GEOMETRIC CRITERIA FOR GOUGE–FREE THREE–AXIS MILLING OF SCULPTURED SURFACES

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Abstract

The paper presents a geometric investigation of collision-free 3-axis milling of surfaces. We consider surfaces with a global shape condition: they shall be interpretable as graphs of bivariate functions or shall be star-shaped with respect to a point. If those surfaces satisfy a local millability criterion involving curvature information, it is proved that this implies globally gouge-free milling. The proofs are based on general offset surfaces. The results can be applied to tool-motion planning and the computation of optimal cutter shapes.

INTRODUCTION AND FUNDAMENTALS

Although NC machining as one of the key manufacturing technologies has received a lot of attention both from the technological and algorithmic point of view, the mathematical foundations of this field have seen comparatively little progress. Collision-free tool path planning is mostly based on time-consuming interference checking algorithms, since mathematical results which would simplify this task are not available. The need for theoretical research on the mathematical foundation of 3-axis machining of sculptured surfaces has recently been pointed out by Choi, Kim and Jerard (1997).

As a contribution in this direction, the present paper deals with a geometric study of collision-free 3-axis milling of surfaces. Considering the accessibility of the design surface Φ with the tool, it seems to be justified to restrict our study at first to those surfaces Φ which can be interpreted as "function graphs". This means that we can find at least one parallel projection, where all projection rays intersect the surface in at most one point, without being tangent to it. If such a projection direction is given by the z-axis of a Cartesian (x, y, z)-coordinate system, the design surface Φ may be written as $z = f(x, y)$, i.e., as graph of a bivariate function f, defined over some domain D in the (x, y)-plane β. The z-direction will also be identified with the axis direction of the cutting tool.

During the milling process, the cutting tool is spinning around its axis. At a fixed axis location, it generates a surface of revolution Σ, which will be called the *cutter*. During 3-axis milling, the axis of the cutter does not change its direction, which shall be the z-direction. Moreover, we assume the cutter to be a *strictly convex C^2 surface*. Strict convexity means that its tangent planes touch the surface just at one point; here we require in addition that all points are elliptic surface points. For concepts from elementary differential geometry, see (do Carmo, 1976). We will later allow curvature discontinuities and even tangent plane discontinuities along some coaxial circles of the cutter.

In order to mill a surface Φ, the cutter has to touch Φ in all positions of a milling course. If the cutter touches Φ at the point p, we denote this position of the cutter by $\Sigma(\mathsf{p})$. Since the design surface is graph of a bivariate function, the part of the cutter that becomes active at some instant may also be written as graph of a function. In an initial position Σ of the cutter, this function shall be $z = s(x, y)$. This initial position is defined by the condition that a previously fixed reference point r of the cutter axis is situated at the origin. Both surfaces Φ and Σ shall be oriented such that their normals have positive z-coordinate. The position $\mathsf{r}(\mathsf{p})$ of the reference point at cutter position $\Sigma(\mathsf{p})$ will be called *cutter location point*. We define a mapping $\mu : \Phi \mapsto \Sigma$, which shall map a point $\mathsf{p} \in \Phi$ to the unique point $\mu(\mathsf{p}) \in \Sigma$, whose oriented surface normal is parallel to the oriented normal of Φ at p. Then translation of Σ by the vector

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$\mathsf{p} - \mu(\mathsf{p})$ transforms Σ into $\Sigma(\mathsf{p})$.

If $\mathsf{x} = (x, y, z)$ is a point, we denote its projection $(x, y, 0)$ onto the (x, y)-plane β by $\widetilde{\mathsf{x}}$. Also, the projection of the mapping μ onto β is denoted by $\widetilde{\mu} : \widetilde{\mathsf{p}} \mapsto \widetilde{\mu(\mathsf{p})}$. We say that the surface Φ is *locally millable with* Σ at p, if and only if (i) there is a neighbourhood of p such that Φ and the convex hull of $\Sigma(\mathsf{p})$ (cutter seen as a solid) have no point in common in this neighbourhood except p; and (ii) that the two surfaces $\Sigma(\mathsf{p}), \Phi$ do not have second order contact at p. This is not necessary for a locally collision free position, but checking local collision would require a higher order analysis if we allowed second order contact. For the sake of efficient algorithms, we therefore make this restriction, which is negligible from the practical point of view.

A surface Φ is said to be *globally millable with* Σ, if and only if for all points p of Φ the corresponding cutter $\Sigma(\mathsf{p})$ does not intersect the interior of the solid which lies below Φ. In this case *no gouging can occur.* For smooth Φ and Σ this means for example that they must not intersect transversally in a curve. The existence of points where they touch each other is allowed.

Local millability at all points is obviously a necessary condition for global millability, but it is not sufficient in general. We will study *conditions under which local millability implies global millability.* Note that local millability needs to be checked anyway. We will see in the following section that this test can be performed efficiently by looking at curvatures, whereas checking for global millability is much more time consuming. There is one known special case for such a global millability result, namely if cutter and design surface are convex (Pottmann, 1997). It is a minor extension of a theorem of W. Blaschke (1956), which says that a sphere Σ can roll freely inside a closed convex C^2 surface Φ, if and only if the radius of the sphere does not exceed the smallest principal curvature radius (reciprocal value of principal curvature) of Φ.

LOCAL MILLABILITY

Since our approach to gouge-free milling is based on local millability, we need to discuss this criterion in detail. We will split the discussion according to the smoothness of cutter and design surface.

Second Order Differentiability

Let us first study the case where design surface and cutter are G^2 in CAGD terminology (Hoschek and Lasser, 1993). This is equivalent to f and s being C^2. We will always assume that the cutter is milling the surface from its upper side, i.e., the side which the positive z-axis is pointing to. Now, the second directional derivatives of the function s are positive everywhere. Thus, local millability at $\mathsf{p} = (x, y, f(x, y))$ means that the second directional derivatives of f at x are less than the corresponding second directional derivatives of s at $\widetilde{\mu}(\mathsf{x})$. The second directional derivative $f_{,\mathsf{vv}}$ of f in direction v ($\|\mathsf{v}\| = 1$) is computed with the matrix of second partial derivatives, the *Hessian* H_f of f,

$$H_f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix},$$

as

$$f_{,\mathsf{vv}} = \mathsf{v}^T \cdot H_f \cdot \mathsf{v}.$$

For local millability, all second directional derivatives of s are greater than the corresponding second directional derivatives of f, i.e.,

$$\mathsf{v}^T \cdot H_s \cdot \mathsf{v} - \mathsf{v}^T \cdot H_f \cdot \mathsf{v} > 0.$$

We have now deduced the fact that *local millability is equivalent to the matrix $M = H_s - H_f$ being positive definite* (Marciniak, 1991). It is well known that this is equivalent to

$$\det M > 0, \quad \text{and} \quad s_{xx} - f_{xx} > 0. \tag{1}$$

Here, f_{xx} and s_{xx} are the second partial derivatives of f and s with respect to x. The condition $s_{xx} - f_{xx} > 0$ can be replaced by $f_{,\mathsf{vv}} > 0$ for an arbitrary direction v.

