Spline Orbifolds

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Abstract. In order to obtain a global principle for modeling closed surfaces of arbitrary genus, first hyperbolic geometry and then discrete groups of motions in planar geometries of constant curvature are studied. The representation of a closed surface as an orbifold leads to a natural parametrization of the surfaces as a subset of one of the classical geometries S^2, E^2 and H^2. This well known connection can be exploited to define spline function spaces on abstract closed surfaces and use them e. g. for approximation and interpolation problems.

§1. Geometries of Constant Curvature

We are going to define three geometries consisting of a set of *points*, a set of *lines*, and a group of *congruence transformations*: The geometry of the euclidean plane E^2, the geometry of the unit sphere S^2 of euclidean E^3, and the geometry of the hyperbolic plane H^2. The geometries of E^2 and S^2 are well known: the hyperbolic plane will be presented in the next subsections. For more details, see for instance (Alekseevskij et al., 1988).

It is possible to define hyperbolic geometry in a completely synthetic way. We could use a system of axioms for euclidean geometry and then negate the parallel postulate or one of its equivalents. Any structure satisfying the axioms would be called a *model* of hyperbolic geometry. We would have to verify that all models, including the classical ones, the *Poincaré* and the *Klein* model, are isomorphic. We start from a different point of view: We first define a set of points, lines and congruence transformations, as linear as possible, and then show some structures isomorphic to it. The reader then will see the difference to euclidean or spherical geometry.

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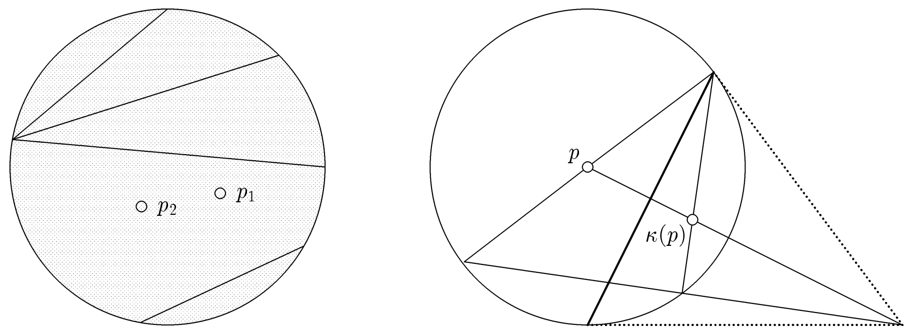

Fig. 1. Projective model of H^2: (a) points and non-intersecting lines, (b) hyperbolic reflection κ.

1.1 The Projective Model of Hyperbolic Geometry

Consider the real projective plane P^2 equipped with a homogeneous coordinate system, where a point with homogeneous coordinates $(x_0 : x_1 : x_2)$ has affine coordinates $(x_1/x_0, x_2/x_0)$. We will not distinguish between the point and its homogeneous coordinate vectors. Every time when a coordinate vector of a point appears in a formula, it is tacitly understood that any scalar multiple of this coordinate vector could be there as well.

We define an *orthogonality relation* between points: Let β be a symmetric bilinear form defined in $\mathbb{R}^3$, and let β have two negative squares, for instance

$$\beta(x, y) = x_0 y_0 - x_1 y_1 - x_2 y_2.$$

An equivalent formulation is $\beta(x, x) = x^T J x$, J being the diagonal matrix with entries 1, -1 and -1. We call x and y *orthogonal*, if $\beta(x, y) = 0$. Points with $\beta(x, x) = 0$ are called *ideal* points. The set of all ideal points is a conic and will be called the *ideal circle*. If we choose β as above, the ideal circle is nothing but the euclidean unit circle.

Now a point $x \in P^2$ shall belong to the *hyperbolic plane* H^2 if it is contained in the interior of the ideal circle,

$$x \in H^2 \iff \beta(x, x) > 0.$$

The *lines* of the hyperbolic plane are the intersections of projective lines with H^2. We define two lines to be parallel if they have no point in common. It is now obvious that for all lines l and all non-incident points p, there are a lot of lines parallel to l and containing p. A picture of the projective model can be found in Figure 1.

So far we have dealt with the incidence structure of the hyperbolic plane. We now come to metric properties. We define the *hyperbolic distance* $d(x, y)$ between points x and y of H^2 by

$$\cosh d(x, y) = \frac{|\beta(x, y)|}{\sqrt{\beta(x, x)\beta(y, y)}}.$$

We leave the verification of the fact that always $\beta(x,x)\beta(y,y) \le \beta(x,y)^2$ to the reader. This metric satisfies the triangle inequality and is compatible with the definition of lines, in the sense that they are precisely the geodesic curves with respect to this metric.

Hyperbolic congruence transformations will be those projective transformations, which map H^2 onto H^2 and preserve hyperbolic distances. For this reason and also because it is shorter, we will call them *isometries* or *motions*. We express the isometric property in matrix form: for each projective transformation κ there is a matrix such that in homogeneous coordinates

$$\kappa(x) = A \cdot x.$$

It is easy to see that the condition $d(x,y) = d(\kappa(x), \kappa(y))$ for all $x \in H^2$ is equivalent to

$$A^T J A = \lambda J \text{ with } \lambda > 0,$$

and that there are the following types of hyperbolic isometries:

1) the identity transformation;
2) *hyperbolic reflections*, which leave the points on a hyperbolic line fixed and reverse orientation (see Figure 1b);
4) *hyperbolic translations*, which preserve orientation and leave no point of H^2 fixed, but a hyperbolic line is mapped onto itself;
5) *hyperbolic rotations*, which leave one point of H^2 fixed and preserve orientation (for a picture in a different model, see Figure 3);
6) *ideal hyperbolic transformations* which leave no point of H^2 fixed, and no line is mapped to itself, but orientation is preserved;
7) the remaining hyperbolic isometries reverse orientation and are the product of a hyperbolic reflection by one of the above.

The model of the hyperbolic plane just described is called the *projective* or *Klein* model. In this model hyperbolic geometry appears as a subset of projective geometry: the point set is a subset of the projective point set, the lines are the appropriate subsets of projective lines, and hyperbolic isometries can be expressed in matrix form.

What remains to be defined is the *hyperbolic angle*. We will do this in a different model, which will also explain the name "hyperbolic".

1.2 The Hyperboloid Model of Hyperbolic Geometry

In $\mathbb{R}^3$, $\beta(x,x) = 0$ is the equation of a quadratic cone with apex at the origin, and $\beta(x,x) = 1$ is the equation of a two-sheeted hyperboloid, which can be seen as the unit sphere with respect to the *pseudo-euclidean* scalar product β. We call the "upper sheet" of this unit sphere the *hyperbolic plane*:

$$x \in H^2 \iff \beta(x,x) = 1 \text{ and } x_0 > 0.$$

There is an obvious one-to-one correspondence between the hyperbolic plane defined in Section 1.1 and the hyperbolic plane defined in this subsection.

