that its normals point to the *outside*, then there are no auto-normal chords and the offset surface $M_o(r)$ is regular and one-to-one as long as $r$ is less than the principal curvature radii in all points of $M$.

- If a *convex surface* $M$ is oriented such that its normals point to the *inside*, consider a sphere of maximum radius $R$ inscribed in $M$. If $r < R$, then obviously there is no auto-normal chord of length $r$, so the offset $M_o(r)$ is regular and free of self-intersections if $r$ is less than all principal curvature radii of $M$. Note that when reversing the orientation of $M$, all principal curvatures and also their reciprocals, the radii, are multiplied by $-1$.

  The maximum inscribed sphere, in principle, either touches $M$ in two points or its center equals a principal curvature center. Blaschke has shown that the former case is not possible [2], so there remains the latter case which shows that $M_o(r)$ has a singularity before having self-intersections.

- If a *star-shaped surface* $M$ is oriented such that its normals point to the *inside*, consider the convex core $\mathbf{c}$ of $M$. It is defined as the set of points $\mathbf{p}$ such that $M$ is star-shaped with respect to $\mathbf{p}$. It is easily seen to be convex. Every auto-normal chord of $M$ gives rise to two parallel tangent planes of $M$ with $C$ lying between them.

  Second, consider a sphere of maximum radius $R$ inscribed into $C$. Clearly no auto-normal chord of $M$ has length less or equal $r$. Thus the offset surfaces $M_r$ with $r < R$ are nonsingular and free of self-intersections if $r$ is less then all principal curvature radii of $M$.

# 5 Algorithm

## 5.1 Singular Points on Offset Surface

Local self-intersections of the offset surface occur when the positive offset distance exceeds the maximum absolute value of the negative minimum principal curvature on the generator surface or the absolute value of the negative offset distance exceeds the maximum value of the positive maximum principal curvature on the generator surface. A detailed formulation and a robust method for finding extrema of principal curvatures can be found in [7].

## 5.2 Auto-Normal Chords

The shortest auto-normal chord problem is to find two different points on the generator surface whose surface normals point in opposite directions and that have a minimum distance. The absolute minimum of the distance function is trivially attained for all pairs of equal points, but their normal vectors point to the same direction. All pairs of points where the distance function attains a local minimum have the property that their normal vectors point either to the same or to the opposite direction. This means that we can look for all stationary points of the distance function, thereby avoiding the trivial solutions, afterwards single out those pairs whose normal vectors behave in the desired way, and choose the one which minimizes the distance.

We assume that the generator surface is given by a NURBS surface, which can be split into rational Bézier surfaces by knot insertion. The minimum distance problem can be decomposed into the minimum distance between two points on different surfaces and the minimum distance between two points on the same surface. The first problem is solved by Zhou et al. [14], so we focus on the second problem here.

Let the generator surface be given by $\mathbf{g}(u,v) = (g_1(u,v), g_2(u,v), g_3(u,v))$. Assume the surface is nonsingular, i.e. $|(\partial \mathbf{g}/\partial u) \times (\partial \mathbf{g}/\partial v)| \neq 0$, and that $\partial \mathbf{g}/\partial u$ and $\partial \mathbf{g}/\partial v$ are continuous.

The squared distance function between two points $\mathbf{p} = \mathbf{g}(u,v)$ and $\mathbf{q} = \mathbf{g}(s,t)$ on the generator surface with parameters $(u,v)$ and $(s,t)$ is given by

$$D(s,t,u,v) = |\mathbf{g}(s,t) - \mathbf{g}(u,v)|^2 = (\mathbf{g}(s,t) - \mathbf{g}(u,v)) \cdot (\mathbf{g}(s,t) - \mathbf{g}(u,v)) \quad (4)$$

where $(s,t) \neq (u,v)$. The stationary points of $D(s,t,u,v)$ satisfy the following equations