If $\det M > 0$ and $s_{xx} - f_{xx} < 0$, the two surfaces Φ and $\Sigma(\mathsf{p})$ also do not intersect locally except at p, but Φ locally is contained in the convex hull of $\Sigma(\mathsf{p})$ and thus interferes with the cutter as a solid.

If $\det M < 0$ at some point, the matrix $M = H_s - H_f$ defines an *indefinite* quadratic form $\mathsf{x}^T \cdot M \cdot \mathsf{x}$. Such a form has two independent null directions $\mathsf{v}_1, \mathsf{v}_2$. For those, the directional derivatives of s and f agree. The corresponding cutter $\Sigma(\mathsf{p})$ and the design surface Φ possess two different tangents t_1, t_2 in the common tangent plane at p, whose normal curvatures of both surfaces agree (by Meusnier's theorem). In other words, their Dupin indicatrices in the tangent plane intersect in 4 points. The line segments which join them with p are the tangents t_1, t_2. It is well known that locally the surfaces intersect transversally in two regular curves which intersect at p, having tangents t_1, t_2 there.

Parametrized Surfaces

Instead of working with second order directional derivatives one can compare (by Meusnier's theorem) the signed Euclidean normal curvatures at corresponding points $\mathsf{p} \in \Phi$ and $\mu(\mathsf{p}) \in \Sigma$. We do not elaborate this, but make an important comment on the use of (1). In practical applications, one cannot assume that the surface Φ is given as graph of a known bivariate function f. More likely, one will have some parametrization $\mathsf{f}(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2))$. We would like to compute the Hessian H_g of the function g, which is defined by the condition that both f and $(x, y, g(x, y))$ define the same surface. It can be computed with well known formulae from isotropic differential geometry of surfaces (Sachs, 1990). There, the desired second directional derivatives have the meaning of normal curvatures. We just state the result (which could also be verified directly without knowledge of isotropic geometry). Define $\widetilde{\mathsf{f}} := (x(u_1, u_2), y(u_1, u_2), 0)$ and the isotropic counterparts to the coefficients of the first and second fundamental form via

$$G_{ij} := \widetilde{\mathsf{f}}_{,i} \cdot \widetilde{\mathsf{f}}_{,j}, \quad H_{ij} := \frac{\det(\mathsf{f}_{,1}, \mathsf{f}_{,2}, \mathsf{f}_{,ij})}{\det(\widetilde{\mathsf{f}}_{,1}, \widetilde{\mathsf{f}}_{,2})}.$$

Here, $\cdot_{,i}$ denotes partial differentiation with respect to u_i. Because of the partial reduction to $\widetilde{\mathsf{f}}$, these formulae are computationally somewhat less expensive than the corresponding formulae of Euclidean surface theory. A direction vector $\mathsf{u} = (\dot{u}_1, \dot{u}_2)$ in the (u_1, u_2)-parameter domain defines a tangent vector $\mathsf{t} = \mathsf{f}_{,1}\dot{u}_1 + \mathsf{f}_{,2}\dot{u}_2$ of the surface. Given its projection $\mathsf{v} = \widetilde{\mathsf{t}}$, we can reconstruct u and then the corresponding isotropic normal curvature

$$\kappa_n^I(\mathsf{v}) = \frac{H_{11}\dot{u}_1^2 + 2H_{12}\dot{u}_1\dot{u}_2 + H_{22}\dot{u}_2^2}{G_{11}\dot{u}_1^2 + 2G_{12}\dot{u}_1\dot{u}_2 + G_{22}\dot{u}_2^2} = \frac{\mathsf{u}^T \cdot H \cdot \mathsf{u}}{\mathsf{u}^T \cdot G \cdot \mathsf{u}}. \quad (2)$$

Let $P = (\widetilde{\mathsf{f}}_{,1}, \widetilde{\mathsf{f}}_{,2})$. Then $G = P^T \cdot P$ and $\mathsf{v} = P \cdot \mathsf{u}$. If the parametrizations $\mathsf{f}(\mathsf{u}) = (f_1(\mathsf{u}), f_2(\mathsf{u}), f_3(\mathsf{u}))$ and $(x, y, g(x, y))$ describe the same surface, the isotropic normal curvatures of a tangent must be the same, when computed with (2) in both ways. This leads to $(\mathsf{v}^T \cdot H_g \cdot \mathsf{v})/(\mathsf{v}^T \cdot \mathsf{v}) = (\mathsf{u}^T \cdot H \cdot \mathsf{u})/(\mathsf{u}^T \cdot G \cdot \mathsf{u})$, which implies

$$H_g = (P^{-1})^T \cdot H \cdot P^{-1}. \quad (3)$$

Now we have computed the Hessian of Φ. Analogously, we can compute the Hessian of the cutter Σ, and then we are able to apply (1).

Remark

To visualize the test for local millability, we may use the isotropic Dupin indicatrices in the x, y-plane. At the point $\mathsf{m} = \widetilde{\mathsf{p}}$ they are defined by

$$\begin{aligned} i_\Phi: &\quad (\mathsf{x} - \mathsf{m})^T \cdot H_f \cdot (\mathsf{x} - \mathsf{m}) = 1, \\ i_\Sigma: &\quad (\mathsf{x} - \mathsf{m})^T \cdot H_s \cdot (\mathsf{x} - \mathsf{m}) = 1. \end{aligned} \quad (4)$$

These are obviously radial diagrams with center m for $1/\sqrt{f_{,\mathsf{vv}}}$ and $1/\sqrt{s_{\mathsf{v},\mathsf{v}}}$, radially plotted over the corresponding directions $\mathsf{v} = (\mathsf{x}-\mathsf{m})/||\mathsf{x}-\mathsf{m}||$. Due to our assumptions, the indicatrix of the cutter Σ is always a real ellipse. A tangent direction of the design surface, however, yields a real point on i_Φ only for a positive second directional derivative. Note that nonpositive directional derivatives are not interesting anyway, since they are always exceeded by the corresponding second directional derivative of the cutter. The indicatrix of Φ is either the empty set (p elliptic or parabolic and Φ, Σ locally at different sides of the tangent plane τ; or p flat), a real ellipse (p elliptic and both Φ, Σ locally at the same side of τ), a pair of parallel lines (p parabolic and both surfaces locally at the same side of τ) or a hyperbola (p hyperbolic). Let us define the interior of i_Φ as the open set of the plane with boundary i_Φ, which contains the center m. (For an empty i_Φ, the interior then is the entire plane $\mathbb{R}^2$). Then, *we have local millability if and only if i_Σ lies in the interior of i_Φ.*

Alternatively, we can use the Euclidean Dupin indicatrices. Note that these indicatrices have to be defined by *signed* normal curvatures, where the positive sign is given by the direction of the normal vector of the surface.

