Variational Interpolation

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Abstract

We consider the problem of variational interpolation of subsets in Euclidean spaces and, more generally, in a Riemannian manifold. In the general case we derive a differential equation which solution curves must fulfill — in the case of Euclidean space we discuss existence and uniqueness of solutions (which are then piecewise cubic curves). We give an algorithm for the actual computation of a variational interpolant of convex polytopes.

1 Introduction

Cubic B-spline curves are frequently used for interpolation in Computer-Aided Geometric Design. One reason is that their definition and handling is rather simple, but they are flexible enough to allow interpolation with C^2 curves.

If we are given points $x_0, \ldots, x_n$, and parameter values $u_0 < \ldots < u_n$, then the unique cubic C^2 spline curve $c(u)$ with $c(u_i) = x_i$ and $c'''(u_0) = c'''(u_n) = 0$ is at the same time the C^2 curve with $c(u_i) = x_i$ and $\int \|c''(u)\|^2 du$ minimal.

In this paper we study a more general problem: the variational interpolation of subsets. Assume that X is a Riemannian manifold (e.g., Euclidean space), and that we are given real parameter values $u_0 < \ldots < u_n$ and *subsets* $X_0, \ldots, X_n$ of X. Consider curves $c : [u_0, \ldots, u_n] \to X$ such that

(i) c is smooth, c' is absolutely continuous (which implies that c'' exists almost everywhere), and $\|c''\|^2$ is integrable,

(ii) c assumes values $s(u_i) \in X_i$ for $i = 0, \ldots, n$.

(iii) c minimizes the functional $F_{u_0}^{u_n}(c) = \int_{u_0}^{u_n} \|c''\|^2$.

If X is Euclidean space, the prime denotes the ordinary derivative. If X is a general Riemannian manifold, the prime denotes the tangent vector of a curve (in the case of c'), or the covariant derivative of a vector field along a curve (in the case c'').

It is not surprising that this question is simpler in Euclidean space than it is for a general manifold. In the latter case, we will adress a slightly different problem and derive a differential equation which the solution curves must fulfill.

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In Euclidean space however, it is clear that any solution of the problem must be a cubic C^2 spline curve. Here we show that the problem is solvable for certain classes of subsets X. For some classes of subsets, we are able to discuss uniqueness of the solution, and also to compute the solutions explicitely.

2 Riemannian manifolds and the first variation

2.1 Definitions

First we introduct some notation. If X is a Riemannian manifold, the tangent space in $p \in X$ is denoted by T_pX. Tangent vectors are denoted by $U_p, V_p, \ldots$. The scalar product in T_pX is denoted by $\langle\, , \rangle_p$. We will omit the index p in the future. If $c : I \to X$ is a curve, c' or dc/dt denotes the tangent vector field of c. A mapping $V : t \mapsto V(t)$, with $V(t) \in T_{c(t)}X$ is called a vector field along c. Analogously, if $s(u,v)$ has values in X (i.e., s is a 2-surface in X), then a vector field $Z(u,v)$ along s is a mapping $(u,v) \mapsto Z(u,v)$ with $Z(u,v) \in T_{s(u,v)}X$. We use the symbols $\partial s/\partial u$ and $\partial s/\partial v$ to denote the tangent vectors of the u and v parameter lines.

We use the symmetric Riemannian connection for covariant differentiation of vector fields. The covariant derivative of a vector field $V(t)$ along a curve is denoted by DV/dt. Analogously, we use the symbols $DZ/\partial u$ and $DZ/\partial v$ for the covariant derivatives of the vector field $Z(u,v)$ along the u and v parameter lines. Recall that $D(\partial s/\partial u)/\partial v = D(\partial s/\partial v)/\partial u$, and that $d\langle V,W\rangle/dt = \langle DV/dt, W\rangle + \langle V, DW/dt\rangle$ for vector fields V, W along the same curve c.

