EXISTENCE OF SET-INTERPOLATING AND ENERGY-MINIMIZING CURVES

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Abstract. We consider existence of curves $c : [0,1] \to \mathbb{R}^n$ which minimize an energy of the form $\int \|c^{(k)}\|^p$ $(k = 1, 2, \dots,\ 1 < p < \infty)$ under side-conditions of the form $G_j(c(t_{1,j}), \dots, c^{(k-1)}(t_{k,j})) \in M_j$, where G_j is a continuous function, $t_{i,j} \in [0,1]$, M_j is some closed set, and the indices j range in some index set J. This includes the problem of finding energy minimizing interpolants restricted to surfaces, and also variational near-interpolating problems. The norm used for vectors does not have to be Euclidean.

It is shown that such an energy minimizer exists if there exists a curve satisfying the side conditions at all, and if among the interpolation conditions there are at least k points to be interpolated. In the case $k = 1$, some relations to arc length are shown.

1. Introduction

Finding curves $c : I \to \mathbb{R}^n$ which minimize the energy functional

$$E(c) = \textstyle\int_I \|c^{(k)}(t)\|^p dt \quad (1 \le p < \infty,\ k = 1, 2, \dots) \tag{1}$$

under certain side conditions has been of interest in Computer-Aided Geometric Design for a long time, especially in the case $k = p = 2$. In this paper we show an existence result for the case $p > 1$ and general set-interpolation side conditions which may involve derivatives. The norm used in $\mathbb{R}^n$ does not have to be Euclidean. For the case $k = 1$ we show some additional properties of energy minimizers and consider the case $p = 1$ also. The simplest case of such interpolation conditions is that for each $t \in I$ there is a closed set $M(t) \subset \mathbb{R}^n$ such that

$$c(t) \in M(t). \tag{2}$$

In particular, the condition that the curve we are looking for is to be contained in a surface "M", is expressed by the requirement that $M(t) = M$ for all t.

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More generally, we assume that there is an indexed family of side conditions of the form

$$G_j(c(t_{0,j}), \ldots, c^{(k-1)}(t_{k-1,j})) \in M_j, \quad (j \in J) \tag{3}$$

where J is some index set, $t_{i,j} \in I$, M_j is a closed subset of some space $\mathbb{R}^{n_j}$, and G_j is a continuous function from $(\mathbb{R}^n)^k$ to $\mathbb{R}^{n_j}$. Equ. (3) can be used to express Hermite interpolation conditions: For instance, the equation $\dot{c}(t_0) = v$, for a given parameter value t_0 and a vector v has the required form, as the left hand side "G_j" is a continous function involving the derivatives of the curve, and we may let $M_j := \{v\}$. Hermite Near-interpolation conditions can be formulated by using sets M_j which contain more than 1 point.

The requirement that G_j is continuous for all j is necessary. An example of a non-continuous geometrically meaningful function is the curvature

$$G(c(t_0), \dot{c}(t_0), \ddot{c}(t_0)) = \frac{\Big(\|\dot{c}(t_0)\|^2 \|\ddot{c}(t_0)\|^2 - (\dot{c}(t_0) \cdot \ddot{c}(t_0))^2\Big)^{1/2}}{\|\dot{c}(t_0)\|^3}.$$

In this formula $\|x\|$ denotes the Euclidean norm of a vector x. In the case $k > 2$, a constraint involving curvature could nevertheless be used if the case $\|\dot{c}(t_0)\| = 0$ is excluded, e.g. by additional constraints of the form $\|\dot{c}(t_0)\| \in [\varepsilon, \infty)$, for some $\epsilon > 0$.

2. Previous Work and Applications

We briefly describe the special cases of this problem which we have in mind and also previous work on that topic.