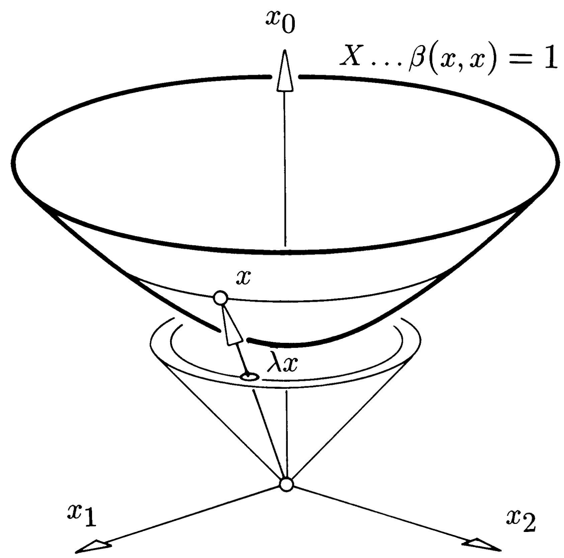

Fig. 2. The hyperboloid model $X \subset \mathbb{R}^3$ of H^2 and the correspondence between hyperboloid and projective model, which appears as a unit disk tangent to X.

Given a projective point, its uniquely defined coordinate vector x with $\beta(x,x) = 1$ and $x_0 > 0$ defines the corresponding point of the hyperboloid model. It is easy to transfer lines and hyperbolic isometries to the hyperboloid model: Hyperbolic lines have linear equations and therefore are intersections of H^2 with two-dimensional linear subspaces of $\mathbb{R}^3$. A picture of the hyperboloid model is given in Figure 2.

In the projective model, a hyperbolic isometry given by its matrix A is equivalently described by any scalar multiple of A. Now scale A such that

$$A^T J A = J.$$

Then the unit hyperboloid $\beta(x,x) = 1$ is invariant under multiplication by A. Conversely, as scalar products can be expressed in terms of distances, the invariance of the unit hyperboloid implies $A^T J A = J$. If A interchanges the two sheets of the unit hyperboloid, multiply A by -1. Thus, without loss of generality, we call all linear automorphisms of $\mathbb{R}^3$ which map H^2 onto itself *hyperbolic isometries* and this definition is compatible with the definition given in Section 1.1.

A scalar product β always defines an angle between vectors x and y: In $\mathbb{R}^3$ the *pseudo-euclidean angle* $\angle(x,y)$ is partially defined by

$$\begin{aligned} \lambda = \frac{\beta(x,y)}{\sqrt{\beta(x,x)\beta(y,y)}} \quad & \text{if } \beta(x,x)\beta(y,y) > 0 \\ \cos\angle(x,y) = \lambda \quad & \text{if } |\lambda| \leq 1 \\ \cosh\angle(x,y) = |\lambda| \quad & \text{if } |\lambda| \geq 1 \end{aligned}$$

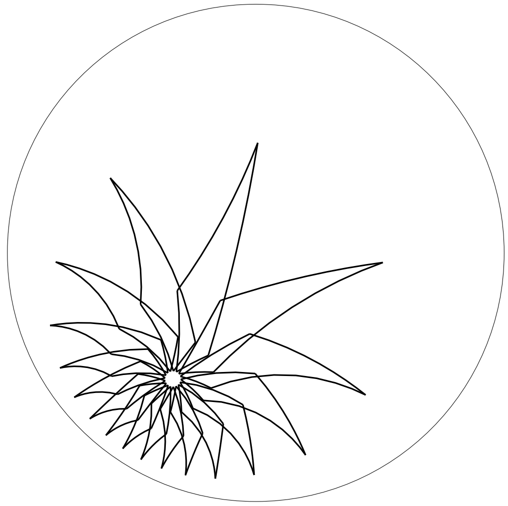

Fig. 3. Conformal model: hyperbolic rotation.

In every case where we will calculate an angle it has to be verified that $\beta(x,x)\cdot\beta(y,y) > 0$. In most cases we leave this verification to the reader. It is clear that the pseudo-euclidean angle of vectors x and y corresponds to the hyperbolic distance defined in Section 1.1. The *hyperbolic angle* between lines meeting in x is defined as the pseudo-euclidean angle of tangent vectors at x. Because all vectors v tangent to H^2 in a point x satisfy $\beta(v,v) < 0$, the angle between them is defined.

We define the *geodesic distance* between points x and y on a smooth surface X in $\mathbb{R}^n$ as the infimum of the arc-lengths of smooth curves c joining x and y in X. Arc-lengths are measured by means of the scalar product β: We can define the *norm* of a vector by $\|x\|^2 = |\beta(x,x)|$ and measure the arc length by $\int \|\dot{c}(t)\| dt$. It is easy to see that for H^2 we can explicitly find the curves for which the infimum, actually then the minimum, is attained: The geodesic distance is the arc-length of the unique hyperbolic line joining x and y and equals the hyperbolic distance $d(x,y)$.

1.3 The Conformal Model of Hyperbolic Geometry

Distorting the projective model leads to a new model of hyperbolic geometry with some other special metric properties: Let H^2 be the interior of the unit circle and define $\sigma : H^2 \to H^2$ in affine coordinates by

$$(x,y) \mapsto \frac{1}{1-\sqrt{1-x^2-y^2}}(x,y).$$

Thus points will be moved a bit towards the origin. Hyperbolic lines will be σ-images of hyperbolic lines defined in Section 1.1. If κ is a hyperbolic isometry as defined in Section 1.1, then $\sigma\kappa\sigma^{-1}$ shall be a congruence transformation. This geometry which is obviously isomorphic to the projective and the hyperboloid model is called the *conformal* or *Poincaré* model of the hyperbolic plane. It has the following interesting properties:

1) Hyperbolic lines appear as euclidean circular arcs or straight line segments which intersect the ideal circle orthogonally.
2) The hyperbolic angle appears as the euclidean angle between circular arcs or straight line segments. This is why the model is called conformal (see Figure 3).
3) Hyperbolic reflections appear as *inversions*. The group of hyperbolic isometries is generated by the hyperbolic reflections, so in the conformal model it appears as the subgroup of Möbius transformations which map H^2 onto itself.

Because the euclidean radius of hyperbolic distance circles with center in the origin is smaller in the conformal model than it is in the projective model, usually the conformal model is used for illustrations. In Figure 3b you can see an iterated hyperbolic rotation in the conformal model.

The conformal properties of this model have also been exploited by the Dutch artist *M. C. Escher* in some of his famous drawings. One of them is depicted in Figure 4.

1.4 An Overview

We can assume that the reader is familiar with the geometry of the euclidean plane E^2 and the unit sphere S^2. In this section we will present these two together with hyperbolic geometry from a unified point of view. S^2 and H^2 will in some places be dual to each other, whereas euclidean geometry does sometimes not fit so nicely into the description. Also the generalizations of S^2, E^2 and H^2 to higher dimensions are obvious: E^n is euclidean n-space, S^n and H^n are the unit spheres with respect to a scalar product in $\mathbb{R}^{n+1}$ with zero or n negative squares, respectively. It may be stated that almost everything in this paper, except, of course, the classification of surfaces in Section 2.5, holds for any dimension with only slight notational changes.