$$D_s(s,t,u,v) = D_t(s,t,u,v) = D_u(s,t,u,v) = D_v(s,t,u,v) = 0 \quad (5)$$

which can be rewritten using (4) as

$$(\mathbf{g}(s,t) - \mathbf{g}(u,v)) \cdot \mathbf{g}_s(s,t) = 0 \quad (6)$$

$$(\mathbf{g}(s,t) - \mathbf{g}(u,v)) \cdot \mathbf{g}_t(s,t) = 0 \quad (7)$$

$$(\mathbf{g}(s,t) - \mathbf{g}(u,v)) \cdot \mathbf{g}_u(u,v) = 0 \quad (8)$$

$$(\mathbf{g}(s,t) - \mathbf{g}(u,v)) \cdot \mathbf{g}_v(u,v) = 0 \quad (9)$$

The geometrical interpretation of equations from (6) to (9) is that the line connecting the two points $\mathbf{p} = \mathbf{g}(u,v)$ and $\mathbf{q} = \mathbf{g}(s,t)$ is *orthogonal* to the generator surface at both points. Without loss of generality we may assume that $\mathbf{g}(s,t)$ is given as a rational Bézier surface, that is

$$\mathbf{g}(s,t) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}\mathbf{P}_{ij}B_{i,m}(s)B_{j,n}(t)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}B_{i,m}(s)B_{j,n}(t)} \equiv \frac{\mathbf{p}(s,t)}{w(s,t)}. \tag{10}$$

Substituting (10) into (6) gives

$$\left[\frac{\mathbf{p}(s,t)}{w(s,t)} - \frac{\mathbf{p}(u,v)}{w(u,v)}\right] \cdot \left[\frac{\mathbf{p}_s(s,t)w(s,t) - \mathbf{p}(s,t)w_s(s,t)}{w^2(s,t)}\right] = 0. \tag{11}$$

Multiplying by its own denominator we finally obtain

$$\begin{aligned}
&\mathbf{q}\cdot[\mathbf{p}_s(s,t)w(s,t) - \mathbf{p}(s,t)w_s(s,t)] = 0\\
&\mathbf{q}\cdot[\mathbf{p}_t(s,t)w(s,t) - \mathbf{p}(s,t)w_t(s,t)] = 0\\
&\mathbf{q}\cdot[\mathbf{p}_u(u,v)w(u,v) - \mathbf{p}(u,v)w_u(u,v)] = 0\\
&\mathbf{q}\cdot[\mathbf{p}_v(u,v)w(u,v) - \mathbf{p}(u,v)w_v(u,v)] = 0\\
\text{with}\quad &\mathbf{q} = \mathbf{p}(s,t)w(u,v) - \mathbf{p}(u,v)w(s,t)
\end{aligned}$$

where $(s,t) \neq (u,v)$. This system of equations consists of four nonlinear polynomial equations with four unknowns $s$, $t$, $u$, $v$. To find all the stationary points, we need to employ global solution techniques which are designed to compute all the roots in the area of interest. One such global method is provided by the Bernstein subdivision-based Interval Projected Polyhedron algorithm [1, 3, 6]. The trivial solutions $(s,t) = (u,v)$ must be excluded from the system, otherwise a Bernstein subdivision-based interval projected polyhedron method would attempt to solve for an infinite number of roots. Unfortunately, the system does not involve the factors $s-u$ and $t-v$ and hence we cannot factor out these factors from the system. Thus, the polynomial system is first solved by the Bernstein subdivision-based polynomial solver at a coarse subdivision level (e.g. $10^{-1} \sim 10^{-2}$) in a global manner. The two rectangular sub-patches on the surface for each set of roots using the de Casteljau subdivision algorithm are extracted. Then the normal rectangular pyramids, which bound normal vectors of Bézier patches, are constructed [5]. If the two pyramids intersect, the associated parameter boxes are considered as representing trivial roots and excluded from the list of roots. Finally we restart the polynomial solver with boxes that include the solutions and solve for them with strict accuracy (e.g. $10^{-8}$).

