CONVERGENCE ANALYSIS OF SUBDIVISION PROCESSES ON THE SPHERE

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Abstract. We analyse the convergence of nonlinear Riemannian analogues of linear subdivision processes operating on data in the sphere. We show how for curve subdivision rules we can derive bounds guaranteeing convergence, if the density of input data is below that threshold. Previous results only yield thresholds that are several magnitudes smaller and are thus useless for a priori checking of convergence. It is the first time that such a result could be shown for a geometry with positive curvature and for subdivision rules not enjoying any special properties like being interpolatory or having nonnegative mask. refinement algorithm; approximation theory; differential geometry.

1. Introduction

Subdivision schemes are iterative refinement algorithms used to produce smooth curves and surfaces. For data in linear spaces they are well-studied and find applications in various areas from approximation theory to computer graphics – see Dyn [1992], Cavaretta et al. [1991] and Peters and Reif [2018] for an introduction and overview.

This paper contributes new results to the convergence analysis of subdivision schemes defined via operations that are natural in nonlinear geometries. Several different methods exist to transfer linear schemes to the nonlinear situation, see [Wallner, 2020] for a survey. We mention the log-exp-analogue which uses the exponential map defined in Lie groups, Riemannian manifolds and in symmetric spaces, cf. [Donoho, 2001, Ur Rahman et al., 2005]. The projection analogue can be applied to embedded surfaces, see [Xie and Yu, 2010, Grohs, 2009]. Generally, such constructions are only locally well-defined. Arguably the most natural of the available intrinsic definitions is the Riemannian analogue of a subdivision rule, obtained by replacing every occurrence of an affine average by a weighted geodesic average, also known as the Riemannian center of mass. It can be made globally well-defined on a complete Riemannian manifold with nonpositive sectional curvature, cf. [Hüning and Wallner, 2019, Wallner et al., 2011, Ebner, 2013, 2014]. Well-definedness has been studied in various contexts, see [Karcher, 1977, Sander, 2016, Dyer et al., 2016a,b, Pennec, 2018].

While *smoothness* of limits produced by such geometric subdivision rules has been successfully investigated, see e.g. [Grohs, 2010], the existence of these limits poses different problems. Early results were limited to 'dense enough' input data. Results concerning convergence for all input data, or at least input data which obey a sensible density property, are available for univariate interpolatory schemes [Wallner, 2014], multivariate schemes with nonnegative mask [Ebner, 2013, 2014] or schemes expressed in terms of multiple binary averages [Dyn and Sharon, 2017b,a]. In Hüning and Wallner [2019] we proved that the Riemannian analogue of a univariate linear scheme on complete Riemannian manifolds with nonpositive sectional curvature converges if the linear scheme converges uniformly, without any restriction on the sign of the schemes's mask coefficients.

Date: 08/2020.

The present work for the first time treats convergence results for refinement algorithms on a space with positive curvature. It turns out that even for the sphere the situation is appreciably different from the nonpositive curvature case.

The paper is organised as follows. We start by repeating some basic facts about subdivision and introduce our notation. Next we discuss Riemannian analogues of linear subdivision schemes, in particular their well-definedness on the sphere. Our strategy to prove convergence is rather technical and is introduced by means of an example first. The last section contains further examples, including ones which could not be treated by earlier methods.

2. Linear subdivision and its Riemannian analogue

Consider a sequence of input data points $(x_i)_{i\in\mathbb{Z}}$ in some linear space. A linear binary *subdivision rule* S defined by the mask $(a_i)_{i\in\mathbb{Z}}$ maps the input data to a new sequence $(Sx_i)_{i\in\mathbb{Z}}$, where

$$Sx_i = \sum_{j\in\mathbb{Z}} a_{i-2j} x_j .$$

Throughout this paper we assume that only finitely many coefficients a_i are nonzero. A *subdivision scheme* is the repeated application S, S^2, S^3, … of the subdivision rule. We consider only translation invariant (*affine invariant*) rules, meaning that $\sum_{j\in\mathbb{Z}} a_{2j} = \sum_{j\in\mathbb{Z}} a_{2j+1} = 1$. This is because others do not make sense geometrically, and even for functional data $x_j \in \mathbb{R}$, translation invariance is known to be a necessary condition for convergence.

We are going to adapt the subdivision rule S so as to act on data contained in a Riemannian manifold M. For basics on Riemannian geometry, the reader is referred to [do Carmo, 1992]. The adaptation replaces the weighted affine averages $Sx_i = \sum_{j\in\mathbb{Z}} a_{i-2j}x_j$ by geodesic averages. Since Sx_i can be equivalently described by $Sx_i = \arg\min_x \sum_{j\in\mathbb{Z}} a_{i-2j}\|x - x_j\|^2$, a natural extension of this construction to data in M is performed by replacing the Euclidean distance by the Riemannian distance. The subdivision rule

$$Tx_i = \arg\min_x \sum_{j\in\mathbb{Z}} a_{i-2j}\, \mathrm{dist}\,(x, x_j)^2\,, \quad i \in \mathbb{Z} \tag{1}$$

is called the *Riemannian analogue* of S. The minimiser occurring in this definition is the *Riemannian center of mass* of points x_j with respect to weights a_{i-2j}. We use 'Riemannian center of mass' and 'weighted geodesic average' synonymously. Note that weights are allowed to be negative.