- *Linear incidence structure:* For each of the three geometries there is a model as a subset X of $\mathbb{R}^3$ such that lines in the geometry are intersections of two-dimensional linear subspaces with X. For X we can choose the unit sphere, the plane with coordinate $x_0 = 1$, and the upper sheet of the hyperboloid described in Section 1.2.
- *Linear model and metric:* Given a scalar product β in $\mathbb{R}^3$, then dependent on the number of negative squares, the unit sphere will be an ellipsoid, a one-sheeted hyperboloid, a two-sheeted hyperboloid, or empty. If β is positive definite, the unit sphere carries the structure of a spherical geometry. If β has two negative squares, then each of the two connected components (sheets) of the unit sphere carries the structure of a hyperbolic geometry. Distances of points are given in terms of angles between the corresponding vectors, as are angles between tangent vectors. The geodesic distance in X equals the distance previously defined.

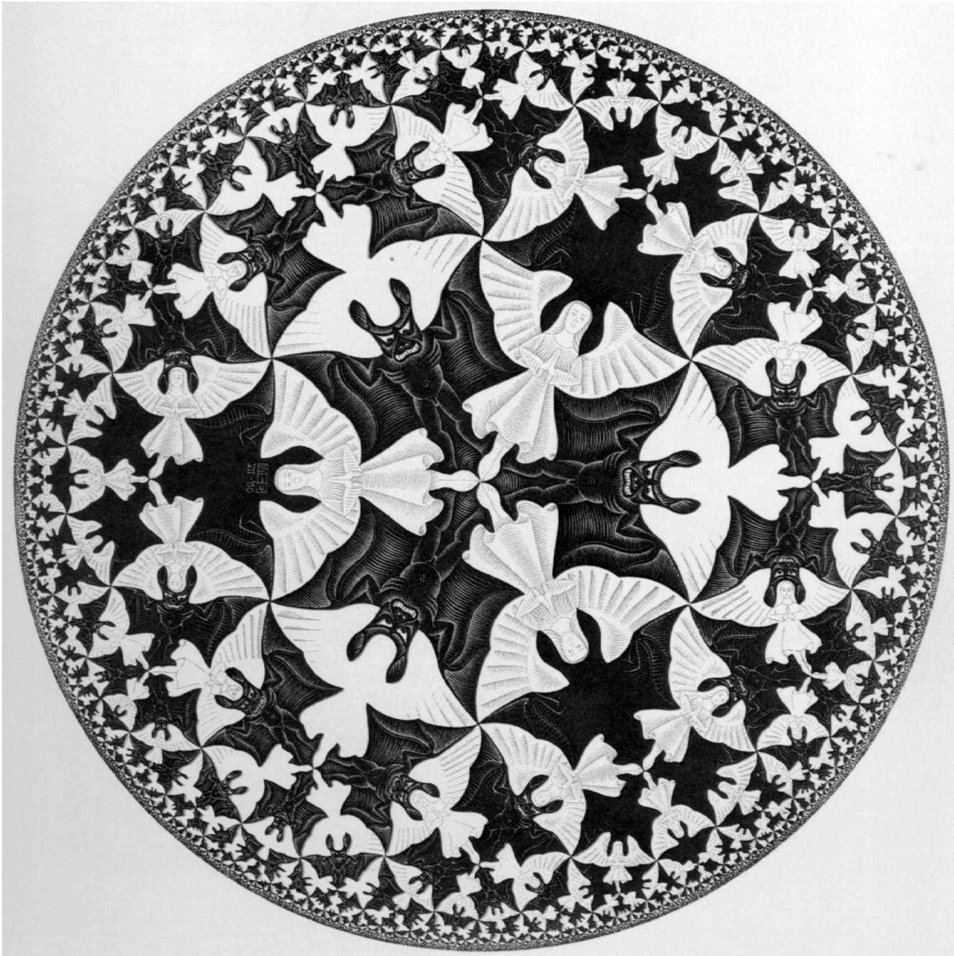

Fig. 4. M. C. Escher's "Circle Limit IV", (c) 1997 Cordon Art – Baarn – Holland. All rights reserved.

- *Congruence transformations:* In the linear models $X \subset \mathbb{R}^3$ of S^2 and H^2, the group of motions or isometries consists of the restrictions $L|_X$ of those linear automorphisms L of $\mathbb{R}^3$ which map X onto itself.
- *Curvature:* The sphere, the euclidean plane and the hyperbolic plane are Riemannian manifolds of constant Gaussian curvature, the value of which is 1, 0 and -1, respectively. From the Gauss-Bonnet theorem it then follows that the angle sum in a triangle is greater than, equal to, or less than π, respectively. Moreover, the absolute value of the difference is the *area* of the triangle as of a Riemannian manifold.

§2. Discrete Motion Groups and Orbifolds

In this section we define the factor orbifold X/H where X is one of E^2, S^2 or H^2, and H is a discrete transformation group acting on X. X will always denote one of the three geometries, and its motion group will be denoted by

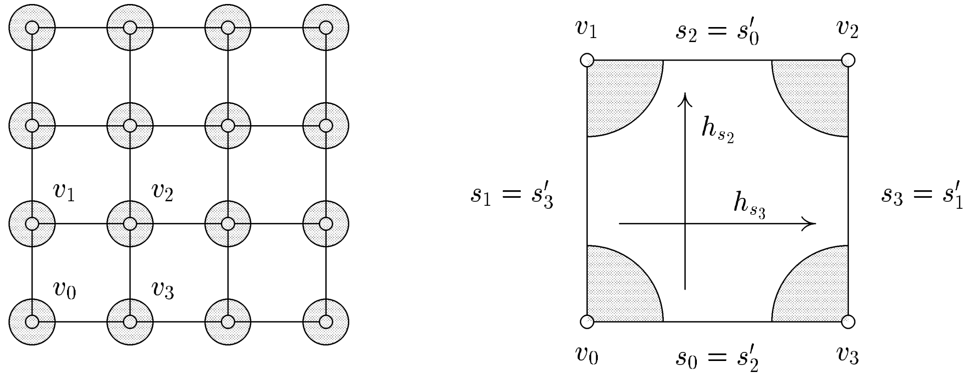

Fig. 5. The torus as an orbifold.

G. We will not be able to present a complete theory, and we simplify some notions in some places.

For a detailed presentation, see for instance (Ratcliffe, 1994), (Vinberg and Shvartsman, 1988) or (Zieschang et al., 1980). For a well illustrated book which is easy to read, see for instance (Week, 1985).

2.1 Discrete Transformation Groups

We will consider groups H of motions *acting* on X, which means that each $h \in H$ is an isometry $h : X \to X$ and $h_1(h_2(x)) = (h_1 \cdot h_2)(x)$. The identity transformation will always be denoted by e. We write $h(x)$ for the h-image of an $x \in X$ and $h(K)$ for the h-image of a subset $K \subset X$. We call a group H acting on X *discrete* if for every compact set K the intersection $K \cap h(K)$ is nonempty only for finitely many $h \in H$. This implies that the *orbit* $\{h(x), h \in H\}$ of a point x is discrete, i. e., it has no accumulation point. An example for this is the group $H = \mathbb{Z}^2$ acting as a group of translations on the euclidean plane: The pair (i, j) of integers acts on $X = E^2$ by $(x, y) \mapsto (x + i, y + j)$. It is only a change in notation if we consider H as a subgroup of the euclidean motion group. A picture can be seen in Figure 5.