The Riemannian curvature tensor R assigns to $u, v \in T_pX$ a linear endomorphism $R(u,v)$ of T_pX. Among other geometric properties, it measures the extent to which covariant differentiation of vector fields obeys Schwarz' rule: With the notations above,

$$\frac{D}{\partial u}\frac{D}{\partial v}Z - \frac{D}{\partial v}\frac{D}{\partial u}Z = R\left(\frac{\partial s}{\partial v}, \frac{\partial s}{\partial u}\right) Z. \tag{1}$$

For all $V, W \in T_pX$, the mapping $R(V,W)$ is linear and skew-symmetric with respect to the scalar product $\langle\, , \rangle_p$. R fulfills the relation $R(U,V) = -R(V,U)$, which is clear from the definition. Further,

$$R(U,V)W + R(V,W)U + R(W,U)V = 0, \quad \langle R(U,V)W, X\rangle = \langle R(W,X)U, V\rangle. \tag{2}$$

If X is a 2-surface, we have

$$R(U,V)W = \det(U,V) \cdot K \cdot W^{\perp}, \tag{3}$$

where $W^{\perp}$ means rotation of W about 90 degrees (the positive sense being defined in the same way as for the determinant of U and V), and K denotes Gaussian curvature.

Further, we use the notation $\Delta_{u_0}(f)$ to denote the jump of a piecewise continuous function f at $u = u_0$.

Definition 1 *We denote the class of curves $c : [u_0, \ldots, u_n] \to X$ which fulfill (i) and (ii) above with $\mathscr{C}$. The class of curves which are C^4 in the intervals $[u_i, u_{i+1}]$, but otherwise fulfill (ii) above will be denoted by $\mathscr{C}^4$.*

An *admissible variation* of a curve within the class $\mathscr{C}^4$ is a 2-surface $s(u,v)$ with domain $[u_0, u_n] \times (-\varepsilon, \varepsilon)$ in X such that $s(u,0) = c(u)$, such that all curves $s(u, v_0)$, $v_0 = \text{const}$, are contained in $\mathscr{C}^4$, and such that the mixed partial derivatives up to order three exist and are continuous.

The *variation vector field* of an admissible variation is the vector field $V(u) = \partial s/\partial v(u,0)$, and the *first variation* of the functional F we want to minimize is the expression

$$\frac{d}{dv}\Big|_{v=0} F(c_v), \quad \text{where } c_v(u) = s(u,v) \tag{4}$$

We will see that the first variation depends linearly on the variation vector field.

2.2 Computing the first variation

Lemma 1 *Assume that s is an admissible variation of the curve c within the class $\mathscr{C}^4$. If $V(u)$ is the variation vector field, then $F_{u_0}^{u_n}(c) = \int_{u_0}^{u_n} \|c''(u)\|^2 du$ is given by*

$$\begin{aligned} \frac{1}{2}\frac{d}{dv} F_{u_0}^{u_n}(c_v)\Big|_{v=0} &= \int_{u^0}^{u_n} \left\langle R(c', c'')c' + c'''', V \right\rangle + \left\langle V'(u_n), c''(u_n) \right\rangle - \left\langle V'(u_0), c''(u_0) \right\rangle \\ &- \left\langle V(u_n), c'''(u_n) \right\rangle + \left\langle V(u_0), c'''(u_0) \right\rangle + \sum_{i=1}^{n-1} \left\langle \Delta_{u_i} c''', V(u_i) \right\rangle . \end{aligned} \tag{5}$$

Proof: We compute the first derivative of $F_{u_i}^{u_{i+1}}$, using the fact that $c'' = D(\partial s/\partial u)/\partial u)$.

$$\begin{aligned} &\frac{1}{2}\frac{d}{dv}\int \left\langle \frac{D}{\partial u}\frac{\partial s}{\partial u}, \frac{D}{\partial u}\frac{\partial s}{\partial u} \right\rangle = \frac{1}{2}\int \frac{d}{dv}\left\langle \frac{D}{\partial u}\frac{\partial s}{\partial u}, \frac{D}{\partial u}\frac{\partial s}{\partial u} \right\rangle = \int \left\langle \frac{D}{\partial v}\frac{D}{\partial u}\frac{\partial s}{\partial u}, \frac{D}{\partial u}\frac{\partial s}{\partial u} \right\rangle \\ &= \int \left\langle \frac{D}{\partial u}\frac{D}{\partial v}\frac{\partial s}{\partial u} + R\left(\frac{\partial s}{\partial u}, \frac{\partial s}{\partial v}\right)\frac{\partial s}{\partial u}, \frac{D}{\partial u}\frac{\partial s}{\partial u} \right\rangle \\ &= \int \left\langle \frac{D^2}{\partial u^2}\frac{\partial s}{\partial v}, \frac{D}{\partial u}\frac{\partial s}{\partial u} \right\rangle + \int \left\langle R\left(\frac{\partial s}{\partial u}, \frac{D}{\partial u}\frac{\partial s}{\partial u}\right)\frac{\partial s}{\partial u}, \frac{\partial s}{\partial v} \right\rangle . \end{aligned} \tag{6}$$