Existence of the minimiser in (1) has been studied by several authors, see e.g. [Hardering, 2015, Sander, 2016, Karcher, 1977]. It always exists locally. In case of nonpositive sectional curvature K completeness and simple connectedness of the manifold together imply existence for all data points. As to more general classes of manifolds, completeness remains necessary. One can get rid of simple connectedness if data are joined by a path, see [Hüning and Wallner, 2019]. It is the topic of the present paper to investigate what happens when the sectional curvature is no longer required to be nonnegative.

We also introduce in this place two properties we are going to need in our analysis. We say that T is contractive, with *contractivity factor* μ, if

$$\mathrm{dist}(T^k x_{i+1}, T^k x_i) \le \mu^k \cdot \sup_\ell \mathrm{dist}(x_\ell, x_{\ell+1}), \quad \text{for all } i \in \mathbb{Z},\ k \in \mathbb{N} \quad (\mu < 1). \tag{2}$$

Further, T is *displacement-safe*, if there is a constant $C > 0$ such that

$$\mathrm{dist}(Tx_{2i}, x_i) \le C \cdot \sup_\ell \mathrm{dist}(x_\ell, x_{\ell+1}), \quad i \in \mathbb{Z}. \tag{3}$$

We say that T *converges* for input data x if iteratively refined data $x, Tx, T^2x, \ldots$ become denser and approach a continuous limit curve. Formally, we treat convergence in a coordinate chart, linearly interpolating points $T^k x_i$ by a piecewise linear function h_k with $h_k(2^{-k}i) = (T^k x)_i$, and observing convergence of functions h_k (uniformly on compact subsets). It has been shown in various situations that displacement-safe rules admitting a contractivity factor $\mu < 1$ are convergent, see e.g. [Dyn and Sharon, 2017a,b, Wallner et al., 2011]. For us it will be convenient to use the following result by Hüning and Wallner [2019].

Theorem 1. *Consider the Riemannian analogue T of a linear binary subdivision rule S in a complete Riemannian manfold M. Assuming T is well-defined for input data x, it converges to a continuous limit $T^\infty x$ if it admits a contractivity factor $\mu < 1$ and is displacement-safe.*

3. Riemannian center of mass on manifolds exhibiting positive sectional curvature

3.1. **Well-definedness of Riemannian averages.** In our study of Riemannian subdivision we deal with weighted geodesic averages of finitely many points x_j w.r.t. weights α_j. The average is defined as the minimiser of

$$f_\alpha(x) = \sum_j \alpha_j \operatorname{dist}(x_j, x)^2 \quad \text{with} \quad \sum_j \alpha_j = 1. \tag{4}$$

Existence and uniqueness of the Riemannian average has been studied by various authors. We are going to make use of the following result by [Dyer et al., 2016a], which uses the notation $B_r(x) = \{y \in M \mid \operatorname{dist}(x, y) < r\}$ for the geodesic ball of radius $r > 0$ centered in $x \in M$.

Lemma 2. *Consider a complete Riemannian manifold with sectional curvature bounded from above by $K > 0$. Consider also finitely many data points $x_j \in B_r(x)$, for some $x \in M$ and $r > 0$, and weights α_j with $\sum \alpha_j = 1$. Letting*

$$\alpha_- := \sum_{\alpha_j<0} |\alpha_j|,$$

the function $f_\alpha = \sum \alpha_j \operatorname{dist}(\cdot, x_j)^2$ has a unique minimiser in $B_{r^*}(x)$, if
(i) $r < r^* < \min\{\frac{\iota_M}{2}, \frac{\pi}{4\sqrt{K}}\}$, where ι_M denotes the injectivity radius of M,
(ii) $r^* > (1 + 2\alpha_-)r$,
(iii) $r^* < \frac{\pi}{4\sqrt{K}}(1 + (1 + \frac{\pi}{2})\alpha_-)^{-1}$.

A convergence result for nonlinear subdivision rules depends on the capability to control the distances of points of the sequence $(T^k x_i)_{i\in\mathbb{Z}}$ from each other as well as their distance to the input data. Unfortunately, Lemma 2 cannot directly be used to control those distances.

3.2. **The Riemannian analogue of a linear subdivision rule on the unit sphere.** From now on, we restrict ourselves to the *unit sphere* $\Sigma^n \subseteq \mathbb{R}^{n+1}$ for $n \geq 2$. In particular the sectional curvature $K = 1$, and the injectivity radius equals π. The conditions on radii r and r^* specified in Lemma 2 reduce to

$$r^* > (1 + 2\alpha_-)r \geq r, \quad r^* < \frac{\pi}{4}\Big(1 + \big(1 + \frac{\pi}{2}\big)\alpha_-\Big)^{-1}. \tag{5}$$

In the special case of nonnegative coefficients we have $\alpha_- = 0$, and conditions reduce further to $r^* > r$ and $r^* < \frac{\pi}{4}$. Applying Lemma 2 to the averages occurring in a Riemannian subdivision rule yields the following result:

Proposition 3. *Let T be a Riemannian subdivision rule in the sphere according to Equ.* (1)*. For any point Tx_i, consider the sum of negative coefficients which contribute to the definition of Tx_i, namely, $\alpha_- = \sum_{a_{i-2j}<0} |a_{i-2j}|$. Then we distinguish two cases:*

$\alpha_- = 0$**:** *Assume the input data points x_j contributing to the computation of Tx_i lie within a ball of radius $r < \frac{\pi}{4}$. Then Tx_i is well defined by Equ.* (1) *as a minimiser within that ball.*

$\alpha_- > 0$**:** *Assume the data points x_j contributing to the computation of Tx_i lie within a ball $B_r(m)$, and find $r^* > r$ satisfying* (5)*. Then Tx_i is well defined by Equ.* (1) *as a minimiser within $B_{r^*}(m)$.*

4. A strategy to prove convergence of Riemannian subdivision schemes

We show a strategy to prove convergence results for the Riemannian analogue T of a linear rule S on the unit sphere. To be able to apply Th. 1 we need bounds on both $\operatorname{dist}(Tx_i, Tx_{i+1})$ and $\operatorname{dist}(x_i, Tx_{2i})$. For those, we join the points involved by curves, and estimate their length. This procedure involves technical details such as the second order Taylor approximation of squared distance functions which have been computed by [Pennec, 2018] and which are summarized by Section 4.1.