### 5.3 Coincident Boundary Points

Let us consider the case of coincident points at the boundary. In the self-intersection point, the boundary curve of the offset surface tangentially self-intersects the boundary curve of the offset surface. In the generic case the self-intersection point is an interior point of the offset surface. Since the offset surface is parallel to the generator surface, we can reduce this condition to the orthogonality of the tangent vector of the boundary curve of the generator surface and the normal vector of the generator surface at its corresponding point. Therefore we have the following equations:

$$\mathbf{g}(s,t) + d\mathbf{n}(s,t) = \mathbf{g}(u,v) + d\mathbf{n}(u,v) \tag{12}$$

$$[\mathbf{g}_s(s,t) \times \mathbf{g}_t(s,t)] \cdot \mathbf{g}_v(u,v) = 0 \tag{13}$$

This system consists of four scalar equations with four unknowns, namely $v$, $s$, $t$ and $d$ when the boundary curve is an isoparametric curve $u$ =0 or 1, or $u$, $s$, $t$ and $d$ when the boundary curve is an isoparametric $v$ =0 or 1. In the latter case $\mathbf{g}_v(u,v)$ in Equation (13) is replaced by $\mathbf{g}_u(u,v)$. We can formulate the four scalar equations in terms of polynomials by splitting the rational B-spline surface into rational Bézier patches and introducing the auxiliary variable to avoid the square roots [5]. However we cannot factor out the trivial solution $(s,t) = (u,v)$ from the system. Maekawa *et al.* [5] developed a method to handle such a case. But in this case we can employ Newton's method to solve the system (12) and (13), as we can provide all the initial approximations to the roots by the following global method which does not involve factoring out trivial solutions.

We are able to compute all the stationary points of the squared distance function [14] between a rational Bézier boundary curve and a rational Bézier boundary curve using the Interval Projected Polyhedron algorithm. For the case of the stationary points of the squared distance function between a rational Bézier boundary curve and a rational Bézier patch, we need to consider two situations. The first situation is the case when the boundary curve is not extracted from the iso-parametric line of the same patch, while in the second case the boundary curve is an iso-parametric line of the patch. The first case does not involve trivial solutions and can be solved easily by the method described in [14]. The second case involves trivial solutions, therefore we split the patch into two patches at the isoparametric line very close to the boundary, i.e. $0 + \epsilon$ or $1 - \epsilon$, where $\epsilon$ is a very small positive number. Then extract the boundary curve from the small patch and treat it as the first case. It is implied that we use a more strict accuracy than $\epsilon$ in the polynomial solver. After all the stationary points are evaluated we need to classify each stationary point as a local maximum, local minimum or saddle point. Then we solve the system (12) and (13) by the Newton's method for all the local minima as an input. The minimum $d$ among the solutions becomes the maximum possible upper limit of the offset distance such that boundary point to boundary point or in the generic case boundary point to interior point will not self-intersect.

### 5.4 Self-intersections of the Offset Strip

As discussed in Section 3.1.5, computation of the self-intersection of the offset strip reduces to computing the self-intersection of a ruled surface. We have four ruled surfaces for a four-sided patches of a tensor-product surface. Without loss of generality, we work with the ruled surface having $\mathbf{c}_1(u) = \mathbf{g}(u, 0)$ as a directrix. In the self-intersection point two different generators of the ruled surface intersect each other. This leads to the geometric condition expressed by Equation (3). Thus, we have one equation with two unknowns, namely $u$ and $u'$. When one of the boundary generators is touching the interior of the ruled surface, the additional condition is obtained by setting $u = 0$ or 1. In general, if two different interior generators intersect each other, it is difficult to find the minimum offset distance for self-intersection of the offset strip.