2.1. **Near-Interpolation.** A well-known instance of the problem of minimizing energy under side conditions as described above is that $M(t) = \mathbb{R}^n$ for almost all t, but there is a finite number of values $t_1, \ldots, t_r$ such hat $M(t_j)$ is a proper subset of $\mathbb{R}^n$. Those subsets may be seen as tolerance zones of points to be interpolated. This is the topic of the papers [1, 2, 3, 4, 5], and [6]. Without going into details we mention that the more difficult problem of subjecting the parameter values $t_1, \ldots, t_r$ to optimization, with the side condition that $t_1 \le t_2 \le \cdots \le t_r$, has been solved in [2] and [3].

2.2. **Variational Interpolation and set-interpolation in a surface.** Assume that M is a closed surface, possibly with boundary. A typical instance of the variational interpolation problem would be that $M(t) = M$ for almost all t, and $M(t)$ is a proper subset of M for finitely many t. If $\mathbb{R}^n$ is equipped with the standard Euclidean norm, then in the case $k = 1$ minimizing energy means finding the shortest path which fulfills the side conditions. This relation of energy minimizers to arc length is discussed below.

For $k = p = 2$, minimizing energy means finding smooth interpolants and near-interpolants on surfaces which minimize the linearized thin beam bending energy. This problem has been considered e.g. in [7] and [8]. Side conditions involving derivatives can be included if they have the form (3).

The main contribution concerning existence of minimizers is contained in the thesis [9], which considers energy functionals which involve derivatives up to some finite order k, and curves whose arclength parametrization lies in a certain Sobolev space. The author looks for set-interpolating minimizers among curves which are contained in a parametric and compact surface in $\mathbb{R}^3$.

The present paper uses an approach which is not restricted to parametric or compact surfaces, and we use slightly different function spaces. The side conditions which may be imposed on curves so that the existence of a curve with minimal energy can be shown are more general on the one hand (the restriction to finitely many side conditions in [9] is not necessary), but more restrictive on the other hand, as we require that k points are to be interpolated, not 2.

2.3. **Applications and computation of energy-minimizing curves in surfaces.** Applications of the concept of an energy-minimizing curve (or "spline curve") in a surface are given e.g. in [8]. Besides the obvious possibility of variational interpolation and near-interpolation, this includes variational motion design (where the Euclidean motion group serves as a surface where curves have to be contained in) and variational interpolation in the presence of obstacles (via barrier manifolds).

The actual numerical computation of energy-minimizing curves works with a discretization, where a curve is replaced by a sequence of points, and k-th derivatives by appropriate k-th order differences of that sequence. If we use the standard Euclidean norm for vectors in $\mathbb{R}^n$, and let $p = 2$, the discretized energy function becomes a quadratic function which takes point sequences as arguments. This is also described in the paper mentioned above.

3. Overview of the existence proof

The method of proof is the following: We consider curves which have a k-th derivative almost everywhere, so that the definition of energy is meaningful. We consider a suitable space of curves (i.e., vector-valued functions defined in an interval) and define a norm for such functions whose p-th power actually equals the energy. Thus finding curves of minimal energy is equivalent to finding functions of minimal norm. It turns out that this space of curves is isomorphic to a well known L^p space. The collection of side conditions defines a certain subset "Z" of admissible curves.

We now have to show the existence of an element of minimal norm in Z. If Z were contained in a finite-dimensional space, closedness of Z would be sufficient for existence, because in finite dimension unit balls are compact. There is a well-known procedure in functional analysis which allows to generalize such an argument to reflexive Banach spaces: There it is sufficient that Z is closed with respect to the weak topology. This is verified below. A standard argument now shows that Z indeed has an element of minimal norm (Th. 1).

In [9] a compactness argument in the domain of a parametrized surface is used in order to show existence.