4.1. **The Riemannian distance function on the unit sphere and its derivatives.** Recall that the tangent space $T_x\Sigma^n$ of the sphere in a point x equals the orthogonal complement $x^\perp$. The geodesic distance of points x, y is given by $\operatorname{dist}(x, y) = \arccos(\langle x, y\rangle)$. The exponential map at $x \in \Sigma^n$ is given by

$$\exp_x : T_x\Sigma^n \to \Sigma^n \quad w \mapsto \cos\|w\|\, x + \frac{\sin\|w\|}{\|w\|} w.$$

Here $\sin s/s$ means an analytic function which evaluates to 1 for $s = 0$. The expression $\exp_x(w)$ denotes that point on Σ^n which is reached by the geodesic line (great circle) starting in x in direction w, travelling the length of $\|w\|$. By restricting to $\|w\| < \pi$ we make $\exp_x$ one-to-one. The inverse $\exp_x^{-1}(y)$ is well defined except for antipodal points x, y.

We will need first and second derivatives of a real-valued function g in the sphere. While the gradient ∇g is well defined in Riemannian geometry, the Hessian occuring in the 2nd order Taylor expansion of a function $\tilde{g} : \mathbb{R}^n \to \mathbb{R}$,

$$\tilde{g}(x) = \tilde{g}(x^*) + (x - x^*)^T \cdot (\nabla\tilde{g})(x^*) + \frac{1}{2}(x - x^*)^T \cdot (H\tilde{g})(x^*) \cdot (x - x^*) + o(2), \tag{6}$$

does not immediately carry over in a way which is independent of a nonlinear change of coordinates. For this reason, we use the particular coordinate representation $\tilde{g}$ of g defined by $\tilde{g} = g \circ \exp_x$. Since the differential of $\exp_x$ at the contact point x itself is the identity, the gradients of g and $\tilde{g}$ coincide:

$$(\nabla g)(x) = (\nabla\tilde{g})(0). \tag{7}$$

The domain of $\tilde{g}$ is a linear space, so the Hessian $H\tilde{g}$ is well defined. For purposes of this paper, we define the Hessian of g itself by

$$(Hg)(x) := (H\tilde{g})(0). \tag{8}$$

For any fixed $y \in M$ the gradient of the squared distance from y is given by

$$\nabla(\operatorname{dist}(\cdot, y)^2)(x) = -2\exp_x^{-1}(y). \tag{9}$$

With I as the identity matrix, the Hessian of the square of distance of y is expressible as

$$y = \exp_x(\rho v), \|v\| = 1 \implies \big(H\operatorname{dist}(\cdot, y)^2\big)(x) = 2(vv^T + \psi(\rho)(I - xx^T - vv^T)), \tag{10}$$

$$\text{where } \psi(s) = \frac{s}{\tan(s)} \text{ and } \big(H\operatorname{dist}\big(\cdot, x\big)^2\big)(x) = 2(I - xx^T),$$

cf. [Pennec, 2018]. This notation requires an explanation: Firstly, $\psi(s)$ is considered as an analytic function with $\psi(0) = 1$. Secondly, the tangent space $T_x\Sigma^n$ is n-dimensional, but is canonically embedded into $\mathbb{R}^{n+1}$ as the subspace $x^\perp$. We describe its elements unambiguously by vectors $v \in \mathbb{R}^{n+1}$. The Hessian therefore is conveniently described by a matrix of size $(n+1) \times (n+1)$, even if it is only ever applied to elements of the n-dimensional tangent space.

The explicit formula also yields the eigenstructure of H: The null eigenspace orthogonal to the tangent space is not geometrically relevant. The eigenvalue 2 has a 1-dimensional eigenspace spanned by v, and the eigenspace of $2\psi(\rho)$ is the orthogonal complement of v inside the tangent space. The null eigenspace in orthogonal direction occurs only because we describe the Hessian via canonical coordinates in $\mathbb{R}^n$. Considered as the matrix of a symmetric linear mapping of the tangent space, H is invertible whenever the eigenvalue $2\psi(\rho)$ is nonzero.

Returning to the function f_α defined by (4) via data points $x_j = \exp_x(\rho_j v_j)$, $\|v_j\| = 1$, its first and second derivatives are computed by an appropriate linear combination of the above:

$$(\nabla f_\alpha)(x) = -2\sum\nolimits_j \alpha_j \exp_x^{-1}(x_j) = -2\sum\nolimits_j \alpha_j \rho_j v_j, \tag{11}$$

$$(Hf_\alpha)(x) = 2\sum\nolimits_j \alpha_j \big(v_j v_j^T + \psi(\rho_j)(I - xx^T - v_j v_j^T)\big). \tag{12}$$

4.2. **Variable mask and estimating distances.** We now consider weighted averages of data points x_j with variable coefficients $\alpha_j(t)$, $t \in [0,1]$. As t changes, the minimiser traverses the curve

$$\gamma(t) = \arg\min_x f_{\alpha(t)} \tag{13}$$

with length $\int_0^1 \|\dot\gamma(t)\|\, dt$. The idea is to choose coefficient functions such that the initial point $\gamma(0)$ of the minmiser's path is known, and via the size of derivatives we obtain an upper bound for the length of the curve γ, and thus an upper bound for the distance between points $\gamma(0)$, $\gamma(1)$.