Piecewise Curvature Continuity

In practical applications, the design surface is often just C^1, but composed of patches of higher smoothness. We will therefore extend our investigation to surfaces which are C^1 and composed of C^2 patches and refer to them as *C^1, piecewise C^2 surfaces.* At a point p, at which k patches meet, there are usually k different curvature behaviours. Local millability is defined by local millability for all k patches. For such a cutter it is possible to use criterion (1) to check local millability. At a contact point, every patch of the cutter has to be in locally millable position against every patch of the design surface.

Edges

Let us first study edges of the *design surface*, which shall be curves e, where two patches are meeting with different tangent planes. In order to apply the previous methods to surfaces with edges, we smooth them, for example with pipe surface parts of radius ϵ (as in the construction of blending surfaces in geometric design (Hoschek and Lasser, 1993)) and then let ϵ tend to zero.

Choose a point p in an edge e, which has two different tangent planes τ_1, τ_2, which belong to two adjacent surfaces Φ^1 and Φ^2. They shall be represented by $z = f^i(x, y)$, $i = 1, 2$. Let us pick any direction v in the (x, y)-plane, which points from the Φ^1-side to the Φ^2-side of the edge tangent's projection (Given by the unit tangent vector e). Let us compute the difference

$$\delta := f^2_{\mathsf{v}} - f^1_{\mathsf{v}}.$$

of derivatives. Because v, e are inearely independent, we have $\delta \neq 0$. Two different cases have to be distinguished:

1. $\delta > 0$. In this case, any ϵ-pipe surface smoothing the edge would have at least a region of positive second directional derivatives across the edge whose inverses (curvature radii) tend to 0 for $\epsilon \to 0$. This means that the edge can never be millable with a smooth cutter and there will necessarily occur an uncut (often called a "concave uncut" in the NC machining literature (Choi et al., 1997)). Throughout this paper, we will therefore assume that for those edges admissible blend surfaces are constructed or uncut is allowed. Thus, we may exclude these edges from further considerations.

2. $\delta < 0$. Now, the critical second directional derivatives are negative, such that there is a chance to mill such an edge with a smooth cutter. To check for millability at p, the following has to be done: Define a set of *admissible tangent planes* at p, for which the first directional derivative f_{v} lies between f_{v}^1 and f_{v}^2.

   For any admissible oriented tangent plane one has to consider the point of the cutter Σ with a parallel oriented tangent plane. We may view μ as a multi-valued correspondence between parallel oriented surface elements (points plus oriented tangent planes) of Φ and Σ.

   Then, these points $\{\mu(\mathsf{p})\}$, lying on the contour of Σ for parallel projection in direction of the edge tangent, are just the μ-images of the design surface elements, defined by the admissible tangent planes at p. Moreover, for any of those tangent planes, we can compute a normal curvature of Φ at p in direction of the edge tangent (Meusnier's theorem). Millability at p then requires that these normal curvatures are less than the corresponding normal curvatures of the cutter at points of $\mu(\mathsf{p})$. Analogously, one can work with second directional derivatives $f_{\mathsf{e},\mathsf{e}}$ (isotropic counterpart of Meusnier's theorem (Sachs, 1990)):

   $$f_{\mathsf{e},\mathsf{e}} = \kappa\alpha.$$

   Here, κ is the Euclidean curvature of the projection $\widetilde{e}$ of the edge at $\widetilde{\mathsf{p}}$. With $z = g(x, y)$ as osculating plane of the edge curve at p, $z = f(x, y)$ as admissible tangent plane there, and n as unit normal vector of $\widetilde{e}$, α is defined by

   $$\alpha := g_{\mathsf{n}} - f_{\mathsf{n}}.$$

   Now, *millability at* p *requires*

   $$s_{\mathsf{e},\mathsf{e}} > f_{\mathsf{e},\mathsf{e}} \tag{5}$$

   for all admissible tangent planes τ *at* p. For arbitrary cutter shapes, this test of local millability at a point of an edge is computationally as expensive as testing local millability of a smooth surface along a curve segment. It is simple, however, for the frequently applied ball cutters, where one will compare Euclidean normal curvatures. If the Euclidean curvature of the edge curve is less than the normal curvature of the ball cutter, local millability is guaranteed.

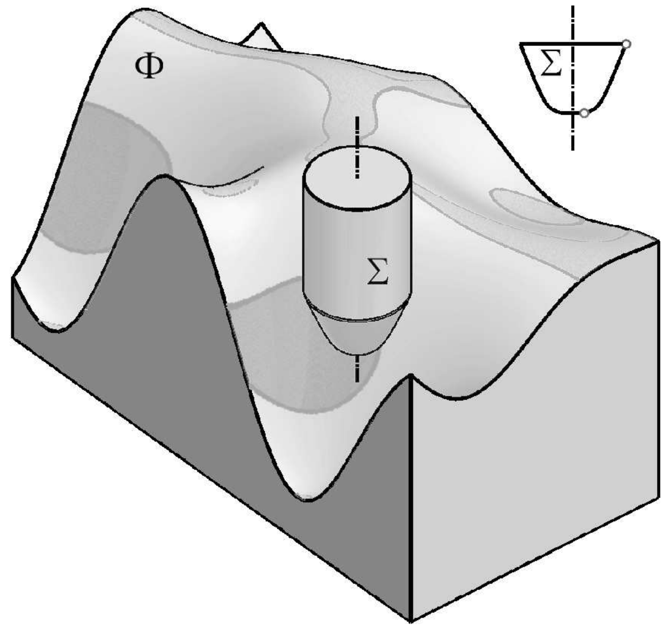

Figure 1: Regions (dark) of Φ which correspond to edges of the cutter

Analogously, we may allow *edges of the overall strictly convex cutter.* Along such an edge, which must be a coaxial circle c, we have two different coaxial tangent cones. The points of the design surface, at which the tangent planes are parallel to those of these cones, are two *isophotes* for z-parallel projection (see Fig. 1). They lie on the boundary of the design surface region, at which the cutting edge c becomes active. If the design surface is smooth, we already know how to test for local millability.