4. Definitions and preliminary results

We use the symbols $C^k(X)$, $AC(X)$, $L^p(X)$ for spaces of real-valued functions defined in X. $f \in L^p(X)$ means that f is defined almost everywhere and that $\|f\|_p := (\int_X |f|^p)^{1/p} < \infty$. $C^k(X)$ denotes the space of k-times continuously differentiable functions. $f \in AC(I)$, where I is an interval, means that f is absolutely continuous, so that the first derivative f' exists almost everywhere ("[a.e.]"), $f' \in L^1$, and the fundamental theorem of calculus is valid, i.e., $f(b) - f(a) = \int_{[a,b]} f'$. We further consider the space $L^p(X, \mathbb{R}^n)$ of vector valued L^p functions $f(t) = (f_1(t), \dots, f_n(t))$, where each component f_i is an L^p function. For facts about these function spaces, the reader is referred to [10] and [11].

We let $I = [0,1]$ henceforth and note that by Hölder's inequality, $\|f\|_1 \le \|f\|_p$ for all $f \in L^1(I)$. Thus the following definition makes sense:

Def. 1. *For any strictly increasing sequence $T = (t_1, \dots, t_k)$ of real numbers and for any $p > 1$ we consider*

$$V_T = \{f \in C^{k-1}(I) \mid f^{(k-1)} \in AC, f^{(k)} \in L^p, \quad f(t_1) = \dots = f(t_k) = 0\}, \tag{4}$$

which is endowed with the norm $\|f\|_V := \|f^{(k)}\|_p$.

Lemma 1. *The mapping $D : V_T \to L^p(I)$, $f \mapsto f^{(k)}$ is a norm isomorphism.*

Proof: We construct an inverse D^{-1}. Choose $g \in L^p$. As $L^p(I) \subset L^1(I)$, we may let

$$h_{k-1}(t) = \int_{[0,t]} g, \;\; h_{k-2}(t) = \int_{[0,t]} h_{k-1}, \;\; \dots, \;\; h_0(t) = \int_{[0,t]} h_1. \tag{5}$$

If $Df = g$, then necessarily $D^{-1}g(t) = h_0(t) + \sum_{i=0}^{k-1} C_i t^i$. The coefficients C_i can be found from the condition $f(t_1) = \cdots = f(t_k) = 0$:

$$C_0 + t_i C_1 + \cdots t_i^{k-1} C_{k-1} = -h_0(t_i), \quad (i = 1, \ldots, k). \tag{6}$$

The matrix A of this system of linear equations is regular, as it consists of the powers t_i^j ($i = 1, \ldots, k$, $j = 0, \ldots, k-1$). It follows that $D^{-1}g$ is uniquely defined, and therefore D is an isomorphism. □

A vector-valued L^p function $f : [0,1] \to \mathbb{R}^n$ may be identified with a real-valued L^p function $g : [0,n] \to \mathbb{R}$, if we define

$$f_i(t) = g|_{[i-1,i]}(t - i + 1). \tag{7}$$

This isomorphism of $L^p(I, \mathbb{R}^n) \cong L^p([0,n])$ introduces a norm, which is denoted by $\|\cdot\|'$:

$$\|f\|' := \sum_j \|f_j\|_p. \tag{8}$$

We fix a norm "$\|x\|$" in $\mathbb{R}^n$ and define a second norm for vector-valued L^p functions by

$$\|f\| := (\textstyle\int \|f(t)\|^p dt)^{1/p}. \tag{9}$$

Lemma 2. *The norms $\|\cdot\|$ and $\|\cdot\|'$ in $L^p(I, \mathbb{R}^n)$ are equivalent, i.e., there are constants α, β such that for all f, $\alpha\|f\| \le \|f\|' \le \beta\|f\|$.*

The proof is given in the appendix.

Def. 2. *Consider the linear space*

$$W_T = \{f : I \to \mathbb{R}^n \mid f_i \in V_T\}. \tag{10}$$

Norms are defined by $\|f\|_W = \|f^{(k)}\|_{L^p(I,\mathbb{R}^n)}$, $\quad \|f\|'_W = \|f^{(k)}\|'_{L^p(I,\mathbb{R}^n)}$.