For a group H acting on X, the *stabilizer* H_x of x is the subgroup of all those $h \in H$ with $h(x) = x$. If H is discrete, obviously H_x is finite. The *order* of x is the cardinality of its stabilizer. In the example given above all stabilizers are trivial. We call such actions *free.*

If x and y are not antipodal points of the sphere, the unique shortest segment joining them is called their convex hull, and a set C is called *convex*, if for all $x, y \in C$ the convex hull of x and y is in C. Then a *convex polygon* is defined as the convex hull of a finite non-collinear set of points. *Edges* and *vertices* are defined in the obvious way. Note that a convex polygon is always the closure of its interior.

2.2 Fundamental Domains

A *fundamental domain* F of a discrete motion group H is a set which is the closure of its interior and fulfills the following conditions: 1) the sets

$h(F)$, $h \in H$ cover X, and 2) if $h_1(F)$ and $h_2(F)$ have an interior point in common, then $h_1 = h_2$. There are discrete groups of motions which have no convex polygons as fundamental domains, for instance the discrete group of translations along integer multiples of one fixed vector in E^2. We will not try to generalize the notion of polygon such that it covers all discrete motion groups (which is possible), but we restrict ourselves to groups which possess convex fundamental polygons.

We denote the edges of the fundamental polygon F by $s_0, \ldots, s_{n-1}, s_n = s_0$. The intersection of edges $s_i \cap s_{i+1}$ is a vertex v_i. By subdividing finitely many edges and introducing new vertices it is possible to achieve that the intersection of F with any $h(F)$ is either empty or an edge. We call the uniquely defined motion $h \in H$ for which $F \cap h(F) = s_i$ the *adjacency transformation* of the edge s_j. We call a sequence $h_1(F), \ldots, h_n(F)$ a chain of polygons, if the intersection $h_i(F) \cap h_{i+1}(F)$ is an edge. Because any two $h \in H$ can be connected by a chain, the group H is entirely generated by the finitely many adjacency transformations of one fundamental polygon.

If an adjacency transformation maps s_i to s_j, then we write $s_i = s'_j$. Obviously then the inverse adjacency transformation maps s_j to s_i, so $s'_i = s_j$. For the example given above, the adjacency transformations are indicated in Figure 5.

2.3 Defining Relations

We write h_s for the adjacency transformation with $F \cap h_s(F) = s$. A sequence $h_{s_1}, \ldots, h_{s_n}$ of adjacency transformations with $h_{s_1} \cdot \ldots \cdot h_{s_n} = e$ corresponds to a chain $F_0 = F, F_1 = h_{s_1}(F), F_2 = h_{s_1}(h_{s_2}(F)), \ldots, F_n = F_0$ of polygons. Such a chain is called a *cycle*.

Let s, s' be edges with $h_s(s') = s$ and $h_{s'}(s) = s'$. Then obviously $h_s h_{s'} = e$ and $F, h_s(F), h_s h_{s'}(F) = F$ is a cycle. Formally, we write

$$ss' = e.$$

Also for all vertices v there is a cycle of polygons consisting of all polygons containing v in the order in which they are encountered when cycling v. The corresponding sequence $h_{s_1}, \ldots, h_{s_n}$ of adjacency transformations gives the formal relation

$$s_1 s_2 \ldots s_n = e,$$

which is called a *Poincaré relation*. The importance of the Poincaré relations is shown by the following

Theorem. *Let H be a group with a convex fundamental polygon. Denote its set of edges by S and the set of relations $ss' = e$ together with all Poincaré relations with R. Then the abstract group with generator set S and relations R is isomorphic to H.*

In the example given above, all adjacency transformations are translations. They correspond to the edges s_0, s_1, s_2, s_3 and $s'_0 = s_2$, $s'_1 = s_3$

(see Figure 5). The four Poincaré relations are $s_0s_1s_2s_3 = e$, $s_1s_2s_3s_0 = e$, $s_2s_3s_0s_1 = e$ and $s_3s_0s_1s_2 = e$. Obviously $s_2s_0 = e$ and $s_1s_3 = e$. So we can eliminate s_2 and s_3. Each Poincaré relation implies the other three. It follows that H as an abstract group is isomorphic to the group with generators s_0, s_1 and the single relation $s_0s_1s_0^{-1}s_1^{-1} = e$, or, equivalently, $s_0s_1 = s_1s_0$. This means that H is a free abelian group with free generators s_0 and s_1.

A natural question to ask now is: Given a convex polygon F and for each edge s an adjacency transformation h_s, such that (a) $F \cap h_s(F) = s$, (b) $h_s(s') = s$ implies $h_{s'}(s) = s'$, and (c) $h_s h_{s'} = e$. Suppose further that (d) for each vertex v of F there are adjacency transformations $h_{s_1}, \ldots h_{s_n}$ such that their product equals e and the polygons $h_{s_1}h_{s_2}\ldots h_{s_i}(F)$ form a "circuit" around v. Does there exist a discrete group of motions having F as fundamental polygon and h_s as adjacency transformations? The answer, due to Poincaré, is yes.

2.4 Orbifolds

Let H be a discrete group of motions in one of the three geometries E^2, S^2 or H^2. By identifying all points $h(x)$, $h \in H$, we get the points of the *orbifold* X/H. This definition, however, gives only the orbifold as a set, without additional structures. They are to be defined by means of the *canonical projection* $p : X \to X/H$ which maps an $x \in X$ to its orbit. The topology on X/H is defined as the final topology of p: U is open if and only if $p^{-1}(U)$ is open. The incidence structure is directly mapped by p: A line segment in X/H is the p-image of a line segment of X. The distance between x, y in X/H is the minimum of distances of points in $p^{-1}(x)$ and $p^{-1}(y)$ measured in X.

An example for an orbifold which is very well known, but, in some sense is not typical, is the torus. It appears as the orbifold X/H if $X = E^2$ and H is the discrete group of translations along integer multiples of two basis vectors e_1 and e_2 (see Figure 5). The order of all points x equals 1, and so for all y in $p^{-1}(x)$ there is a neighborhood of y which is mapped isometrically (and, of course, homeomorphically) to X/H. This need not be the case, and happens if and only if some $h \in H$ has a fixed point. These orbifolds have metric singularities and could also be used for modeling surfaces, but we will omit them in order to keep the presentation simple.

2.5 Surfaces

Our aim is to find discrete groups H in a geometry X of constant curvature such that the corresponding orbifold X/H is a compact surface, orientable or nonorientable, of arbitrary genus g. It is well known that the compact surfaces without boundary are precisely the spheres with g handles and the spheres with g crosscaps. For the classification of surfaces from the topological, differentiable, or combinatorial viewpoint, see textbooks of algebraic topology, differential topology or combinatorial topology, for instance (Hirsch, 1976) or (Kinsey, 1993).