Finally, assume that we have an admissible surface element (p, τ) on an edge of the design surface (with $\delta < 0$) and the extended mapping μ leads to a point $\mu(\mathsf{p})$ on a cutter edge c. Now, only in the case of parallel edge tangents at p and $\mu(\mathsf{p})$ a millability test has to be made: it is required that (5) holds.

Example

A circular cutting edge c occurs for a *flat end mill.* This cutter shape is contained in our study if the flat part (circular disk defined by c) does not become active during milling. If there are points of the design surface with tangent plane

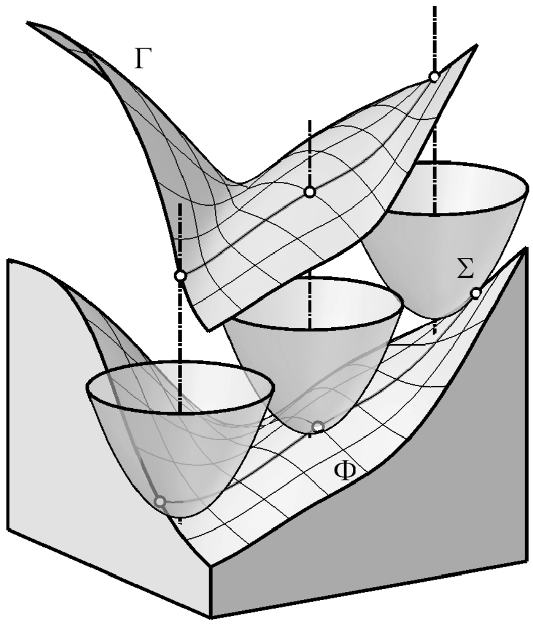

Figure 2: Design surface Φ (bottom), cutter Σ and general offset surface Γ (top)

orthogonal to the cutter axis, we may use an auxiliary cutter, where the circular disk is covered by a spherical cap and apply our results. The sphere radius can be arbitrarily large, which gives an arbitrarily good approximation to the actual cutter.

GENERAL OFFSET SURFACES

Consider the touching positions $\Sigma(\mathsf{p})$ of the cutter for all points p of the design surface Φ. Then, the set of cutter location points $\mathsf{r}(\mathsf{p})$ (reference point positions) is a surface Γ, which shall be referred to as *general offset surface* of Φ with respect to Σ (see Fig. 2). A parametrization of Γ using Φ's surface parameters $\mathsf{x} = (x, y) \in D \subset \mathbb{R}^2$ is given by

$$\mathsf{g}(x, y) = (\mathsf{x}, f(\mathsf{x})) - (\tilde{\mu}(\mathsf{x}), s(\tilde{\mu}(\mathsf{x}))). \tag{6}$$

¿From this, we easily see that oriented tangent planes at corresponding points of Φ, Σ and Γ are parallel. Moreover, the general offset appears as Minkowski sum of the *oriented* surfaces Φ and $-\Sigma$. The symbol $-\Sigma$ is used for the *reflected cutter*, which may be described by $(-x, -y, -s(x, y))$ in the sense of (Pottmann, 1997). This extension of the Minkowski sum of two domains is also referred to as *convolution surface* (see (Lee et al., 1998), where the curve case is treated and trimming of the convolution curve to the boundary of the classical Minkowski sum is studied). General offset surfaces have first been introduced by Brechner (1992) and studied more carefully from a differential geometric and algebraic point of view by Pottmann (1997). Here we will need a close relation between local millability and regularity of general offset surfaces. This result has so far been formulated only within the framework of relative differential geometry (Pottmann, 1997) and therefore we will present here an elementary derivation and extensions to practical situations. Again we will first treat the case of second order differentiability and later weaken this assumption.

C^2 Surfaces

Let us parametrize the active part of Σ by the first partial directional derivatives of s,

$$\mathsf{u} = (u, v), \quad u = \frac{\partial s}{\partial x} =: s_x, \; v = \frac{\partial s}{\partial y} =: s_y.$$

This is possible locally because the Jacobian matrix of the parameter transform equals the Hessian H_s, which is invertible. If the function s is only C^1, but piecewise C^2, this is also possible because the strict convexity of Σ forces strict monotonicity of the partial derivatives. The resulting parametrization shall be

$$\mathsf{s}(\mathsf{u}) = (s_1(\mathsf{u}), s_2(\mathsf{u}), s_3(\mathsf{u})).$$

Then, the general offset is

$$\begin{aligned} &\mathsf{g}(x, y) = (x, y, f(x, y)) - \mathsf{s}(\mathsf{u}(x, y)) \\ &\text{with } \mathsf{u}(x, y) = (f_x(x, y), f_y(x, y)). \end{aligned} \tag{7}$$

The projection of this surface onto β, i.e.,

$$\begin{aligned} &\tilde{\mathsf{g}}(\mathsf{x}) = \mathsf{x} - \tilde{\mathsf{s}}(\mathsf{u}) = (x - s_1(\mathsf{u}), y - s_2(\mathsf{u})) \\ &\text{with } \mathsf{u}(\mathsf{x}) = (f_x(\mathsf{x}), f_y(\mathsf{x})), \end{aligned} \tag{8}$$

is a mapping from D to $\mathbb{R}^2$, whose Jacobian matrix is computed by the chain rule as

$$J_{\tilde{\mathsf{g}}}(\mathsf{x}) = I - J_{\tilde{\mathsf{s}}}(\mathsf{u}(\mathsf{x})) \cdot H_f(\mathsf{x}).$$

Here, I denotes the 2×2 identity matrix. Differentation of the identities $x = s_1(s_x, s_y), y = s_2(s_x, s_y)$ with respect to x, y yields

$$I = J_{\tilde{\mathsf{s}}}(\mathsf{u}(\mathsf{x})) \cdot H_s(\mathsf{x}).$$

Since Σ is strictly convex, the Hessian H_s is invertible and we arrive at the important relation

$$J_{\tilde{\mathsf{g}}}(\mathsf{x}) = I - H_s^{-1}(\mathsf{x}) \cdot H_f(\mathsf{x}). \tag{9}$$

If a tangent plane of the general offset is well defined, it is parallel to the corresponding tangent planes of Φ and Σ and therefore it can never be parallel to the z-axis. This means that a singularity of the offset surface Γ at some point is characterized by a singular Jacobian $J_{\tilde{\mathsf{g}}}$. Note that we are speaking of local singularities which must be reflected in singular parametrizations; we are not yet speaking of self-intersections. At a local singularity there exists a nonzero vector $\mathsf{v} \in \mathbb{R}^2$ with $J_{\tilde{\mathsf{g}}} \cdot \mathsf{v} = \mathsf{o}$, i.e.,

$$H_s \cdot \mathsf{v} = H_f \cdot \mathsf{v}. \tag{10}$$

This can nicely be interpreted geometrically when using curvature theory of surfaces. Conjugate surface tangents of Φ belong to vectors v, w in the parameter domain with $\mathsf{w}^T \cdot H_f \cdot \mathsf{v} = 0$ and analogously for Σ. Thus, at corresponding points p and $\mu(\mathsf{p})$ of Φ and Σ there exist parallel tangents whose conjugate directions are the same. Of course also the corresponding signed normal curvatures agree.