## 6 Examples

### 6.1 Singular Points on Offset Surface

A shaded image of an offset of a bumpy wave-like surface is shown in Figure 10. The generator surface, which is in wireframe, is a bicubic integral B-spline surface with uniform knots which consists of 4×4 Bézier patches. The global minimum value of the minimum principal curvature is -21.018 and located at (0.5, 0.1079), (0.5, 0.8921), (0.1079, 0.5) and (0.8921, 0.5) respectively in the $uv$-parametric space. Thus maximum limit of offset distance without self-intersection is $d < 0.04758$. The offset image in Figure 10 has this limit, and we can observe the four locations where the the minimum principal curvature has its global minimum in the concave regions.

### 6.2 Auto-Normal Chords

The generator surface (wireframe), together with its offset (shaded image), shown in Figure 11 is a sextic-quadratic Bézier patch. The surface has a global minimum along the auto-normal chord. We have adopted the method described in [5] to solve the system (6) to (9), which provides $(s, t)$ = (0.10593, 0.5), $(u, v)$ =(0.89407, 0.5) and $d = 0.8082$.

### 6.3 Coincident Boundary Points

The generator surface is a quintic-cubic Bézier patch and has a global minimum distance between the boundary curve $u$=1.0 and its interior point. First all the initial approximations to the Newton's method to solve the system (12) and (13) are obtained in a global manner. The stationary points of the squared distance function between a boundary curve and an interior point of the surface, and between two boundary curves are computed using the Interval Projected Polyhedron algorithm. After all the stationary points are evaluated we classify each stationary point as a local maximum, local minimum or saddle point. Then we solve the system (12) and (13) by the Newton's method for all the local minima as an input. Finally the global minimum is found to be $d = 0.09346$ between a point on the boundary curve $(u, v)$ =(1, 0.5) and an interior point $(s, t)$ = (0.05319, 0.5). The initial approximation $d = 0.08793$ $(u, v)$ =(1, 0.5), $(s, t)$ = (0.04594, 0.5) was used, which resulted from the minimum distance computation between the boundary curve and the interior point of the surface. The Figure 12 shows the generator surface (wireframe), its offset (shaded image) with $d = 0.08793$, and the straight line connecting the minimum distance between the boundary curve $u$=1.0 and its interior point.

### 6.4 Self-intersections of the Offset Strip

The generator surface is a quartic Bézier ruled surface. Since the generator surface is a four-sided patch, we need to check for all the four ruled surfaces for possible self-intersections. In this example the ruled surface, having $\mathbf{P}(u, 0)$ as a directrix, self-intersects. Thus, we have $t$=$v$=1. In this case one of the boundary generators $u = 1$ touches the interior of the ruled surface. Newton's method converges to $u'$= 0.12319324. Now we can easily find $\lambda$ and $\lambda'$ to be -0.244291, -0.226883.. In this case, an initial $s$ close to the final result is critical, otherwise Newton's method may converge to different solutions. The Figure 13 shows the generator surface (wireframe), its offset (shaded image) with $d = -0.244291$.

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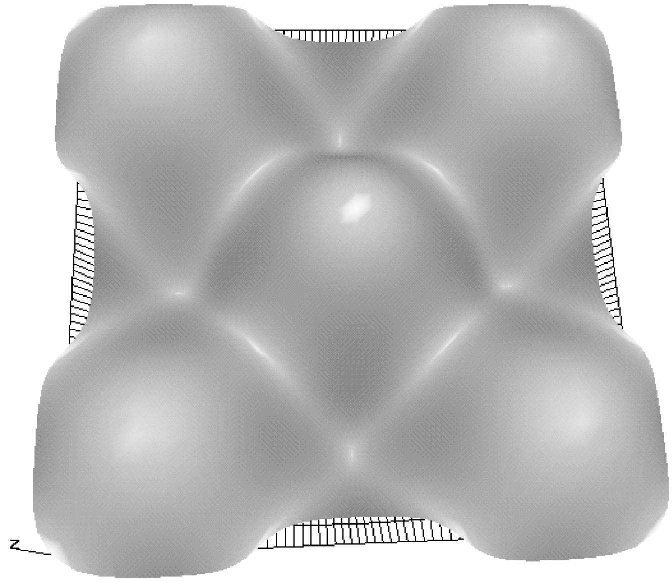