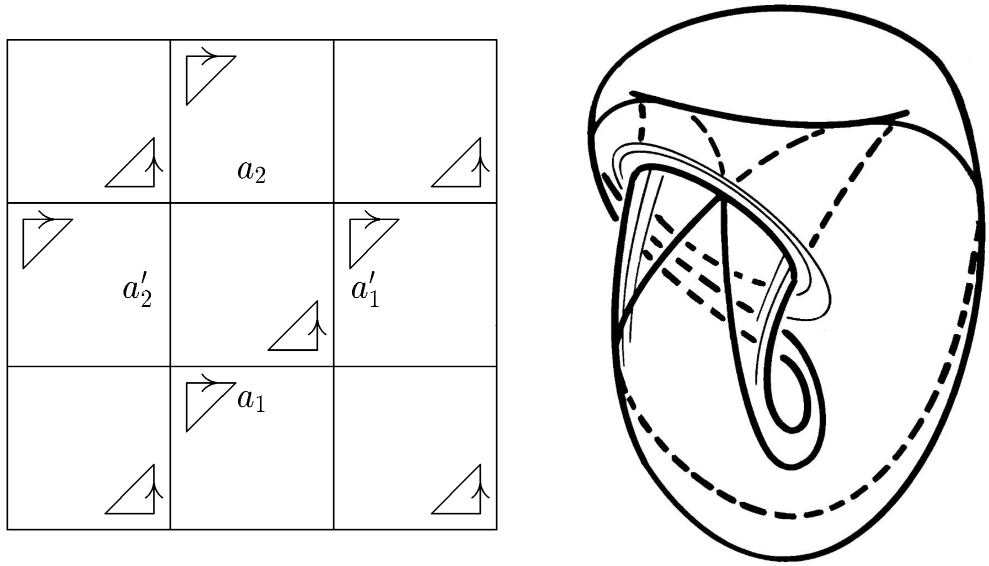

Fig. 8. Klein bottle.

It is well known that the following discrete transformation groups H in various geometries X lead to all compact surfaces:

- *The sphere:* S^2 itself as the orientable surface of genus 0 is one of the primitive geometries. Formally, let $X = S^2$ and $H = \{e\}$.
- *The projective plane:* P^2 is obtained by identifying antipodal points in S^2. If s denotes the antipodal map, then $P^2 = S^2/H$ with $H = \{e, s\}$. A fundamental polygon is the upper hemisphere.
- *The torus:* Letting $X = E^2$ and H equal the group generated by the translations along two linearly independent vectors gives the torus, which is the orientable surface of genus 1. A fundamental polygon is the parallelogram spanned by a lattice basis.
- *The Klein bottle:* Letting $X = E^2$ and H equal the group generated by the adjacency transformations depicted in Figure 8 gives the nonorientable surface of genus 2, which is called the Klein bottle.
- *Orientable surfaces of higher genus:* In the conformal model of the hyperbolic plane, consider the points $(r\cos(2k\pi/l), r\sin(2k\pi/l))$ with $l = 4g$, $g \geq 2$ and $k = 0, \ldots, l-1$. The convex hull F of these points is a regular $4g$-gon, with interior hyperbolic angles α depending on the value of r (see Figure 9). It is easily seen that α tends to 0 as r tends to 1, and α tends to $\pi - 2\pi/l$ as r tends to 0. By continuity, for all l there is a value of r such that the interior angle α equals $2\pi/l$. Now denote the edges of F by $a_1, b_1, a_1', b_1', \ldots, a_g, b_g, a_g', b_g'$ and define orientation-preserving adjacency transformations which map a_k to a_k', b_k to b_k' and vice versa, for all k. Then the Poincaré relations will be equivalent to the relation

$$a_1 b_1 a_1' b_1' a_2 b_2 a_2' b_2' \ldots a_g b_g a_g' b_g' = e.$$

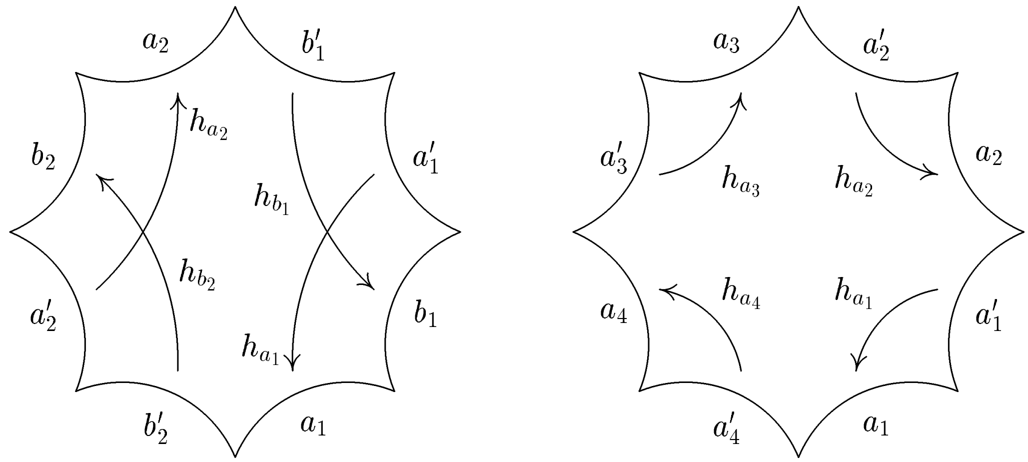

Fig. 9. Regular octogon as fundamental domain (a) of a group whose orbifold is an orientable surface of genus 2 (b) of a group whose orbifold is a non-orientable surface of genus 4.

This shows that the group H generated by the adjacency transformations defined above is, as an abstract group, isomorphic to the group with generators $a_1, b_1, \ldots, a_g, b_g$ and the single relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = e.$$

The order of all vertices is 1, and therefore X/H is a manifold. Gluing those edges of the fundamental polygon together which are mapped onto each other by adjacency transformations, gives precisely X/H. From the gluing construction it is clear that X/H is a sphere with g handles. A picture of the gluing for $g = 2$ can be seen in Figure 11a. This shows that the orientable surface of genus g with $g > 1$ is an orbifold, even a manifold, of the form X/H where X is the hyperbolic plane and H is the group generated by the adjacency transformations defined above. The torus fits into this description, if we set $g = 1$ but instead of a polygon in H^2 use a euclidean square.

- *Nonorientable surfaces of higher genus:* In analogy to the previous construction, construct a regular $2g$-gon ($g \geq 3$) in the hyperbolic plane with angles π/g. Denote the edges by $a_1, a'_1, \ldots, a_g, a'_g$. To the edge a_k corresponds the uniquely determined adjacency transformation which reverses orientation and maps a_k to a'_k, and for a'_k vice versa (see Figure 9). Then all Poincaré relations are equivalent to the relation $a_1^2 a_2^2 \ldots a_g^2 = e$. Thus the discrete motion group H generated by these adjacency transformations is, as an abstract group, isomorphic to the group with generators $a_1, \ldots, a_g$ and the single relation

$$a_1^2 \ldots a_g^2 = e.$$

The order of all vertices equals 1, and therefore X/H is a manifold. It is nonorientable because H contains orientation reversing motions. From the gluing construction it is clear that X/H is a sphere with g crosscaps.