The following is a direct consequence of Lemma 1:

Lemma 3. *The mapping $f \mapsto f^{(k)}$ is a norm isomorphism of the spaces W_T and $L^p(I, \mathbb{R}^n)$, both with respect to the norms $\|\cdot\|$, $\|\cdot\|_W$ and the norms $\|\cdot\|'$, $\|\cdot\|'_W$.*

5. Closedness and Weak Closedness of Side Conditions

Lemma 4. *The evaluation mappings*

$$v_{t,j} : W_T \to \mathbb{R}^n, \quad f \mapsto f^{(j)}(t) \quad (0 \le j < k) \tag{11}$$

are bounded (i.e., continuous with respect to the norm topology in W_T).

Proof: We are going to show the existence of constants γ_j such that

$$\|v_{t,j}(f)\| \leq \gamma_j \|f\|'_W. \tag{12}$$

We use the 1-norm in $\mathbb{R}^n$, but this specific choice is not relevant for continuity of $v_{t,j}$. We first consider the case that $n = 1$, i.e, $W_T = V_T$. Recall that $\|f\|_1 \leq \|f\|_p$. We compute upper bounds for the functions h_j defined by (5):

$$\begin{aligned} |h_{k-1}(t)| &= |\textstyle\int_{[0,t]} f^{(k)}| \leq \int_{[0,t]} |f^{(k)}| \leq \|f^{(k)}\|_1 \leq \|f^{(k)}\|_p, \\ |h_{k-2}(t)| &= |\textstyle\int_{[0,t]} h_{k-1}| \leq \int_{[0,t]} |h_{k-1}| \leq t\|f^{(k)}\|_p \leq \|f^{(k)}\|_p, \ \ldots \\ |h_0(t)| &\leq \|f^{(k)}\|_p. \end{aligned}$$

We consider the linear system of equations (6), whose matrix is denoted by A. There is $\sigma > 0$ such that $\|A^{-1}x\|_\infty \leq \sigma \|x\|_\infty$ for all $x \in \mathbb{R}^n$, so

$$|C_i| \leq \sigma \|h_0\|_\infty \leq \sigma \|f^{(k)}\|_p \quad i = 0, \ldots, k-1. \tag{13}$$

It follows that

$$\begin{aligned} |f^{(j)}(t)| &= |\frac{d^j}{dt^j}(h_0(t) + C_0 + \cdots + C_{k-1}t^{k-1})| \\ &\leq |h_j(t)| + j!\,|C_j| + \cdots + \tfrac{(k-1)!}{(k-1-j)!}\,|C_{k-1}|\,t^{k-1-j} \\ &\leq \gamma_j \|f^{(k)}\|_p, \quad \text{with } \gamma_j = 1 + \sigma \textstyle\sum_{i=j}^{k-1} \frac{i!}{(i-j)!}. \end{aligned} \tag{14}$$

for all $t \in I$ and $0 \leq j < k$. By definition of the norm in W_T, this shows (12) in the case $n = 1$. As to the case $n > 1$, we consider the component functions $f_1, \ldots, f_n$ of f and argue as follows:

$$\|v_{t,j}(f)\|_1 = \textstyle\sum_i |f_i^{(j)}(t)| \leq \gamma_j \sum_i \|f_i^{(k)}\|_p = \gamma_j \|f\|'_W. \tag{15}$$

This shows (12) in the general case. □

Lemma 5. *$v_{t,j}$ is weakly continuous in W_T.*

Proof: Continuity of $v_{t,j}$ means that all functionals $f \mapsto f_i^{(j)}(t)$ are continuous, and vice versa; regardless of the topology we choose in W_T. For linear functionals, however, norm-continuity is the same as weak continuity. Thus the statement follows immediately from the previous lemma. □

Lemma 6. *Suppose that for each index j in some index set J there is a continuous function $H_j : \mathbb{R}^{nk} \to \mathbb{R}^{n_j}$, parameter values $t_{0,j}, \ldots, t_{k-1,j}$, and a closed set $N_j \subseteq \mathbb{R}^{n_j}$. Then*

$$Z := \{f \in W_T \mid H_j(c(t_{0,j}), \ldots, c^{(k-1)}(t_{k-1,j})) \in N_j \ \forall\ j \in J\} \tag{16}$$

is closed with respect to both norm and weak topology in W_T.