The derivative of the minimiser's path can be computed from the condition $(\nabla f_{\alpha(t)})(\gamma(t)) = 0$ by differentiation. With $g(x) = \mathrm{dist}(x, x_j)^2$ we have $f_\alpha = \sum \alpha_j g_j$. We switch to a coordinate representation via the chart $\exp_{\gamma(t)}^{-1}$. Then $\tilde f_\alpha$ and its gradient have the Taylor approximation $\tilde f_\alpha(x) = \tilde f_\alpha(0) + (\nabla \tilde f_\alpha)(0)\cdot x + \frac{1}{2}x^T(H\tilde f_\alpha)(0)x + o(2)$ and $(\nabla \tilde f_\alpha)(x) = (\nabla \tilde f_\alpha(0)) + (H\tilde f_\alpha)(0)\cdot x + o(1)$. This yields

$$\frac{d}{dt}\sum \alpha_j(t)(\nabla g_j)(\gamma(t)) = \sum \dot\alpha_j(t)(\nabla g_j)(\gamma(t))) + \sum \alpha_j (Hg_j)(\gamma(t)) \cdot \dot\gamma(t) = 0$$
$$\Longrightarrow \dot\gamma(t) = -\big(\sum \alpha_j (Hg_j)\big)^{-1}\big(\sum \dot\alpha_j (\nabla g_j)\big) = -(Hf_\alpha)^{-1} \cdot \big(\sum \dot\alpha_j \exp_{\gamma(t)}^{-1} x_j\big). \tag{14}$$

In accordance with (7) and (8), we are dropping the tilde symbol indicating a coordinate representation. Before we describe a more general procedure, we show some first steps by means of an example.

Example 4. (*cubic Lane-Riesenfeld subdivision rule*). Consider the linear subdivision rule

$$(Sx)_{2i} = \frac{1}{8}x_{i-1} + \frac{6}{8}x_i + \frac{1}{8}x_{i+1} \quad \text{and} \quad (Sx)_{2i+1} = \frac{1}{2}x_i + \frac{1}{2}x_{i+1}.$$

Assuming $\sup_i \mathrm{dist}(x_i, x_{i+1}) < r$, Prop. 3 ensures that the Riemannian version T of S is well defined, if $r < \pi/4$. Further, the result of subdivision is close to the original data, since $Tx_{2i}, Tx_{2i+1} \in B_{x_i}(r)$.

Our aim is to estimate the distance of Tx_{2i}, Tx_{2i+1} from the given data more precisely. Since Tx_{2i+1} is the geodesic midpoint of x_i, x_{i+1}, its distance to the input data is bounded by $r/2$. To obtain a similar estimate for Tx_{2i} we define time-dependent weights

$$\alpha_{i-1}(t) = \alpha_{i+1}(t) = \frac{t}{8}, \quad \alpha_i(t) = 1 - \frac{t}{4}. \tag{15}$$

The path $\gamma(t)$ defined as the average of data points w.r.t. weights $\alpha_j(t)$ yields $\gamma(0) = x_i$ and $\gamma(1) = Tx_{2i}$.

We are interested in the speed $\dot\gamma(0)$ and therefore have a look at the Hessian of the function $f_{\alpha(0)}$. Since at $t = 0$ only one weight is nonzero, Equ. (12) reduces to $(Hf_{\alpha(0)})(\dot\gamma(0)) = 2(I - \gamma(0)\gamma(0)^T)$, i.e., within the tangent space, the Hessian is twice the identity. By (14),

$$\dot\gamma(0) = -\frac{1}{8}\big(\exp_{x_i}^{-1}(x_{i-1}) + \exp_{x_i}^{-1}(x_{i+1})\big) \implies \|\dot\gamma(0)\| < \frac{r}{4}.$$

In order to estimate the arc length of the path γ, we need to give a similar bound for all t. This is postponed until some technical lemmas have been shown. ◇

Proposition 5. *Assume that* $\mathrm{dist}(x_j, x_{j+1}) \le r$ *for some* $r > 0$ *and that* $\|\dot\gamma(t)\| \le C_0 r$ *for* $C_0 > 0$ *and all* $t \in [0, a]$, $a \le 1$. *Let* ℓ_j *be upper bounds for* $\mathrm{dist}(x_j, \gamma(0))$. *Then, for all* $t \in [0, a]$,

$$\|\dot\gamma(t)\| \le \frac{2}{|2 - L(t)|}\sum_j |\dot\alpha_j(t)|(rC_0 t + \ell_j), \quad \textit{where } L(t) = \sum_j |\alpha_j(t)|(2 - 2\psi(C_0 r t + \ell_j)). \tag{16}$$

Proposition 6. *This conclusion holds under the milder condition* $\|\dot\gamma(0)\| \le C_0 r$, *if the inequality*

$$\frac{2}{|2 - L(t)|}\sum_j |\dot\alpha_j(t)|(rC_0 t + \ell_j) < C_0 r \quad \textit{for all } t \in [0, a]$$

can be guaranteed.