The Klein bottle ($g = 2$) and the projective plane ($g = 1$) fit into this formalism, if we choose a euclidean square or a spherical 2-gon (such as the northern hemisphere) instead.

§3. Functions on Surfaces

3.1 Group-Invariant Functions

We call a function $\widetilde{f} : X \to R$ *invariant* with respect to the group H, if

$$\widetilde{f}(h(x)) = \widetilde{f}(x) \quad \text{for all } x \in X, h \in H.$$

If $p : X \to X/H$ denotes the canonical projection, an H-invariant function directly leads to a function f whose domain is the factor orbifold:

$$f : X/H \to R, \; f(p(x)) = \widetilde{f}(x)$$

and vice versa: a function f defined on X/H gives rise to an H-invariant function

$$\widetilde{f} : X \to R, \; \widetilde{f}(x) = f \circ p(x).$$

If the range R is the real number field $\mathbb{R}$ and $\widetilde{f}$ is a function defined on X, then we can build an H-invariant function $\widetilde{g}$ from $\widetilde{f}$ by letting

$$\widetilde{g}(x) = \sum_{h \in H} \widetilde{f}(h(x)).$$

Of course it has to be verified that this sum makes sense. If X is the sphere, every discrete motion group is finite, and the sum above is finite. So every property of f which is invariant with respect to finite sums is preserved, so for instance continuity or differentiability.

If f has compact support, then for all x there is a neighborhood U of x such that the sum defined above is finite in U, by discreteness of H. So all local properties which are invariant with respect to finite sums are preserved, for instance continuity or differentiability.

If X/H is a manifold, it is clear that $f : X/H \to \mathbb{R}$ is continuous (differentiable, of class C^r, of class C^∞) if and only if the corresponding $\widetilde{f} : X \to \mathbb{R}$ has this property. If X/H is an orbifold with metric singularities, we avoid difficulties by *defining* that an f defined on X/H is differentiable (of class C^r, of class C^∞) if the corresponding $\tilde{f}$ has this property.

The above sum can make sense even if f does not have compact support. It is sufficient that f decreases fast enough. An example for a summable function whose sum is of class C^∞ is the Gaussian $\widetilde{f}(x) = \exp(-d(x, m)^2)$ in E^2 and H^2 (note that in S^2 $\widetilde{f}$ is not differentiable everywhere).

3.2 Polynomial and Rational Functions

For each of the three geometries S^2, E^2 and H^2 we have found a model as a subset X of $\mathbb{R}^3$. This enables us to define polynomial or rational functions on X as the restriction of polynomial or rational functions defined in $\mathbb{R}^2$ to X. It is well known that both S^2 and H^2 possess rational parametrizations which can be given by stereographic projections: The mapping σ defined by

$$\sigma : \mathbb{R}^2 \to S^2 \setminus \{(-1,0,0)\},\ (p,q) \mapsto \frac{1}{1+p^2+q^2}(1-p^2-q^2, 2p, 2q)$$

is one-to-one. Also, the mapping σ defined by

$$\sigma : D \to H^2,\ (p,q) \mapsto \frac{1}{1-p^2-q^2}(1+p^2+q^2, 2p, 2q)$$

with D being the interior of the unit circle, is one-to-one. If f is a polynomial defined in $\mathbb{R}^3$, then $f \circ \sigma$ is a rational function defined in the domain of σ.

We want to indicate how modeling of closed surfaces with the aid of piecewise rational functions is possible. First we give an easy example which shows how to proceed in the not so trivial cases: The B-spline basis functions on the real line are well known, and so are tensor product B-splines in E^2. We define a knot sequence on the x_1-axis which is periodic and has period 1. This means that if t is a knot, then $t+k$ is a knot for every integer k. The same we do for the x_2-axis, and then we consider the B-spline basis functions $B_i(x_1)$ and $B_j(x_2)$ which correspond to this knot sequences. Their products $B_{ij}(x_1,x_2)$ defined in the plane form a partition of unity. There are finitely many functions $B_{ij}(x_1,x_2)$ such that all others can be expressed in the form $B(x_1,x_2) = B_{ij}(h(x_1,x_2))$ where h is an element of the translation group H generated by translations along the unit vectors in x_1- and x_2-direction. All B_{ij}'s are compactly supported, so the functions

$$\widetilde{C}_{ij}(x) = \sum_{h \in H} B_{ij}(h(x))$$

are well defined, are group-invariant, and form a partition of unity. Thus there are finitely many functions C_{ij} defined on the torus E^2/H such that

$$C_{ij}(p(x)) = \tilde{C}_{ij}(x) \text{ and } \sum C_{ij}(p(x)) = 1 \text{ for all } x \in E^2,$$

where p is the canonical projection which maps a point $x = (x_1, x_2)$ to its orbit.

The preceding paragraph contained nothing new. It could be said that it is a complicated formulation of the simple fact that “closing” B-spline curves is also possible in the plane, and analogously to closed curves which can be viewed as defined on the circle, this closing operation yields a closed surface defined on the torus. On other surfaces the process of making a function group-invariant may be more complicated, but the principle is the same and has been shown in Section 3.1.

3.3 Simplex Splines and a DMS-Spline Space

It is well known that the restriction of homogeneous B-splines to the sphere leads to spline spaces of functions whose domain are subsets of the surface of the sphere, see e. g. (Alfeld et al., 1996). We want to show that the concept of simplex spline is not restricted to the sphere and that there is a natural generalization to abstract surfaces of higher genus.

Choose a basis $b_1, \dots, b_n \in \mathbb{R}^n$. Then for all $v \in \mathbb{R}^n$ there is a unique linear combination $v_1 b_1 + \cdots + v_n b_n$ equal to v. For all n-tuples $k = (k_1, \dots, k_n)$ of integers we define the function

$$B_k : \mathbb{R}^n \to \mathbb{R}, \; v \mapsto \frac{(k_1 + \ldots + k_n)!}{k_1! \ldots k_n!} v_1^{k_1} \ldots v_n^{k_n},$$

which is called a *homogeneous Bernstein basis polynomial* of *degree* $|k| = k_1 + \cdots + k_n$. Any linear combination of homogeneous Bernstein basis polynomials of the same degree is called a *homogeneous Bernstein polynomial.* For such a polynomial $p = \sum_{|k|=d} c_k B_k$ the equation $p(\lambda v) = \lambda^d p(v)$ holds. If X is the linear model of one of the three geometries S^2, E^2 or H^2, the restrictions $p|X$ are called *spherical*, *planar* or *pseudo-spherical* Bernstein polynomials. Note that the planar Bernstein polynomials are just the well-known triangular Bernstein polynomials in the plane.