Figure 10: Local self-intersection

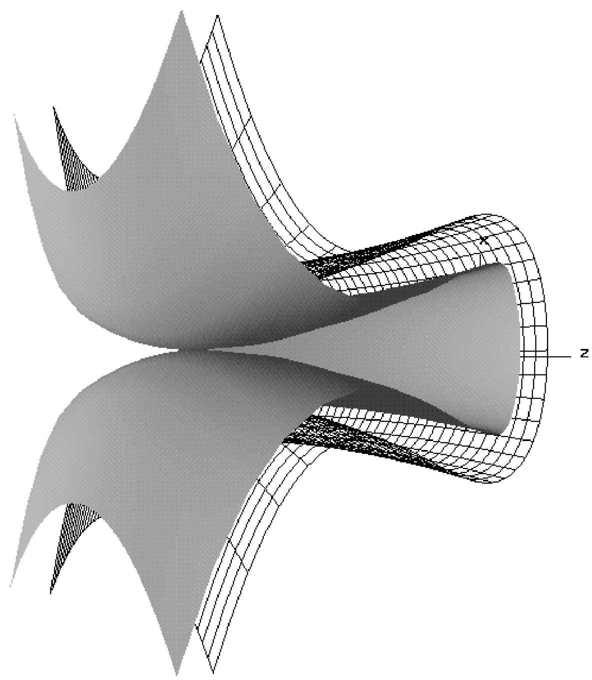


Figure 11: Auto-normal chords

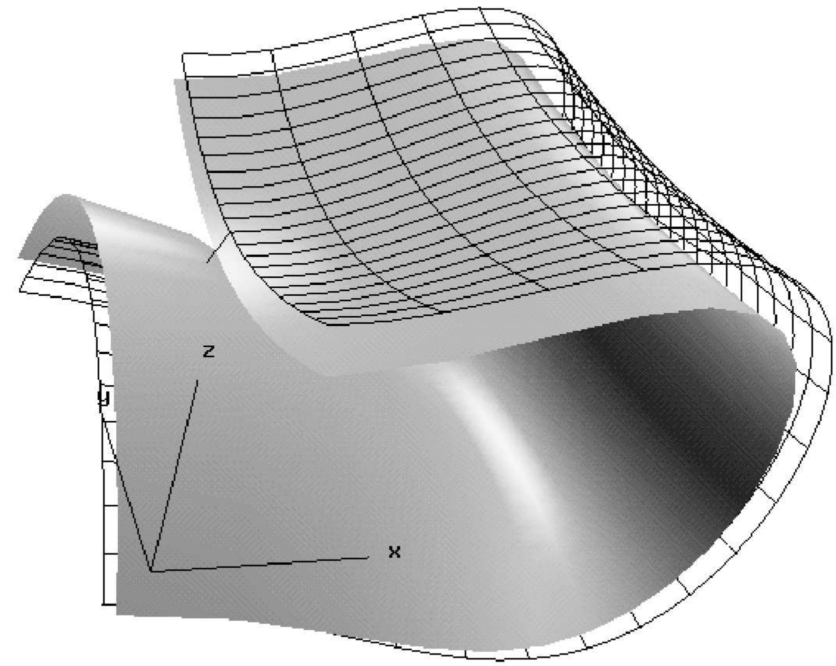


Figure 12: Coincident boundary points

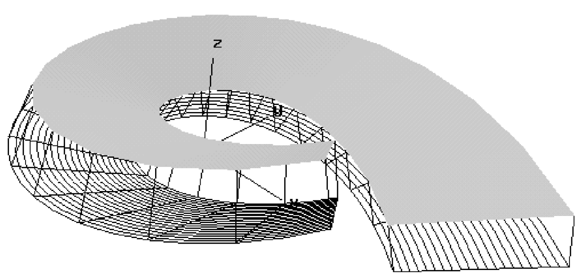


Figure 13: Self-intersection of the offset strip