Proof of Prop. 6. Let t^* be maximal with the property that $\dot\gamma(t) \le C_0 r$ in $[0, t^*]$. The statement of Prop. 5 applies to the interval $[0, t^*]$, showing that $\dot\gamma(t^*) < C_0 r$. If $t^* < a$, this contradicts maximality. □

Proof of Prop. 5. We use the notation $g_j(x) = \mathrm{dist}(x, x_j)^2$. We have $\mathrm{dist}(\gamma(t), x_j) < \frac{\pi}{2}$, since the radius r^* of the ball containing the input data and the minimiser $\gamma(t)$ is smaller than $\frac{\pi}{4}$, see Section 3.2. The eigenvalues $\lambda_{1,j} \ge \lambda_{2,j}$ of the Hessian Hg_j obey $0 < \lambda_{2,j} \le 2$. As to the Hessian of $f_{\alpha(t)}$, observe that

$$\|Hf_{\alpha(t)}\big|_{\gamma(t)}\| = \Big\|\sum_j \alpha_j(t) Hg_j\big|_{\gamma(t)}\Big\| \le 2\sum_j |\alpha_j(t)|,$$

since all eigenvalues $\lambda_{1,j}$ equal 2. In particular, eigenvalues of Hf_α are bounded by $2\sum|\alpha_j(t)|$. Further,

$$\|2I - Hf_\alpha(t)\big|_{\gamma(t)}\| = \Big\|\sum_j \alpha_j(t)\big(2I - Hg_j\big|_{\gamma(t)}\big)\Big\| \le \sum_j |\alpha_j(t)|\Big\|2I - Hg_j\big|_{\gamma(t)}\Big\|.$$

The smaller eigenvalue $\lambda_{2,j}(t) = 2\psi(\mathrm{dist}(\gamma(t), x_j))$ of Hg_j does not exceed 2. From $\|\dot\gamma(t)\| \le C_0 r$ and the fact that ψ is positive and decreasing in $[0, \frac{\pi}{2}]$ we deduce that $\lambda_{2,j}(t) \ge 2\psi(C_0 r t + \ell_j)$. Therefore,

$$\|2I - Hf_{\alpha(t)}\big|_{\gamma(t)}\| \le L(t),$$

i.e., the minimal eigenvalue of $Hf_{\alpha(t)}$ is bounded from below by $|2 - L(t)|$. This implies that

$$\|\big(Hf_{\alpha(t)}\big|_{\gamma(t)}\big)^{-1}\| \le \frac{1}{|2 - L(t)|}.$$

By the assumption $\|\dot\gamma(t)\| \le C_0 r$, the length of the curve $\gamma|_{[0,t]}$ does not exceed rC_0t. Thus

$$\|\exp^{-1}_{\gamma(t)}(x_j)\| = \operatorname{dist}(\gamma(t), x_j) \le \operatorname{dist}(\gamma(t), \gamma(0)) + \operatorname{dist}(\gamma(0), x_j) \le rC_0t + \ell_j,$$

and it follows that

$$\Big\| \sum \dot\alpha_j \nabla g_j\big|_{\gamma(t}\Big\| \le 2\sum_j |\dot\alpha_j(t)|\,(rC_0t + \ell_j)\,.$$

By (14), the speed of the curve γ is bounded by $\|(Hf_\alpha)^{-1}\| \cdot \big\|\sum \dot\alpha_j \nabla g_j\big|_{\gamma(t)}\big\|$, and the statement now follows directly from the inequalities above. □

Example 7. *(Cubic Lane-Riesenfeld rule, part II)* We continue Ex. 4, where we chose weights $\alpha_{i-1}(t) = \alpha_{i+1}(t) = \frac{t}{8}$ and $\alpha_i(t) = 1 - \frac{t}{4}$, resulting in a path $\gamma(t)$ starting in $\gamma(0) = x_i$. Consequently, the bounds ℓ_j needed by Prop. 5 can be chosen as $\ell_{i-1} = r$, $\ell_i = 0$, $\ell_{i+1} = r$. The function $L(t)$ then reads

$$L(t) = \frac{t}{4}(2 - 2\psi(C_0rt + r)) + (1 - \frac{t}{4})(2 - 2\psi(C_0rt)).$$

Choose $r_0 = \frac14$ and $C_0 = 0.53$ (this choice will be justified below). Just for comparision, we are also illustrating a second choice, namely $r_0 = 0.6$ and $C_0 = 0.69$. By Ex. 4,

$$\|\dot\gamma(0)\| < rC_0 \quad \text{for any } r \in (0, r_0), \quad r_0 = \frac14.$$

Since $L(t)$ is strictly increasing in the interval $[0,1]$, and $\psi(s)$ is positive and decreasing on $[0, \frac{\pi}{2}]$, the upper bound for $\|\dot\gamma(t)\|$ provided by Prop. 5 can be estimated as

$$\frac{2}{|2 - L(t)|}\sum_j |\dot\alpha_j(t)|(rC_0t + \ell_j) \le \frac{2}{|2 - L(1)|}\Big(\frac14 r + \frac12 rC_0t\Big)$$
$$= \frac{2}{\frac12\psi(C_0r + r) + \frac32\psi(C_0r)}\Big(\frac{r}{4} + \frac{rC_0}{2}\Big) \le \frac{2}{\frac12\psi(C_0r_0 + r_0) + \frac32\psi(C_0r_0)}\Big(\frac{r}{4} + \frac{rC_0}{2}\Big) \le 0.524\,r$$