We let

$$V(u) = \frac{\partial s}{\partial v}(u,0) \tag{7}$$

and evaluate at $v = 0$:

$$\frac{1}{2}\frac{d}{dv} F_{u_i}^{u_{i+1}}(0) = \int_{u_i}^{u_{i+1}} \left\langle R(c', c''), c', V \right\rangle + \int_{u_i}^{u_{i+1}} \left\langle V'', c'' \right\rangle . \tag{8}$$

We use $\langle V', c''\rangle' = \langle V'', c''\rangle + \langle V', c'''\rangle$ and $\langle V, c'''\rangle' = \langle V', c'''\rangle + \langle V, c''''\rangle$ to partially integrate this equation:

$$\begin{aligned} \frac{1}{2}\frac{d}{dv}F_{u_i}^{u_{i+1}}(0) &= \int_{u_i}^{u_{i+1}} \left\langle R(c', c''), c', V\right\rangle + \left\langle V', c'''\right\rangle \Big|_{u_i}^{u_{i+1}} - \int_{u_i}^{u_{i+1}} \left\langle V', c'''\right\rangle. \\ &= \int_{u_i}^{u_{i+1}} \left\langle R(c', c''), c', V\right\rangle + \left\langle V', c''\right\rangle \Big|_{u_i}^{u_{i+1}} - \left\langle V, c'''\right\rangle \Big|_{u_i}^{u_{i+1}} + \int_{u_i}^{u_{i+1}} \left\langle V, c''''\right\rangle \quad (9) \end{aligned}$$

By summing up we get the derivative of $F_{u_0}^{u_n}$. □

We say that a curve makes the function F *stationary* in the class $\mathscr{C}$, if the derivative of F is zero at $v = 0$ for all admissible variations.

2.3 A differential equation for minimizing curves

Theorem 1 *A curve $c : [u_0, u_n] \to X$ makes the functional $F_{u_0}^{u_n}$ stationary in the class $\mathscr{C}^4$, if and only if the conditions*

(a) $c'''' = -R(c', c'')c'$ in $[u_i, u_{i+1}]$,

(b) $\Delta_{u_i} c'''$ is orthogonal to the tangent space $T_{c(u_i)}X_i$ for $i = 1, \ldots, n-1$, and $c'''(u_0)$, $c'''(u_n)$ are orthogonal to the tangent spaces $T_{c(u_0)}X_0$, $T_{c(u_n)}X_n$, respectively.

(c) $c''(u_0) = c''(u_n) = 0$.

are satisfied. If c minimizes $F_{u_0}^{u_n}$ in the class $\mathscr{C}^4$, then it must make the functional stationary.

Note that in $\mathbb{R}^n$, the Riemannian curvature tensor is zero, so the differential equation $c'''' = -R(c', c'')c'$ simplifies to $c'''' = 0$. This is equivalent to c being picewise cubic.

Proof: (of Th. 1) It is obvious that conditions (a)–(c) ensure that the derivative of F is zero for all V, i.e, c makes the functional stationary.

It is straightforward to show the converse. Choose a C^∞ function f and a vector field V such that

$$f : [u_0, u_n] \to [0, \infty), \quad f(u_i) = f'(u_i) = f''(u_i) = 0, \quad f(x) = 0 \Rightarrow x = u_i. \qquad (10)$$

$$V(u) = f(u)(R(c', c'')c' + c''''). \qquad (11)$$

Then $s(u, v) = \exp_{c(u)}(v \cdot V(u))$ is an admissible variation with $\partial s/\partial v(u, 0) = V(u)$. By construction, the first variation of F is positive unless (a) is fulfillled.

To show (c), we choose any $V(u)$ such that $V(u_i) = 0$, but $V'(u_0) = c''(u_0)$ and $V'(u_n) = c''(u_n)$. Then the derivative of F is positive if not both $c''(u_0) = c''(u_n)$ vanish.