Proof: The collection $\phi_j = (v_{t_{0,j},0} \times \cdots \times v_{t_{k-1,j},k-1}) : W_T \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ of evaluation mappings is continuous with respect to both the norm and the weak topologies in W_T, as each component is. We can write Z in the form $Z = \bigcap_{j \in J} (H_j \circ \phi_j)^{-1}(N_j)$, so obviously Z is closed. □

6. Existence of curves which respect the side conditions

We are going to show some cases where there always are curves which satisfy the side conditions. The following is well known:

Lemma 7. *For all piecewise C^k curves $c : I \to \mathbb{R}^n$ there is a nondecreasing parameter transform $\gamma : I \to I$ such that $d := c \circ \gamma$ is C^k. We can choose γ such that $c = d$ in any finite union of disjoint closed intervals where c is C^k.*

For the convenience of the reader, a proof is indicated in the appendix.

Def. 3. *Assume that M is a subset of $\mathbb{R}^n$. A finite collection of side conditions, involving values and derivatives of curves at the parameter values $t_1, \ldots, t_r$, is called locally admissible for M, if there is a curve c, defined only locally in a neighbourhood of each t_i and taking values in M, such that the side conditions are fulfilled.*

An important and trivial example of a locally admissible collection of side conditions are interpolation conditions of the form $c(t_j) \in M_j$, for parameter values $t_1 < t_2 < \cdots < t_r$ and $M_j \subseteq M$. Curves which fulfill those interpolation conditions may be defined as locally constant curves.

Lemma 8. *If M has the property that any two points $p, q \in M$ may be connected by a piecewise C^k curve, then for any finite collection of locally admissible boundary conditions there is a C^k curve $c : I \to M$ which is defined in entire I and which fulfills those side conditions.*

Proof: Assume that $t_1 < t_2 < \cdots < t_r$ is the list of parameter values involved in the side conditions, and choose a curve $\widetilde{c}$, defined only locally in disjoint intervals $[t_i - \epsilon, t_i + \epsilon]$ which fulfills those side conditions. Now connect the points $\widetilde{c}(t_i + \epsilon/2)$ and $\widetilde{c}(t_{i+1} - \epsilon/2)$ by C^k curves. By pasting those curves and the curve $\widetilde{c}$ together, we get a piecewise C^k curve $\bar{c}$ which is defined in the entire interval $[0, 1]$, and which fulfills the side conditions. We apply Lemma 7 to $\bar{c}$ and the intervals $[t_i - \epsilon, t_i + \epsilon]$, and get a C^k curve which fulfills the side conditions. □

Lemma 8 applies directly to interpolation of points in surfaces, and interpolation of subsets of $\mathbb{R}^n$. The property of M referred to in Lemma 8 is fulfilled if M is a surface, possibly with boundary, or any connected finite union of curvilinear polyhedra.

7. Existence of energy-minimizing curves

We consider the set $\mathcal{V}^{k,p}$ of curves whose k-th derivatives are L^p functions:

$$\mathcal{V}^{k,p} := \{c : I \to \mathbb{R}^n \mid c \in C^{k-1},\ c^{(k-1)} \in AC,\ c^{(k)} \in L^p\}. \tag{17}$$