(the value of the denominator is 1.966...). The right hand side is just a bit smaller than C_0r, and Prop. 6 applies. The second choice of parameters yields a right hand side of $0.689\,r$, with 1.728911... in the denominator. In both cases Prop. 6 applies, but the auxiliary parameter C_0 in both cases has been chosen so that we come close to failing the assumptions of Prop. 6. Applying Prop. 6, we compute

$$\operatorname{dist}(x_i, Tx_{2i}) \le \int_0^1 \|\dot\gamma(t)\|\,dt \lessapprox \int_0^1 \frac{2}{1.966\ldots}\Big(\frac14 + t\frac{C_0}{2}\Big) r\,dt \le 0.39r.$$

This inequality amounts to the displacement-safe condition needed by Th. 1. In order to apply Th. 1 and show convergence, we also need T to be contractive:

$$\operatorname{dist}(Tx_{2i}, Tx_{2i+1}) \le \operatorname{dist}(Tx_{2i}, x_i) + \operatorname{dist}(x_i, Tx_{2i+1}) \le 0.39r + \frac{r}{2} = 0.89r,$$

which by symmetry is also an upper bound for $\operatorname{dist}(Tx_{2i}, Tx_{2i-1})$.

The second choice of parameters yields a right hand side of $0.49r$ in the displacement-safe condition and $0.99r$ in the contractivity condition. If r_0 is any higher, we can no longer show contractivity in this way. The true contractivity factor enjoyed by the Riemannian Lane-Riesenfeld scheme is closer to 1/2 than these bounds suggest. This is because we do not measure $\operatorname{dist}(Tx_{xi}, Tx_{2x+i})$ directly, but we go via a broken path consisting of the curve γ and a geodesic. We summarize: *Let $(x_i)_{i\in\mathbb{Z}}$ be a sequence of points on the unit sphere. If $\sup_\ell \operatorname{dist}(x_\ell, x_{\ell+1}) < 0.6 \approx 0.19\pi$, then the Riemannian analogue of the linear cubic Lane-Riesenfeld scheme converges to a continuous limit function.* The limit is even C^2 smooth, as shown by Grohs [2010]. ♢

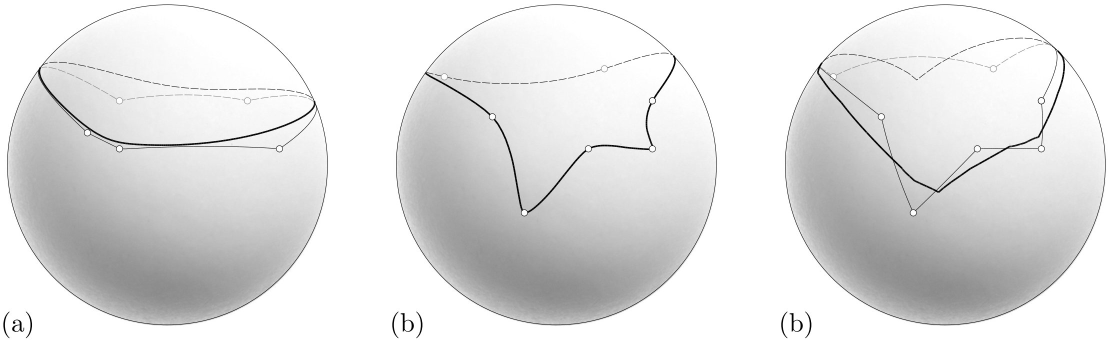
(a) (b) (b)

FIGURE 1. Initial polygon and limit curve of (a) the cubic Lane- Riesenfeld scheme on the unit sphere; (b) the 4-point scheme; (c) the Riemannian subdivision rule (19).

4.3. **The Strategy.** In a more general situation, the strategy to show convergence of a Riemannian subdivision rule T can be summarized as follows.

- Use Prop. 3 to give a bound r such that T is well defined whenever the distance of successive input data points x_i does not exceed r.
- By linear interpolation of weights, describe a path γ connecting $Tx_{2i} = \gamma(1)$ with a point $\gamma(0)$ whose location relative to the input data is well known, e.g. $\gamma(0) = x_i$, or $\gamma(0)$ located on the geodesic segment joining x_i, x_{i+1}. Employ Propositions 5 and 6 to estimate $\operatorname{dist}(\gamma(0), Tx_{2i})$.
- If necessary, provide similar distance estimates for neighbours Tx_{2i+1}, Tx_{2i-1}.
- Using the triangle inequality, estimate $\operatorname{dist}(Tx_{2i}, x_i)$, establishing the so-called displacement-safe condition. Again using the triangle inequality, estimate distances of successive points in the sequence Tx_j, showing that indeed $\sup_j \operatorname{dist}(Tx_j, Tx_{j+1}) \leq Cr$, where $C < 1$.

Applying this method, one must choose r, the constant C_0 referred to by Prop. 5, and the initial point $\gamma(0)$ of the path. There is no general rule how these choices should be made, but we show two more examples which illustrate the strategy. The next example is an interpolatory scheme where $\gamma(0)$ is the geodesic midpoint of successive input data points. The last example is that of a subdivision rule which does not have any special properties, and where $\gamma(0)$, $\delta(0)$ are chosen on the geodesic segments connecting successive input data points.