It is also clear that $H_s\mathsf{v} = H_f\mathsf{v}$ implies $\mathsf{v}^T H_s \mathsf{v} = \mathsf{v}^T H_f \mathsf{v}$, so we have equality of the second directional derivatives $f_{,\mathsf{vv}}$ and $s_{\mathsf{v},\mathsf{v}}$. Any regular surface can locally be written as graph of a function, and we have not used the fact that Σ is a surface of revolution yet. Therefore, we have proved the following characterization of the existence of a local singularity, which uses only terms of Euclidean geometry, and is invariant against translations and rotations of the surfaces Φ and Σ.

Theorem 1 *The general offset surface of a regular C^2 surface Φ with respect to a strictly convex C^2 surface Σ possesses a singularity at a point belonging to associated points $\mathsf{p} \in \Phi$ and $\mu(\mathsf{p}) \in \Sigma$, if and only if there is a tangent vector v, whose signed normal curvature and conjugate direction are the same for both surfaces Φ and Σ.*

We want to show that *local millability implies*

$$\det J_{\tilde{\mathsf{g}}} > 0. \tag{11}$$

For a proof, we use the fact $\det H_s > 0$ to see that

$$\begin{aligned} \text{sign} \det J_{\tilde{\mathsf{g}}} &= \text{sign} \det H_s \ \text{sign} \det(I - H_s^{-1} \cdot H_f) \\ &= \text{sign} \det(H_s - H_f) = \text{sign} \det M. \end{aligned}$$

Local millability implies $\det M > 0$, which proves (11). Note, however, that (11) does not characterize local millability. It characterizes a position where the two surfaces $\Sigma(\mathsf{p})$ and Φ do not intersect locally, except at p.

We nicely recognize the situation which implies the singularities of the general offset (Theorem 1) as limit case between local interference of the two surfaces and a locally intersection free position of the surfaces. In the "singular position", the two Dupin indicatrices touch each other in two points whose joining line segment's direction vector is given by the vector v of Theorem 1.

Example

Let us have a look at *ball cutters*, where the active part Σ of the cutter is part of a sphere with radius R. Here, the general offsets are the classical offsets, which have been treated extensively in the CAD literature (see e.g. (Chen and Ravani, 1987; Hoschek and Lasser, 1993)). Theorem 1 is now equivalent to the well known fact that a singularity of the offset belongs to a surface point one of whose signed principal curvature radii equals R. Local millability means that all signed normal curvatures of Φ are less than $1/R$.

Curvature Discontinuities and Edges

We have seen that a general offset surface of a locally millable C^2 surface with respect to a C^2 cutter is free of local singularities. Let us now discuss locally millable surfaces and cutters, which may have curvature discontinuities and edges (C^0, piecewise C^2 surfaces).

Since the tangent planes of a general offset Γ are parallel to both the corresponding tangent planes of Φ and Σ, the tangent planes of Γ are well defined also for a C^1, piecewise C^2 surface Φ. With (11) there cannot occur edges of regression along the curves where the patches of the offset are joined, because this behaviour would stem from a sign change in the Jacobian determinant. Singularities inside the patches are already excluded by Theorem 1. Analogously, we can consider a C^1, piecewise C^2 cutter and get a further segmentation of the general offset according to the patch boundaries of the cutter. Again, this does not introduce local singularities.

Moreover, we may include everywhere locally millable edges in the design surface. This is easiest to see as follows. Construct Euclidean offset surfaces Φ_d and Σ_d of Φ and Σ at distance d, on the positive (upper) side of these surfaces. At points of an edge, use all admissible tangent planes for offset point construction; thereby one constructs a pipe surface smoothly blending the offsets to the adjacent surfaces. At a touching position of Φ and $\Sigma(\mathsf{p})$ at p, also the offsets $\Phi_d, \Sigma_d(\mathsf{p})$ touch each other, namely at $\mathsf{p} + d\mathsf{n}$ (n being the unit normal vector at p). Therefore, the general offset of Φ with respect to Σ agrees with the general offset of Φ_d with respect to Σ_d. If we choose d so small that the cutter remains smooth, both surfaces are smooth and thus the general offset is smooth by the considerations above. The same holds if the cutter has edges and the design surface has none; then we use offsets on the other side.

This, however does not work at edge-edge contacts. Here, if the tangents of the two edges are not parallel, the general offset surface locally equals the surface

$$\mathsf{f}(u, v) = \mathsf{e}_1(u) - \mathsf{e}_2(v) + \text{const.}$$

where $\mathsf{e}_1(u)$ is a parametrization of the edge in Φ and $\mathsf{e}_2(v)$

is a parametrization of the edge in Σ. Such a surface obviously is smooth if both edges are. On the other hand, it is easy to find an example of a non-smooth general offset surface, if the edge tangents are parallel (consider a design surface which is a surface of revolution).

We have arrived at a fundamental tool for the study of global millability.

Theorem 2 *Let Φ be a regular C^0, piecewise C^2 surface, which is locally millable with a strictly convex C^0, piecewise C^2 cutter Σ. If at edge-edge contacts the edge tangents are not parallel and the edges are smooth, then the general offset surface of Φ with respect to Σ is free of singularities.*

Let us point out again that the singularities are local ones in the sense of differential geometry, such as edges of regression and the vertex of a cone. Handling self-intersections of the general offsets is more complicated and tied to global millability, which will be investigated in the next section.

GLOBAL MILLABILITY CONDITIONS

Surfaces Representable as Graphs of Bivariate Functions

Let Φ be a surface, which is represented as graph of a compactly supported C^0, piecewise C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$. This situation occurs frequently in die and mould machining, where 3-axis milling is mostly employed. Of course, only part of the plane will be realized in practice, but this is not important in our context. We can now prove our first global result.