Theorem 1. *Assume that an integer $k > 0$, values $t_1 < \cdots < t_k \in [0,1]$, and points $M(t_1), \ldots, M(t_k) \in \mathbb{R}^n$ are given. Assume that an arbitrary collection of further interpolation conditions of the form (3) (which includes those of the form (2)) is given. If there is a curve $c \in \mathcal{V}^{k,p}$ which fulfills the interpolation conditions at all, there is also one where the energy (1) is minimal.*

Proof: We consider the polynomial function $p : I \to \mathbb{R}^n$ of degree $k-1$ with the property $p(t_i) = M(t_i)$. Subtracting p furnishes a bijection between the set of curves in $\mathcal{V}^{k,p}$ which interpolate the points $M(t_j)$, and the set of functions in W_T. With the definition of energy in (1), we have

$$E(c) = \|c^{(k)}\|_{L^p(I,\mathbb{R}^n)} = \|(c-p)^{(k)}\|_{L^p(I,\mathbb{R}^n)} = \|c-p\|_W. \tag{18}$$

Minimizing energy therefore is equivalent to finding f such that

$$\begin{aligned} & f \in W_T,\ G_j(c(t_{0,j}), \ldots, c^{k-1}(t_{k-1,j})) \in M_j \quad \text{for all } j \in J, \\ & c = f + p,\ \|f\|_W \to \min. \end{aligned} \tag{19}$$

If we define $H_j(x_0, \ldots, x_{k-1}) = G_j((x_0, \ldots, x_{k-1}) + (p(t_{0,j}), \ldots, p^{(k-1)}(t_{k-1,j})))$, and choose Z as in (16), then (19) is equivalent to $f \in Z,\ \|f\|_W \to \min$. Let

$$\beta = \inf\{\|f\|_W \mid f \in Z\} \tag{20}$$

and consider a sequence $f_1, f_2, \ldots$ with $f_i \in Z$ such that $\|f_i\|_W \to \beta$.

$L^p(I,\mathbb{R}^n) \cong L^p([0,n])$ is a reflexive Banach space, so there is a weakly convergent subsequence $f_{i_k} \rightharpoonup f$. $f \in Z$, because Z is weakly closed. As a weak limit, f has the property that $\beta' := \|f\|_W \le \underline{\lim}_{k\to\infty} \|f_{i_k}\|_W$, so $\beta' \le \beta$. The definition of β however shows that $\beta \le \beta'$. Thus we have $\beta = \beta' = \|f\|_W$, and f minimizes the norm $\|\cdot\|_W$ in Z. This is equivalent to $f + p$ being a minimal energy interpolant. □

8. The case $k = 1$

Here we show some relations of energy minimizers with curves which minimize arc length, which are well known in the smooth category.

Def. 4. *The arc length $L(c)$ of a curve is defined by letting $k = 1$ in (1), i.e., $L(c) := \int \|\dot c\|$. We call $\gamma_c(u) = \frac{1}{L(c)} L(c|_{[0,u]})$ the arc length function of c.*

This definition of $L(c)$ is not restricted to the Euclidean norm in $\mathbb{R}^n$. The arc length function maps I to I.

Def. 5. *For any set M and points $u, v \in M$ we use the abbreviation $\mathcal{C}_{u,v,M}$ for $\{c \in \mathcal{V}^{1,p} \mid c(I) \subseteq M, c(0) = u, c(1) = v\}$.*

Lemma 9. *If $c \in \mathcal{V}^{1,p}$ is one-to-one, then the curve $d = c \circ \gamma_c^{-1}$ also is in $\mathcal{V}^{1,p}$, has the same arclength as c, and $\|\dot{d}\| = L(c)$ almost everywhere.*

The proof is given in in the appendix.

Lemma 10. *For all curves $c_0 \in \mathcal{C}_{u,v,M}$ there is another curve in $\mathcal{C}_{u,v,M}$, which is not longer than c_0 and which is one-to-one.*

The proof is given in in the appendix.