4.4. **Example: 4-point scheme.** We consider the well-known interpolatory 4-point scheme defined by

$$(Sx)_{2i} = x_i \quad \text{and} \quad (Sx)_{2i+1} = -\frac{1}{16}x_{i-1} + \frac{9}{16}x_i + \frac{9}{16}x_{i+1} - \frac{1}{16}x_{i+2}, \tag{17}$$

see [Dyn et al., 1987]. Its Riemannian analogue is defined by $Tx_{2i} = x_{2i}$ and

$$Tx_{2i+1} = \operatorname*{arg\,min}_{x\in\Sigma^n} \Big(-\frac{1}{16}\operatorname{dist}(x, x_{i-1})^2 + \frac{9}{16}\operatorname{dist}(x, x_i)^2 + \frac{9}{16}\operatorname{dist}(x, x_{i+1})^2 - \frac{1}{16}\operatorname{dist}(x, x_{i+2})^2\Big).$$

If the distance of successive data points is bounded by r, all input data points which contribute to the computation of Tx_{2i+1} are in a geodesic ball of radius $3r/2$ centered in the geodesic midpoint of x_i, x_{i+1}. With $\alpha_- = \frac{1}{8}$, Proposition 3 resp. Equ. (5) allows us to compute the maximal value $0.317\ldots$ of r such that there is a unique minimiser Tx_{2i+1}. We choose $r_0 = 0.31$ and assume $r < r_0$. Then T is well defined.

In order to check how much Tx_{2i+1} deviates from the input data, consider a path $\gamma(t)$ defined by minimizing $f_{\alpha(t)}$, where the variable coefficients $\alpha(t)$ are given as

$$\alpha_{i-1}(t) = \alpha_{i+2}(t) = -\frac{t}{16}, \quad \text{and} \quad \alpha_i(t) = \alpha_{i+1}(t) = \frac{1}{2} + \frac{t}{16} \quad (0 \le t \le 1). \tag{18}$$

This path connects $\gamma(1) = Tx_{2i+1}$ with $\gamma(0)$, which is the geodesic midpoint of x_i, x_{i+1}. The bounds ℓ_j needed by Prop. 5 are $\ell_{i-1} = \ell_{i+2} = \frac{3}{2}r$, $\ell_i = \ell_{i+1} = \frac{1}{2}r$. Without choosing C_0 yet, we write down the auxiliary function $L(t)$ as

$$L(t) = \frac{2t}{16}\Big(2 - 2\psi(C_0rt + \frac{3}{2}r)\Big) + 2\Big(\frac{1}{2} + \frac{t}{16}\Big)\Big(2 - 2\psi(C_0rt + \frac{1}{2}r)\Big).$$

We observe that $L(0) = 2 - r/\tan(\frac{1}{2}r)$ and that $L(0)$ is a positive and increasing function of r. Thus $L(0)$ is bounded by $2 - r_0/\tan(\frac{1}{2}r_0) \le 1.66 \cdot 10^{-2}$. Prop. 5 then yields

$$\|\dot\gamma(0)\| \le \frac{2}{|2 - L(0)|}\sum_j |\dot\alpha_j(0)|\ell_j \le \frac{2}{|2 - 1.66\cdot 10^{-2}|}\frac{r}{4} \le 0.26\, r.$$

Using the fact that $L(t)$ increases with t, we estimate the bound for $\|\dot\gamma(t)\|$ provided by Prop. 5:

$$\frac{2}{|2 - L(t)|}\sum |\dot\alpha_j(t)|(rC_0t + \ell_j) \le \frac{2}{|2 - L(1)|}\Big(\frac{1}{4}rC_0t + \frac{1}{4}r\Big).$$

It is easy to choose C_0 such that this value does not exceed C_0, say $C_0 = 0.45$. Then Prop. 6 applies, and

$$\operatorname{dist}(Tx_{2i+1}, \gamma(0)) = \int_0^1 \|\dot\gamma(t)\| \le \frac{2}{|2 - L(1)|}\Big(\frac{1}{8}rC_0 + \frac{1}{4}r\Big) \le 0.31r.$$

Contractivity of T is expressed as $\operatorname{dist}(Tx_j, Tx_{j+1}) \le Cr$, for some constant $C < 1$. Because of symmetry it is sufficient to check

$$\operatorname{dist}(Tx_{2i}, Tx_{2i+1}) \le \operatorname{dist}(Tx_i, \gamma(0)) + \operatorname{dist}(\gamma(x), Tx_{2i+1}) \le 0.81r.$$

Since the 4-point rule is interpolatory with $x_i = Tx_{2i}$, the displacement-safe condition is automatic. We have shown: *For input data x_i in the unit sphere, where the distance of successive points does not exceed* 0.31*, the Riemannian analogue of the linear* 4*-point scheme converges to a continuous limit function.* The limit is even C^1 smooth [Grohs, 2009].

4.5. **Example: A scheme with negative coefficients.** We consider the linear subdivision rule defined by the mask $(a_i)_{-4<i<5} = \frac{1}{32}(-1, -1, 13, 21, 21, 13, -1, -1)$. Its Riemannian analogue T is defined by

$$Tx_{2i} = \operatorname*{arg\,min}_{x\in\Sigma^n} \frac{1}{32}\Big(-\operatorname{dist}(x, x_{i-1})^2 + 21\operatorname{dist}(x, x_i)^2 + 13\operatorname{dist}(x, x_{i+1})^2 - \operatorname{dist}(x, x_{i+2})^2\Big) \tag{19}$$

and similar for Tx_{2i+1}. With $\alpha_- = \frac{1}{16}$, Proposition 3 implies that T is well defined if $\sup_j \operatorname{dist}(x_j, x_{j+1}) < r$, for any $r \le r_0$, with $r_0 = 0.4$.