To show (b), we consider the orthogonal projections of the vectors $c'''(u_0)$, $\Delta_{u_1} c'''$, …, $\Delta_{u_{n-1}} c'''$, $c'''(u_n)$ onto the respective tangent spaces $T_{c(u_i)}X_i$, and call them $W_0, \ldots, W_n$. If we can choose an admissible variation whose variation vector field $V(u)$ fulfills $V(u_i) = W_i$, then the derivative of F is nonzero unless (b) is fulfilled. In the appendix it is shown how to choose this variation (Lemma 7). □

3 Euclidean space

We now consider variational interpolation of subsets in Euclidean space. First we show a result analogous to Th. 1 for the case the $X_0, \ldots, X_n$ are affine subspaces. Afterwards we consider different types of subsets — smooth submanifolds, compact subsets, and convex polytopes. In some cases we are able to describe the set of solutions completely.

3.1 Affine subspaces of $\mathbb{R}^n$

If the subsets $X_0, \ldots, X_n$ to be interpolated are affine subspaces (such as points, lines, and planes), the conditions (a)–(c) of Th. 1 actually characterize minimizing interpolants. First we show that (a)–(c) are sufficient for being minimal. The results concerning affine subspaces will be special cases of more general results later, but we also discuss them separately.

Lemma 2 *Assume that $X_0, \ldots, X_n$ are affine subspaces of $\mathbb{R}^d$, and that $c : [u_0, \ldots, u_n] \to \mathbb{R}^d$ is in the class $\mathscr{C}$. Then c minimizes the functional $F_{u_0}^{u_n}$ in $\mathscr{C}$, if and only if*

(a) c is a polynomial function of degree ≤ 3 in the intervals (u_i, u_{i+1}),

(b) $\Delta_{u_i} c'''$ is orthogonal to X_j for $i = 1, \ldots, n-1$, and $c'''(u_0)$, $c'''(u_n)$ are orthogonal to X_0, X_n, respectively.

(c) $c''(u_0) = c''(u_n) = 0$;

Definition 2 *The class of C^2 curves defined in the interval $[u_0, u_n]$, which are piecewise polynomial of degree at most three in the intervals $[u_i, u_{i+1}]$ is denoted by $\mathscr{P}^3$.*

Proof: (of Lemma 2) $\mathscr{C}$ is a linear space, and we endow it with the inner product

$$\langle c, d \rangle = \int_{u_0}^{u_n} \left\langle c'', d'' \right\rangle. \tag{12}$$

If we can show $\langle c, d-c \rangle = 0$ for all $d \in \mathscr{C}$, then we can use $\langle d-c, d-c \rangle \geq 0$ to conclude $\langle d, d \rangle - \langle c, c \rangle \geq 0$. We compute $\langle c, d-c \rangle$ by using partial integration as in the proof of Lemma 1, with the simplification that $R = 0$ and that differentiation obeys Schwarz' rule. The difference $d - c$ plays the role of the vector field V, so we even let $V(u) = d(u) - c(u)$:

$$\begin{aligned} \langle c, d-c \rangle &= \int \left\langle c'''', V \right\rangle du + \left\langle c''(u_n), V'(u_n) \right\rangle - \left\langle c''(u_0), V'(u_0)) \right\rangle \\ &\quad + \left\langle c'''(u_0), V(u_0) \right\rangle - \left\langle c'''(u_n), V(u_n) \right\rangle + \sum_{i=2}^{n-1} \left\langle \Delta_{u_i} c''', V(u_i) \right\rangle. \end{aligned} \tag{13}$$

As $V(u_i)$ is parallel to X_j, conditions (a)–(c) immediately imply that $\langle c, d-c \rangle = 0$.

To show the converse, we use an argument similar to the proof of Lemma 1. Note that for any given $V(u)$ of sufficient differentiability with $V(u_i)$ parallel to X_i, $s(u,v) = c(u) + vV(u)$ is an admissible variation. Further, we find any d with $\langle c, d-c \rangle \neq 0$ then c is not minimizing. By choosing $V(u) = f(u)c''''(u)$ analogous to Lemma 1 we get $\langle c, d-c \rangle > 0$ unless $c'''' = 0$, which shows (a). By choosing appropriate V's we analogously show (b) and (c). □

We now derive a result concerning the existence and uniqueness of the variational interpolant in the case that $X_0, \ldots, X_n$ are affine subspaces. The existence and uniquess theorem in its 'final form' is given by Th. 3.

Lemma 3 *We use the notation of Lemma 2. Assume that the affine subspaces X_j have the form $X_j = p_j + U_j$, with linear subspaces U_j. If there are no $a_0, a_1 \in \mathbb{R}^n$ such that*

$$a_0 + u_i a_i \in U_j, \quad (i = 0, \ldots, n), \tag{14}$$

then the variational interpolation problem within the class $\mathscr{C}$ is uniquely solvable.