Also the notion of *simplex spline* has a natural meaning in the linear model of S^2, E^2 and H^2. We recall that the homogeneous simplex spline $M_B : \mathbb{R}^n \to \mathbb{R}$ is well defined for a set $B = \{b_1, \dots, b_m\}$ of vectors as follows:

$$\begin{aligned} M_B(v) &= \chi_B(v) / \det(b_1, \dots, b_n) && \text{if } |B| = n \\ M_B(v) &= \sum_{b \in T} \lambda_b M_{B \setminus b}(v) && \text{if } |B| > n \text{ and } v = \sum_{b \in T} \lambda_b b. \end{aligned}$$

Here $\chi_B(v)$ equals 1 if all coordinates λ_i with respect to the basis B are positive and zero if at least one is negative. T denotes an arbitrary n-element subset of B which is a basis of $\mathbb{R}^n$. This defines the simplex spline in $\mathbb{R}^n$ except in some subspaces. Now extend the simplex spline continuously. This gives a C^{m-n} function. It is natural to define spherical, planar or pseudo-spherical simplex splines as the restriction $M_B|X$ of simplex splines M_B.

This allows the definition of a spline space analogous to the DMS-spline spaces introduced in (Dahmen et al., 1992). This is of theoretical interest, because it shows the existence of a spline space consisting of piecewise rational functions of arbitrary finite differentiability defined on a surface of genus g over an arbitrary triangulation. The planar and the spherical variant of the DMS-spline space have already been defined, for instance in (Pfeifle and Seidel, 1994).

Simplex splines are most easily made group-invariant if they are defined over a group-invariant triangulation. Here group-invariant means that every motion $h \in H$ maps the triangulation onto itself. One possibility to construct

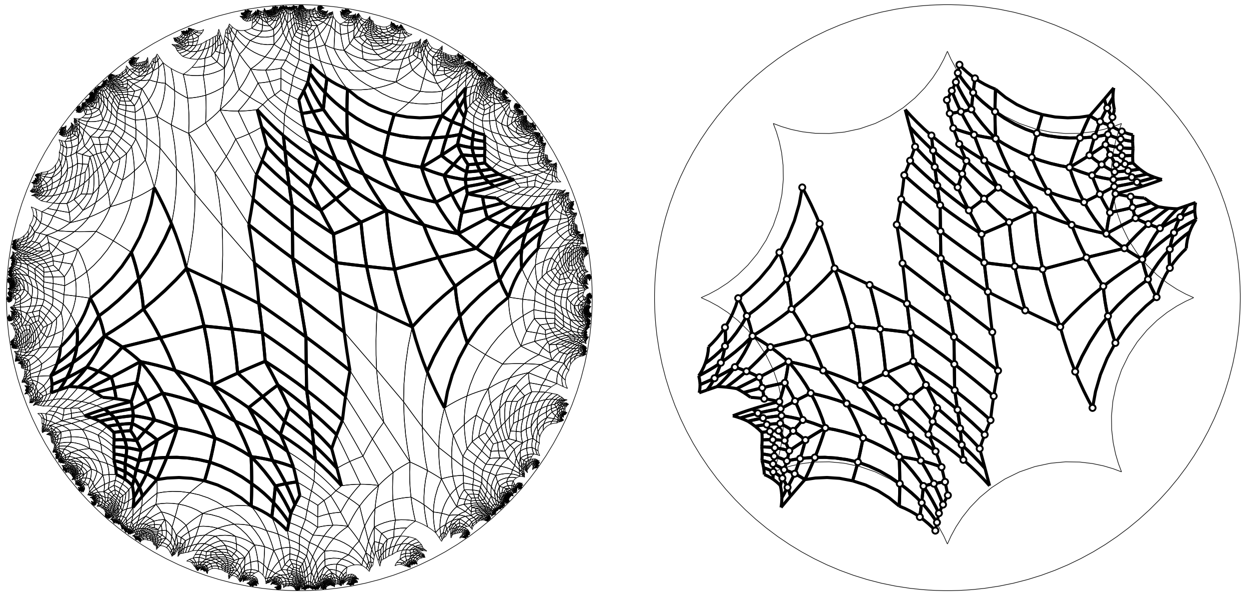

Fig. 10. Group-invariant tesselation.

such a triangulation is the following: Choose a set V of vertices in a fundamental domain of the group H and consider the set $\widetilde{V} = \{h(v), h \in H, v \in V\}$. Then apply an algorithm which finds the edges of a triangulation with vertex set $\widetilde{V}$ and is designed such that it uses only information which can be expressed in terms of the geometry, for instance distance. Then a congruence transformation κ applied to $\widetilde{V}$ must result in edges which are just the κ-images of the previous ones. Now it is clear that if $\widetilde{V}$ is group-invariant, so is the whole triangulation. An example of a triangulation which is invariant with respect to the group corresponding to the orientable surface of genus 2 is shown in Figure 10.

A function defined by means of the triangulation and the geometry alone then is group-invariant. This is especially true for all functions which are defined by means of one of the linear models $X \subset \mathbb{R}^3$ and are linearly dependent on the coordinate vectors of the vertices. One example of this is given by the simplex splines defined above.

3.4 Approximation, Interpolation, Visualization

These tools can be used for approximation and interpolation of functions defined on a compact surface and also for visualizing such surfaces. This has been pointed out by Ferguson and Rockwood (1993). The spherical, euclidean or hyperbolic area $d\widetilde{\mu}$ defines a measure in X. If we assume that the boundary of the fundamental domain F has measure zero, $d\widetilde{\mu}$ naturally defines a measure $d\mu$ on X/H and we can define the space $L^2(X/H)$ with the scalar product

$$(f, g) = \int_{X/H} fg d\mu = \int_F \widetilde{f}\widetilde{g} d\widetilde{\mu}$$