We use a path $\gamma(t)$ to connect $Tx_{2i} = \gamma(1)$ with a weighted geodesic average of x_i, x_{i+1}, located on the geodesic segment connecting x_i and x_{i+1}, and dissecting it according to the ratio $(1-\beta) : \beta$.

At the present time we do not know which value of β works out, but in order to shorten the presentation we reveal in advance that $\beta = 0.65$ will. We compute $\gamma(t)$ as the minimiser of $f_{\alpha(t)}$, with

$$\alpha_{i-1}(t) = \alpha_{i+2}(t) = -\frac{t}{32}, \quad \alpha_i(t) = (1-t)0.65 + t\frac{21}{32}, \quad \alpha_{i+1}(t) = (1-t)0.35 + t\frac{13}{32}.$$

The distances of input data from $\gamma(0)$ are then bounded by $\ell_{i-1} = 1.35\, r$, $\ell_i = 0.35\, r$, $\ell_{i+1} = 0.65\, r$, $\ell_{i+2} = 1.65\, r$. The auxiliary function $L(t)$ needed by Prop. 5 reads $L(t) = \frac{t}{32}(2 - 2\psi(C_0rt +$

$1.35r)) + \cdots$. It depends on the choice of r in a monotonically increasing way, so by evaluating it for $r = r_0$ we observe that $L(0) \leq 0.02$. We use this information to estimate

$$\|\dot\gamma(0)\| \leq \frac{2}{|2 - L(0)|} \sum |\dot\alpha_j(0)\|\ell_j \leq \frac{2 \cdot 0.1325\, r}{|2 - 0.02|} \leq 0.14\, r.$$

Since $L(t) \leq L(1)$, the upper bound for $\|\dot\gamma(t)\|$ provided by Prop. 5 can be estimated by

$$\begin{aligned}\frac{2}{|2 - L(t)|} \sum |\dot\alpha_j(t)|(rC_0 t + \ell_j) &\leq \frac{2}{|2 - L(1)|} \Big(\frac{1}{32}(rC_0 t + 1.35r) + \cdots \Big) \\ &= \frac{2}{|2 - L(1)|} \Big(\frac{1}{8} rC_0 t + \frac{53}{400} r \Big)\end{aligned}$$

for all $t \in [0, 1]$. It is not difficult to pick a value C_0 such that this expression is less than C_0; let us choose $C_0 = 0.16$. Invoking Propositions 5 and 6 we see that indeed the previous bound applies to $\|\dot\gamma(t)\|$, so

$$\operatorname{dist}(\gamma(0), Tx_{2i}) \leq \int_0^1 \|\dot\gamma(t)\|\, dt \leq \frac{2}{|2 - L(1)|} \Big(\frac{1}{16} rC_0 + \frac{53}{400} r \Big) \leq 0.1425\, r.$$

The displacement-safe condition now follows from

$$\operatorname{dist}(x_i, Tx_{2i}) \leq \operatorname{dist}(x_i, \gamma(0)) + \operatorname{dist}(\gamma(0), Tx_{2i}) \leq (0.35 + 0.1425)r = 0.4925\, r.$$

Similarly we use a path $\delta^{(i)}(t)$ connecting Tx_{2i+1} with a weighted geodesic average of x_i, x_{i+1} where this time the weights are 0.35 and 0.65 instead of the previous choice of 0.65 and 0.35. We get the same estimate for the length of those paths. Thus,

$$\begin{aligned}\operatorname{dist}(Tx_{2i}, Tx_{2i+1}) &\leq \operatorname{dist}(Tx_{2i}, \gamma(0)) + \operatorname{dist}(\gamma(0), \delta^{(i)}(0)) + \operatorname{dist}(\delta^{(i)}(0), Tx_{2i+1}) \\ &\leq (2 \cdot 0.1425 + 0.30)r = 0.585\, r, \\ \operatorname{dist}(Tx_{2i-1}, Tx_{2i}) &\leq \operatorname{dist}(Tx_{2i-1}, \delta^{(i-1)}(0)) + \operatorname{dist}(\delta^{(i-1)}(0), x_i) + \operatorname{dist}(x_i, Tx_{2i}) \\ &\leq (0.1425 + 0.35 + 0.4925)r = 0.985\, r.\end{aligned}$$

This shows that T is contractive and concludes the proof that the Riemannian subdivision rule T produces a continuous limit function for all data with $\sup_i \operatorname{dist}(x_i, x_{i+1}) < 0.4 \approx 0.13\pi$.

Concluding Remarks. This paper introduces a procedure to treat convergence of Riemannian subdivison algorithms in spheres. An obvious question in this context is how to analyze more general Riemannian manifolds. Our method generalizes to all situations where the inverse of the Hessian of square-of-distance functions can be estimated. This topic is discussed by Pennec [2017], who goes on to study spherical and hyperbolic geometry as concrete examples. The squared distance is computed by means of the Riemannian log, whose Taylor expansion differs from the Euclidean situation in its terms of degree ≥ 2 (they feature the Riemannian curvature tensor). A differential-geometric tour de force would probably allow us to show results like Prop. 5, only with a modified function ψ, and with statements becoming weaker as curvatures are varying. Further topics of future research include the multivariate case, and alternative definitions of 'average' adapted to special manifolds.

Acknowledgments. The authors acknowledge the support of the Austrian Science Fund (FWF) under grant no. W1230. They also want to thank the reviewers for their comments which improved this article. The results contained in this paper are part of the PhD thesis of the first author.

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