Theorem 3 *Let Φ be a surface, which is represented as graph $z = f(x, y)$ of a compactly supported, C^0, piecewise C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$. If Φ is everywhere locally millable with a strictly convex C^0, piecewise C^2 cutter Σ with z-parallel axis, then Φ is globally millable with Σ.*

Proof: At first, let us treat the case of C^1 surfaces Φ and Σ. We enclose the support of f in a sufficiently large disk $D \subset \mathbb{R}^2$ and call the resulting surface patch again Φ. During milling of the surface, the cutter is always in contact with the surface Φ. Due to our assumptions on Φ, the cutter can start at a gobally interference-free position (touching at a maximum of f or at a point sufficiently far away from the support of f). Before it reaches a position which has a global interference, it touches Φ in at least 2 different points p and q. This means that the reference point on the cutter axis twice is situated at the same point d of the general offset surface Γ, which it is generating. Since the tangent planes of $\Sigma(\mathsf{p}) = \Sigma(\mathsf{q})$ at p and q cannot be parallel, this shows that the general offset has two different tangent planes at d and therefore a self-intersection.

This, however, cannot be the case: The mapping from points on Φ to the corresponding points on Γ has a projection onto the x, y-plane, which is the mapping $\widetilde{\mathsf{g}} : D \to D$ given by (8). Note that the image of D under $\widetilde{\mathsf{g}}$ is D, since at points x outside the support of f, the cutter is touching the x, y-plane and thus the reference point lies on the z-parallel line through x, i.e., $\widetilde{\mathsf{g}}(\mathsf{x}) = \mathsf{x}$. Obviously, if we take D sufficiently large, no other projection of a reference point position can be outside D. Through each point $\mathsf{x} \in D$ we can draw a z-parallel line and place the axis of the cutter there. Moving the cutter along that line from a non-interfering position above Φ in negative z-direction, we must somewhere reach a position where it touches Φ. This means that any point $\mathsf{x} \in D$ occurs as projection of a reference point position and $\widetilde{\mathsf{g}} : D \to D$ is a surjective, C^1 mapping with $\det J_{\widetilde{\mathsf{g}}} > 0$ everywhere.

Because of compactness of D and the absence of singular values (cf. (Milnor, 1965)) the number of points in the preimage $\widetilde{\mathsf{g}}^{-1}(\mathsf{x})$ is constant, and therefore equal to 1, which implies that $\widetilde{\mathsf{g}}$ is indeed bijective and Γ is free of self-intersections.

In the presence of edges, we construct ϵ-pipe surfaces to smoothly round off design surface and cutter and obtain global millability. By equation (6), the error introduced onto the cutter position by smoothing tends to zero for $\epsilon \to 0$. Thus, even in the presence of edges no gouging can occur. □

A surjective, regular C^1 mapping $\widetilde{\mathsf{g}} : \mathbb{R}^2 \to \mathbb{R}^2$ is bijective, if and only if $||\mathsf{x}|| \to \infty$ implies $||\widetilde{\mathsf{g}}(\mathsf{x})|| \to \infty$ (see (Dieudonné), §16.28). This leads to the following result.

Theorem 4 *Theorem 3 also holds, if $f : \mathbb{R}^2 \to \mathbb{R}$ has a bounded gradient, $||\nabla f|| < A$, and if the cutter, represented as graph $z = s(x, y)$ possesses gradients of s, whose norms exceed A.*

Proof: The assumptions on the gradients (which of course have to be applied to each patch) make sure that there are tangent planes of the cutter which are steeper than all tangent planes of the design surface. Again, it is sufficient to treat the C^1 case. As above, we prove that $\widetilde{\mathsf{g}}$ is surjective and regular. Let us consider the circle on the cutter along which the gradients of s have norm A; then all parts of the cutter "above" this circle (of radius R) never become active. Even for a cutter that reaches to infinity (we will see later that this may be a useful thing for practical considerations concerning 5-axis machining), the distance between the projections of the cutter location point and the touching point is bounded by R,

$$||\widetilde{\mathsf{g}}(\mathsf{x}) - \mathsf{x}|| \leq R.$$

Therefore, we have the above mentioned condition for a bijective map g and again see that the surface is globally millable. □

Note that theorem 4 applies to a surface with a planar extension which is not normal to the cutter axis (could also be proved like theorem 4). More generally, the theorem is applicable if a bounded surface has an extension such that the assumptions hold. It is, however, not always possible and, even if posible, in general quite hard to construct such an extension. The problem with boundaries is the following: Even if the general offset is free of self-intersections, i.e., there are no positions where the cutter touches the design surface at two different points, there might be positions $\Sigma(\mathsf{p})$ which collide with the boundary and thus gouging at a region adjacent to the boundary occurs.

Therefore, when dealing with boundaries we make the following assumption, which is useful from the practical point of view. Consider a graph surface $z = f(x, y)$ over some connected, bounded domain D. We now append at the boundary curve of the surface a z-parallel cylindrical surface in negative z-direction. As an example, the extension may be part of the boundary of the final workpiece (solid). This surface plus its extension shall be millable by the cutter.

At this point, it is necessary to say more about the cutter. We have so far focussed only on the active part of it. It may be extended by a coaxial right circular cylindrical shaft, which does not become active during milling, but holds the tool. Thus, this extension cannot cause any trouble in applications for the kind of design surfaces, which we are treating here.

Boundary milling in our sense requires a cutter which has vertical tangent planes. It does not matter, if the actual cutter is not like this. In order to apply our previous considerations, embed it into such a cutter, which is extended in positive z-direction by a right circular cylinder of radius ρ. We now say that our surface boundary c is *locally millable*, if the introduced edge is locally millable. Note that the cylindrical extension of the cutter has to mill locally the cylindrical extension of the surface. In the projection to the x, y-plane this is a local interference check between the boundary curve $\widetilde{c}$ of D and a circle $\widetilde{d}$ of radius ρ, touching D from the outside. Thus, the curvature of $\widetilde{c}$ has to be greater than $-1/\rho$. Obviously, we can allow millable (convex) vertices in the boundary of D.

In order to mill the whole piece without gouging, the (outer) offset of $\widetilde{c}$ at distance ρ, which shall be closed by circular arcs at boundary vertices, needs to be free of self-intersections. We will now show that local millability plus

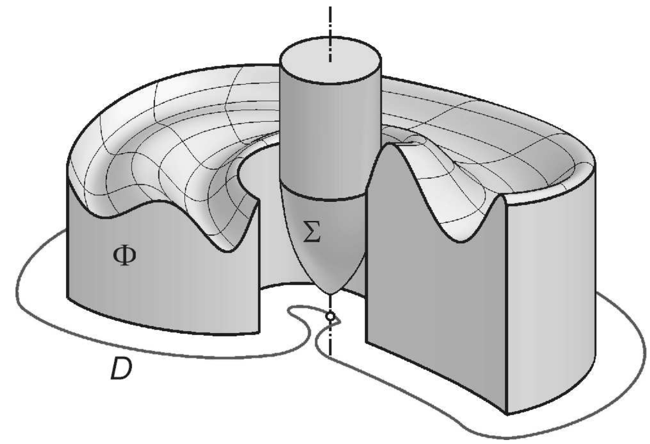

Figure 3: Milling a surface with boundary

this global condition are sufficient for gouge-free milling. A picture of this can be seen in Fig. 3.