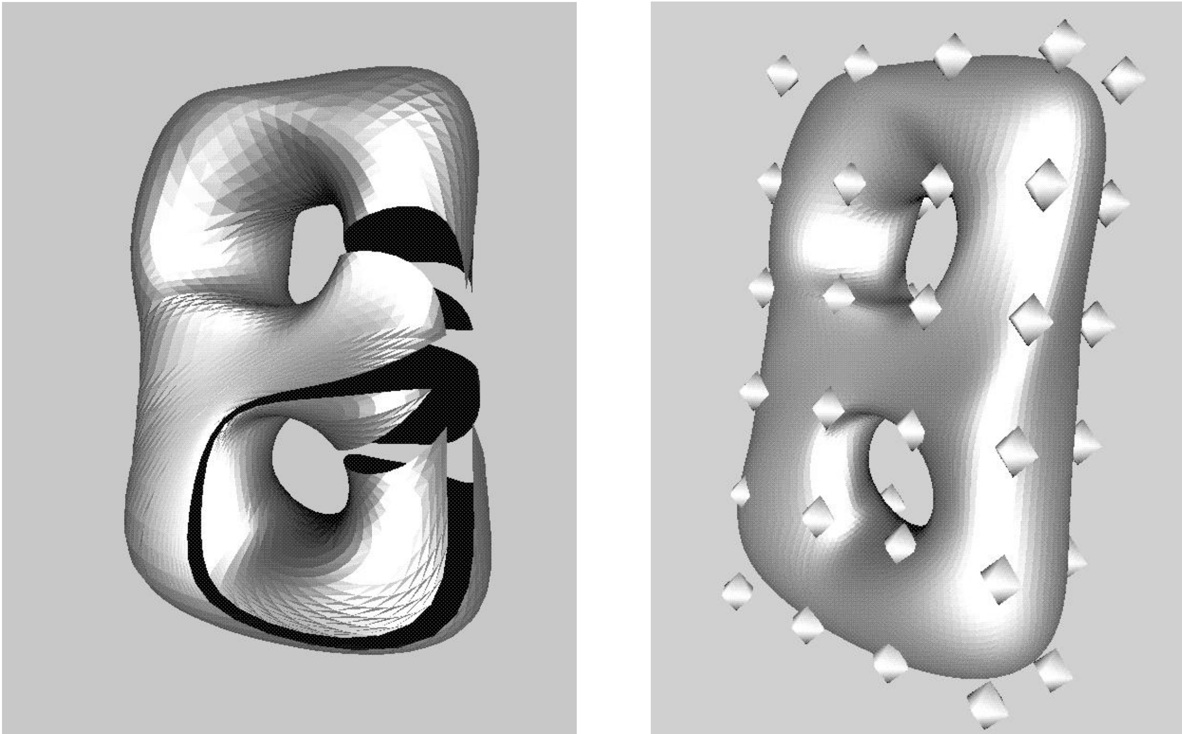

Fig. 11. (a) Gluing the boundary of an octogon together yields a surface of genus 2, (b) C^∞-approximation of a polyhedron.

in the well known way. The resulting norm will be denoted by $\|f\|_2$. One typical problem now is the following: Given a finite set $B = \{b_1, \ldots, b_n\}$ of basis functions and a function f on X/H, we seek a linear combination of the b_i such that

$$\|f - \sum \lambda_i b_i\|_2 \to \min .$$

This is a classical least squares problem and can easily be solved: If the b_i are orthonormal, $\lambda_i = (f, b_i)$ is the solution. If not, apply the Gram-Schmidt orthogonalization process. For *interpolation* we for instance introduce the space $L^2_\Delta(X/H)$ with the scalar product

$$(f, g)_\Delta = \int_{X/H} \Delta f \Delta g d\mu,$$

where Δ is the Laplace-Beltrami operator on X/H which is inherited from X. This allows us to find in the linear solution space of the interpolation problem $\sum \lambda_i b_i(x_j) = c_j$ a solution of *minimal energy*. It is also possible to extend interpolation schemes which have been successfully employed for sphere-like surfaces to the linear models of the S^2, E^2 and H^2, for instant the hybrid patch of (Liu and Schumaker).

Modeling and *visualization* of compact surfaces is possible in the following way: A closed surface in $\mathbb{R}^3$ can be seen as an embedding (or, at least, an immersion) of an abstract closed surface X/H into $\mathbb{R}^3$. For each abstract point $p \in X/H$ three coordinate values $x_1(p)$, $x_2(p)$ and $x_3(p)$ are given. This means that we have three real functions x_1, x_2 and x_3 whose domain is X/H. Equivalently, we have three H-invariant functions $\widetilde{x}_i$ whose domain is X. They have the property that the (x_1, x_2, x_3)-image of X/H is homeomorphic to X/H. Thus approximation, interpolation and modeling of surfaces is nothing but approximation, interpolation and modeling of three separate coordinate functions.

Suppose we are given a polyhedron with vertices $p_1, \ldots, p_n$ and we seek a C^∞ approximation to it. We describe our solution to this problem, which is typical for the sort of problems arising in this context. The algorithm is the following:

1) Cut the polyhedron along four closed curves passing through one fixed base point, such that the resulting surface becomes simply connected. An example of such a cutting is shown in Figure 11a. The cuts are in correspondence to the fundamental polygon which is shown in Figure 9a.
2) Find finitely many points $q_1, \ldots, q_n$ in the fundamental octogon corresponding to the appropriate group H with the following property: If we construct a group-invariant triangulation with vertices $h(q_i)$, $i = 1, \ldots, n$, $h \in H$, then this triangulation, when factored to the orbifold, is combinatorically equivalent to the triangulation of the polyhedron.

   This triangulation need not necessarily consist of triangles, it can also be a tesselation with n-gons of different shape and different number of vertices. The vertices do not necessarily have to lie inside the fundamental polygon. The cuts and the 8-gon are merely a guide where to put the q_i. For instance the tesselation shown in Figure 10 after factoring is combinatorically equivalent to the polyhedron shown in Figure 11 after subdividing each of the squares in four parts.
3) Optimize the triangulation/tesselation with respect to appropriate criteria. For instance we can try to optimize the shape of the faces of the triangulation/tesselation. In our case, the faces of the polyhedron are squares, so we want the faces of the tesselation be as square-like as possible. Because not for all vertices the number of faces containing this vertex equals four, we have to compromise.
4) Find a one-to-one correspondence between X/H and the polyhedron, or, equivalently, find a covering map from X to the polyhedron which is compatible with the triangulation. In our case this is done easily by mapping the 4-gons of the tesselation to the squares of the polyhedron in the obvious way.
5) To each point of X we assign the three coordinate values x_1, x_2, x_3 of its corresponding point on the polyhedron. This defines three continuous H-invariant functions on X.
6) Approximate the x_i by functions y_i which are linear combinations of C^∞ basis functions, for instance Gaussians.
7) Use the three functions y_1, y_2 and y_3 as coordinate functions of a surface

in $\mathbb{R}^3$ whose parameter domain is X/H or X depending on the level of abstraction. This is how Figure 11b was made.

If the correspondence between the points $q_1, \ldots, q_n$ of X/H and the vertices $p_1, \ldots, p_n$ is established, interactive modeling of polyhedra of similar shape is easy. We can construct the correspondence between X/H a further polyhedron P of the same shape by finding a correspondence between the model polyhedron and P. This is especially trivial if P combinatorically is just a *refinement* of the model polyhedron. The approximation problem for P is then just the problem of approximation of three new coordinate functions.

If the basis functions b_i are already orthonormal, approximation can be done very quickly. Moving the vertices of the polyhedron, which can now be seen as *control points* of the surface, influences the approximant surface. Depending on the type of basis function, the influence will be local or global. As Gaussians decrease rapidly, and in addition can be multiplied by compactly supported bump functions to become compactly supported without essentially changing their global shape, we have local control. For implementation purposes, the basis functions with compact support are very convenient, because the handling of infinite sums can be avoided completely.

Modeling of C^r surfaces with polynomial coordinate functions is possible if we choose the b_i as simplex splines or homogeneous DMS-splines. This gives an algorithm which makes it possible to model surfaces of arbitrary differentiability, of arbitrary genus, over an arbitrary triangulation, without any boundary and gluing conditions. A more detailed theory and further examples of this can be found in (Wallner, 1996).

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