Lemma 11. *Minimizers of the energy $E(c)$ for $k = 1$ in $\mathcal{C}_{p,q,M}$ are also minimizers of the arc length $L(c)$, and $\|\dot{c}\|$ is constant almost everywhere. The converse is also true — curves with these two properties minimize $E(c)$. Such curves are one-to-one.*

Proof: This fact is well known in differential geometry, where one considers piecewise regular smooth curves, which are easily shown to possess smooth arc length parametrizations. We add a few technicalities to the existing proofs (see e.g. [12], p. 70) in order to deal with insufficient smoothness. The base of the proof is the Hölder inequality $L(c)^p \le E(c)$, where equality holds if and only if $\|\dot{c}\|$ is constant [a.e.]. The result is trivially true for constant curves, so we consider only non-constant curves now.

Clearly a minimizer of $L(c)$ is one-to-one. The same is true for $E(c)$: Assume that $c(t_0) = c(t_0 + \tau)$ and define $d(t) = c(\frac{t}{1-\tau})$ for $t \le \frac{t_0}{1-\tau}$ and $d(t) = c(\frac{t-\tau}{1-\tau})$ for $t \ge \frac{t_0}{1-\tau}$. It is elementary that $E(d) \le (1-\tau)^{p-1} E(c)$, so $E(c)$ is not minimal.

Suppose now that c minimizes $E(c)$ in $\mathcal{C}_{p,q,M}$. As c is one-to-one, Lemma 9 shows that the curve $d = c \circ \gamma_c^{-1}$ has the property that $\|\dot{d}\|$ is constant and equals $L(c)$ [a.e.]. By construction, we have the inequalities $E(c) \le E(d) = L(d)^p = L(c)^p \le E(c)$, so $L(c)^p = E(c)$ and $\|\dot{c}\|$ is constant [a.e.].

We eventually want to show that c minimizes arc length. Assume now that c minimizes $E(c)$, and that $\bar{c} \in \mathcal{C}_{p,q,M}$. By Lemma 10, it is sufficient to consider only such curves $\bar{c}$ which are one-to-one.

Construct $\bar{d}$ from $\bar{c}$ analogous to the construction of d from c above. Then $L(c)^p = E(c) \le E(\bar{d}) = L(\bar{d})^p = L(\bar{c})^p$. This shows that $L(c) \le L(\bar{c})$, i.e., c does minimize arc length.

The converse statement is easily shown: Assume that c minimizes arc length and $\|\dot{c}\|$ is constant [a.e.]. Then for all $\bar{c} \in \mathcal{C}_{p,q,M}$ we have $E(c) = L(c)^p \le L(\bar{c})^p \le E(\bar{c})$. So c minimizes energy. □

9. Characterization of energy minimizers by infinitesimal conditions

A problem related to the one considered in this paper is to find infinitesimal conditions which characterize the energy minimizers or at least are fulfilled by them. If $p = 2$ and the Euclidean norm is used in $\mathbb{R}^n$, it is not difficult to show that the variational problem of making the energy functional (1) stationary for curves in a given surface results in the Euler-Lagrange condition that the $2k$-th derivative of c is orthogonal to that surface. The well-known case $k = 1$ leads to the differential equation of geodesic lines.

10. Appendix: Proofs

Proof: (of Lemma 2) In $\mathbb{R}^n$, all norms (even the non-convex ones) are equivalent, so there are constants B, D such that $\|x\|_{1/p} \leq B\,\|x\|_1$ and $\|x\|_p \leq D\,\|x\|$ for all $x \in \mathbb{R}^n$. The existence of B means that for all x we have $\sum |x_i|^{1/p} \leq B^{1/p}(\sum |x_i|)^{1/p}$. In order to show that there is β with $\|f\|' \leq \beta\|f\|$, we compute $\|f\|' = \sum(\int |f_i|^p)^{1/p} \leq B^{1/p}(\sum \int |f_i|^p)^{1/p} \leq B^{1/p}(\int D^p \|f(t)\|^p)^{1/p} = B^{1/p} D \|f\|$. A reverse inequality is shown in an analogous way. □