Theorem 5 *Let Φ be a surface, which may be written as graph $z = f(x, y)$ of a C^0, piecewise C^2 function $f : D \to \mathbb{R}$ over a connected domain $D \subset \mathbb{R}^2$ with C^0, piecewise C^2 boundary. Moreover, we assume a strictly convex C^0, piecewise C^2 cutter Σ with z-parallel axis, which is extended by a cylindrical surface of radius ρ. If the boundary of D has an outer offset at distance ρ, which is free of self-intersections, and if Φ, including its boundary, is everywhere locally millable with Σ, it is also globally millable with Σ.*

Proof: It is sufficient to consider the edge-smoothed surfaces. Now, gouge-free milling is guaranteed if the general offset is free of self-intersections. As in the previous proofs of surjectivity we see that $\widetilde{\mathsf{g}}$ maps the connected compact domain D *onto* the compact and connected domain D_ρ, bounded by the outer offset of D at distance ρ.

We want to prove that the parametrization of Γ is regular everywhere. In the interior of D it is because of local millability. In the cylindrical part, the regularity of the outer offset curve implies the regularity of Γ. For a boundary point p there is a neighbourhood $U \subset \Phi$ of p and a plane β_1 such that the surface, including the cylindrical part, has a graph representation over U. Local millability has been characterized by the Euclidean curvatures of Φ and Σ and remains invariant when changing the base plane. So the regularity of Γ in a neighbourhood of p follows from local millability inside D and from the existence of the outer offset curve in the cylindrical part, as before.

The mapping $\widetilde{\mathsf{g}} : D \to D_\rho$ now is the projection of Γ's parametrization to β, and therefore is continuous and surjective. In the interior of D it is C^2 and regular. When restricted to the boundary, it is bijective. Now $\widetilde{\mathsf{g}}$ is a cov-

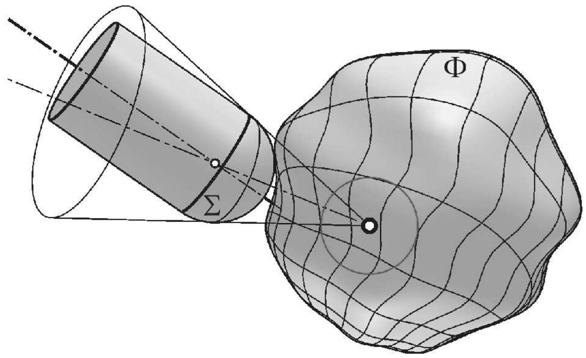

Figure 4: Milling of a star-shaped closed surface and 'shadow' of cutter

ering map, whose restriction to the boundary is injective, which implies the bijectivity of $\widetilde{\mathsf{g}}$ and the global millability of Γ. □

Star-shaped Surfaces

A connected compact domain $D \subset \mathbb{R}^3$ will be called *star-shaped* with respect to a point $\mathsf{c} \in D$, if all line segments $[\mathsf{c}, \mathsf{p}]$ with p in the boundary of D are contained in the interior of D (see Fig. 4). Its closed boundary shall be called a *star-shaped surface*, if it consists of C^2 patches, and all radial lines emanating from c intersect it transversally. This implies that the surface may be viewed as graph of a positive real-valued function f, defined on the unit sphere S^2. Each point $\mathsf{s} \in S^2$ is mapped to the point $\mathsf{c} + f(\mathsf{s}) \cdot \mathsf{s}$.

We are interested in milling the surface from the outside (exterior of D). At first, we assume the cutter Σ to be a compact, strictly convex surface (needs not necessarily be a surface of revolution), which is undergoing a translatory motion in 3-space. We prove the following result on the generated offset:

Theorem 6 *Let both Φ and Σ be C^0, piecewise C^2 surfaces, where Φ is star-shaped and Σ is strictly convex and compact. If Φ is locally millable with Σ from the outside, the corresponding general offset is free of self-intersections.*

Proof: We may assume smooth surfaces and place the reference point r in the interior of Σ. Let us project the mapping between points $\mathsf{p} \in \Phi$ and the corresponding points $\mathsf{r}(\mathsf{p}) \in \Gamma$ radially onto S^2 and call it f. Obviously, f is surjective. At a touching position, the cutter $\Sigma(\mathsf{p})$ and c lie at different sides of the tangent plane at p, and thus the line $\mathsf{cr}(\mathsf{p})$ is never tangent to Γ at $\mathsf{r}(\mathsf{p})$. With local millability this implies regularity of f. S^2 is a compact manifold and $\mathsf{f} : S^2 \to S^2$ is regular and surjective. Therefore, f is a covering map (cf. (Dieudonné)). Because of simple connectedness of S^2 it is bijective, and Γ must be free of self-intersections.

Another argument is the following: Shrinking Σ gives a homotopy of f to the identity map. Equ. 11 shows that f is orientation-preserving and there are only regular values. Therefore the oriented mapping degree counts the number of pre-images of points. This number is a homotopy invariant and therefore equals 1. The second argument is better than the previous one because it does not appeal to the simple connectedness of the sphere, which actually is not essential. □

We did not speak about global millability yet, because this question also involves the position of the axis of the cutter and the rest of the machinery holding the cutter, which in practical applications of course is present all the time. Let us mention just one application in this direction. Think of 5-axis milling with a ball cutter (hemisphere extended by a circular cylinder) and let us ask, where the axis of the tool can be placed. At a position $\Sigma(\mathsf{p})$ of the cutting sphere, we consider the half-cone with vertex c that touches $\Sigma(\mathsf{p})$ and trim away the part between c and the circle along which it touches the sphere. The remaining part of the cone and the part of the sphere which is visible from p, bound a region $S \subset \mathbb{R}^3$. Its points lie in the "shadow" of the sphere for a light source at c (see Fig. 4). It is not hard to see that a tool position which lies in this shadow region has no collisions with the design surface. To check for possible tool positions, one needs not use that actual cutter. Instead we can work with the largest sphere that can mill the surface from the outside. Its radius is $1/\kappa_{\max}$, where $\kappa_{\max}$ is the largest positive principal curvature of Φ (when we orient the surface Φ with normals pointing to the outer side). For a convex design surface and this orientation, all normal curvatures are nonpositive and the largest sphere degenerates to a plane.

Another application of theorem 6 concerns its planar counterpart.

Corollary 1 *Let D be a star-shaped planar domain with a C^0, piecewise C^2 boundary curve c, and let r be the radius of a circle that can locally mill D from the outside. Then, all exterior offset curves of D at distance $d \leq r$ are free of singularities and self-intersections.*

Proof: We extend $D \subset z = 0$ to a spatial domain $\overline{D}$ which is formed by points $\mathsf{x} + \lambda(0, 0, 1)$ for all points $\mathsf{x} \in D$ and $\lambda \in [0, 1]$. This cylindrical solid is star shaped with center $\mathsf{c} + (0, 0, 0.5)$ and millable with any sphere of radius $\leq r$. Its offset at distance $d \leq r$ contains part of a cylinder erected over the offset of c, which, by theorems 2 and 6 is free of singularities and self-intersections. □

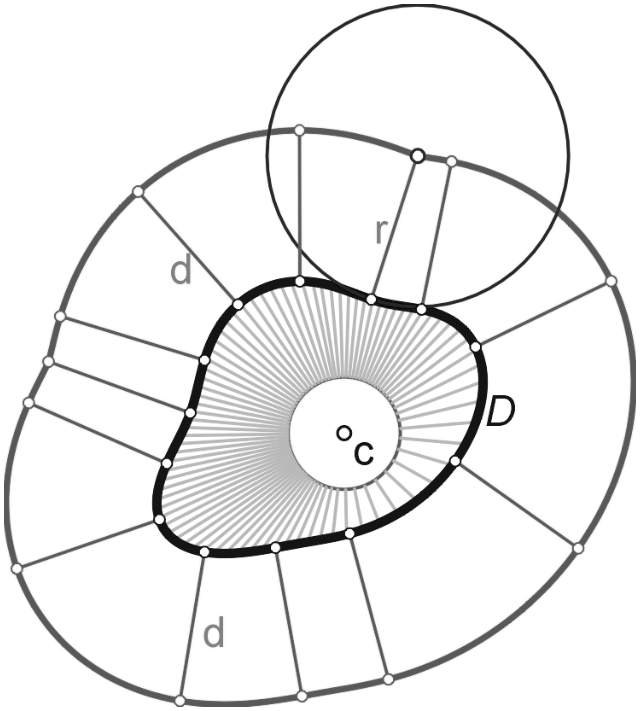

Figure 5: Offset of a star-shaped planar domain

One can prove this result directly in the plane, because the winding number of the analogue to $\tilde{\mathrm{g}}$ is easily found to be 1, and now regularity and surjectivity implies bijctivity. is useful in connection with theorem 5: For a star-shaped D the check for self-intersections of its offset at distance ρ is not necessary. This is illustrated in Figure 5, which shows the offset to the maximum distance for which no self-intersection occurs.

APPLICATIONS AND FUTURE RESEARCH

For three-axis milling of surfaces, representable as function graphs, we may construct an *optimal cutter* as follows. Let I be the interval of all inclination angles of the tangent planes of Φ, measured against the cutter's axis direction. For an $\alpha_i \in I$, we analyze the curvature behaviour at the surface points with tangent plane inclination angle α_i and compute a region in the tangent plane of the cutter, in which the Dupin indicatrix has to lie in order to guarantee local millability. This shall be done for a sufficient number of values α_i. The meridian curve of the cutter, defined as intersection with a plane through the axis, shall be described as a function $z = m(r)$ of the axis distance r. Then, for an admissible r and α (or equivalently $m'(r)$), the indicatrix region allows us to compute a lower bound for the second derivative $m''(r)$. This gives an inequality

$$m''(r) \geq F(r, m'(r)), \tag{12}$$

on which the computation of an optimal cutter can be based. This will be discussed in another paper.

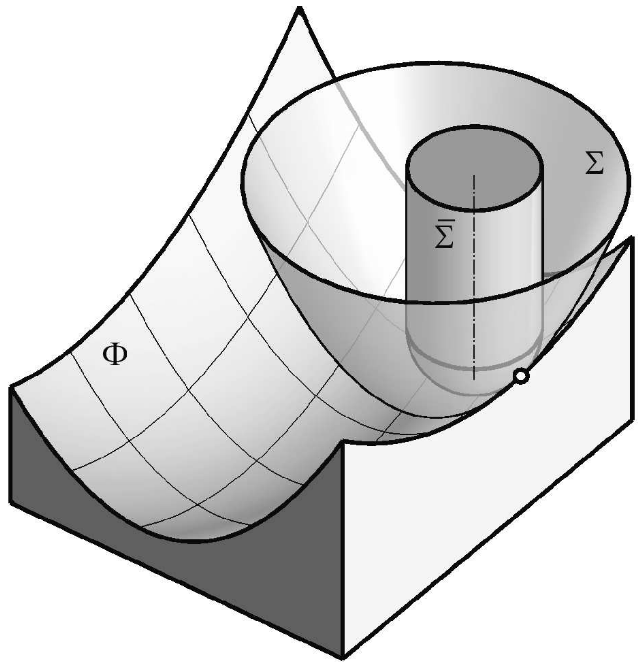

Figure 6: Protecting cutter

Even if an optimal cutter is not available in practice, it can serve as an auxiliary cutter in connection with the following *principle of the protecting cutter: if a surface can be globally milled with a cutter* Σ*, it can also be milled with any cutter* $\overline{\Sigma}$*, which can mill* Σ *from inside* (see Fig. 6). If Σ extends to infinity, we may even change the axis direction of $\overline{\Sigma}$ and get information for *tool-motion planning* for 5-axis machining. Moreover, we may determine how far one can safely go with a flat end mill. More generally, using a protecting cutter we can apply our results to cutter shapes which do not fulfil the requirements of our millability criteria. Thus, we have reduced collision tests against complicated surfaces to those against a convex surface of revolution. For parallel or intersecting rotation axes of actual and protecting cutter, this is just a planar problem.

We also plan to investigate other global millability results in the future. In this paper, we have treated surfaces which are graphs over the plane or the sphere. Other base surfaces seem to be useful as well.

In applications, the final design surface is not produced in one stage. Therefore, it seems to be interesting to investigate millability with respect to varying design surfaces with increasing detail. For that, we can use a *multiresolution approximation* with surfaces

$$z = f_i(x, y), \; i = 1, \ldots, N.$$

The approximation shall proceed from above, i.e.,

$$f_{i+1}(\mathsf{x}) \leq f_i(\mathsf{x}) \tag{13}$$

and N shall belong to the surface of the final part. For increasing i, we have more detail, higher curvatures and decreasing optimal cutter sizes. One easy way to include the restriction (13) in a multiresolution analysis (MRA) is to do the MRA without the restriction and then adding appropriate small constants to the f_i. The constants are easily 'milled away' in the next milling step. In this way, the optimization can simultaneously treat the whole milling procedure.

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