Proof: (of Lemma 7) Assume that the parameter values where c is not C^k are $t_1, \ldots, t_r$, and that the intervals where we want $c = d$ are $[u_1, u_1'], \ldots, [u_s, u_s']$. Now construct a parameter transform γ in the following way: In the quadrangle $[0,1] \times [0,1]$, connect the points (u_i, u_i) and (u_i', u_i') by straight line segments $(i = 1, \ldots, s)$. Further draw small horizontal line segments through the points (t_i, t_i) $(i = 1, \ldots, r)$. The graph of γ as a real function shall contain these segments, and it is constructed by filling in the gaps by means of a nondecreasing C^∞ function $\alpha(x)$ with the property that $\alpha(x) = 0$ for $x \leq 0$ and $\alpha(x) = 1$ if $x \geq 1$. After construction of γ, we let $d = c \circ \gamma$ and the proof is finished.

It remains to show how to find $\alpha(x)$. For this purpose, we let $\beta(x) := \exp(x^{-2})$ if $x > 0$ and $\beta(x) = 0$ for $x \leq 0$ and let $\alpha(x) = \frac{\beta(x)}{\beta(x)+\beta(1-x)}$. □

Proof: (of Lemma 9) It is well known that continuous curves which are one-to-one may be parametrized using the arc length $\int_{[0,t]} \|\dot{c}\|$ as a parameter. The arc length of $c|_{[t',t'']}$ is an upper bound for $\|c(t) - c(t'')\|$, and $L(c|_{[0,t]})$ is strictly increasing. Apart from the factor $1/L(c)$, which ensures that after the parameter transform the curve is still defined in the same interval $[0,1]$, the curve $d = c \circ \gamma_c^{-1}$ is exactly this parametrization by arc length.

As $\|d(t) - d(t')\| \leq L(c)|t - t'|$, d is Lipschitz and so $d \in AC$. The chain rule applies to $c = d \circ \gamma_c$ (cf. (20.5) of [10]), so from $\dot{\gamma}_c(u) = \frac{1}{L(c)}\|\dot{c}(u)\|$

[a.e.] and $\|\dot{d}(\gamma(u))\| \cdot \dot{\gamma}_c(u) = \|\dot{c}(u)\|$ [a.e.], we conclude that $\|\dot{d}(t)\| = L(c)$ [a.e.]. □

Proof: (of Lemma 10) If a curve $c_0 \in \mathcal{V}^{1,p}$ is not one-to-one, we look for t_0, τ_0, such that $c(t_0) = c(t_0+\tau_0)$, but c_0 is not constant in $[t_0, t_0+\tau_0]$. Choose τ_0 maximal with this property. We define $c_1(t) := c_0(t)$ if $t \notin [t_0, t_0+\tau_0]$, and $c_1(t) := c_0(t_0)$ if $t \in [t_0, t_0+\tau_0]$. This procedure is iterated and curves $c_2, c_3, \ldots$ are constructed. Obviously $\widetilde{c} := \lim c_i$ is in AC, has the property that $L(\widetilde{c}) \le L(c)$, and is not necessarily one-to-one, but 'interval-to-one'.

By deleting all intervals $(t_j, t_j']$ from I where $\widetilde{c}$ is constant, we get a subset $J \subseteq I$. Let $\delta(u) = \frac{1}{|J|}|[0,u] \cap J|$. Define $\widetilde{\widetilde{c}}$ by $\widetilde{\widetilde{c}}(\delta(u)) = \widetilde{c}(u)$. Then $\widetilde{\widetilde{c}} \in AC$ (which follows easily from $\widetilde{c} \in AC$, as $\sum |t_i' - t_i| \le 1$), $\widetilde{\widetilde{c}}$ is one-to-one, and $L(\widetilde{\widetilde{c}}) = L(\widetilde{c}) \le L(c)$. □

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