Proof: We show that conditions (a)–(c) of Lemma 2 can be uniquely fulfilled with a curve in $\mathscr{P}^3$. Recall that a cubic curve c is uniquely determined by $c(a)$, $c'(a)$, $c''(b)$, and $c(b)$, if $a \neq b$. If $c \in \mathscr{P}^3$, then c, c', c'' are continuous, so every cubic segment leaves only d degrees of freedom for its successor. Therefore,

$$\dim \mathscr{P}^3 = 4d + (n-1)d = (n+3)d. \tag{15}$$

The interpolation conditions $c(u_i) \in X_i$, the orthogonality conditions (b), and the end conditions (c) represent

$$\sum_{i=0}^{n} (d - \dim X_i), \quad \sum_{i=0}^{n} \dim X_j, \quad 2d \tag{16}$$

inhomogeneous linear equations, respectively. These are, in summa, $(n+3)d$ linear equations. The corresponding homogeneous system is the one defined by U_j instead of X_j for $j = 0, \ldots, n$. If the latter system has a unique solution then so has the inhomogeneous one.

Obviously $c(u) = 0$ is a minimizing interpolant for the 'homogeneous' problem. We have to show that it is the only one. Any minimizing interpolant d must therefore have the property that $\langle d, d \rangle = \langle c, c \rangle = 0$, where $\langle \, , \, \rangle$ is defined by Equ. (12).

$\langle d, d \rangle = 0$ implies that d is piecewise linear. As d is C^2, is actually of the form $d(u) = a_0 + ua_1$, with $a_0, a_1 \in \mathbb{R}^n$. That means $a_0 + u_i a_i \in U_i$, which was forbidden by our assumptions. $\square$

Lemma 3 includes the following well known elementary result as a special case:

Corollary 1 *The variational interpolation problem for points $p_0 = X_0, \ldots,\ p_n = X_n$ is uniquely solvable.*

Further, it is well known that for all points $p_0, \ldots, p_n$, and vectors w_0, w_n, there exists a unique C^2 piecewise cubic curve $\tilde{c} = \tilde{c}(p_0, \ldots, p_n; w_0, w_n)$ with $\tilde{c}(u_i) = p_i$, and $\tilde{c}'(u_0) = w_0$, $\tilde{c}'(u_n) = w_n$.

Definition 3 *The piecewise cubic interpolant determined by the points $p_0, \ldots, p_n$ according to Cor. 1 is denoted by $c(p_0, \ldots, p_n)$.*

The piecewise cubic interpolant of the points $p_0, \ldots, p_n$ with tangent vectors w_0, w_n at u_0, u_n is denoted by $\tilde{c}(p_0, \ldots, p_n; w_0, w_n)$.

Corollary 2 *The mapping $(p_0, \ldots, p_n) \to c(p_0, \ldots, p_n)$ of a list of points to its variational interpolant is a linear mapping of $(\mathbb{R}^d)^{n+1}$ into $\mathscr{P}^3$.*

3.2 Reduction to the cubic case

We use Cor. 1 to reduce the problem of interpolating within the class $\mathscr{C}$ to interpolating within the class $\mathscr{P}^3$. This converts the problem into an algebraic one.

Lemma 4 *For all X_j,*

$$\inf\{F_{u_0}^{u_n}(c) \mid c \in \mathscr{C}\} = \inf\{F_{u_0}^{u_n}(c) \mid c \in \mathscr{P}^3\}. \tag{17}$$

A minimizing interpolant of $X_0, \ldots, X_n$ fulfills, in the notation of Def. 2, $c = c(c(u_0), \ldots, c(u_n))$.

Proof: The '$\leq$' part of (17) is trivial. The '$\geq$' part is shows as follows: If $c \in \mathscr{C}$ interpolates the subsets X_j, Cor. 1 yields a curve $\widetilde{c} \in \mathscr{P}^3$ which interpolates the points $c(u_i)$ (and therefore the sets X_j), and with $F(\widetilde{c}) \leq F(c)$.

The second statement follows immediately from the first one. □

3.3 Minimizing curves

We consider the vector space $V = (\mathbb{R}^d)^{n+1}$ and define a a quadratic form in V by letting

$$q(p_0, \ldots, p_n) = F_{u_0}^{u_n}(c(p_0, \ldots, p_n)). \tag{18}$$

By Lemma 4, finding a minimizing interpolant of subsets $X_0, \ldots, X_n$ is equivalent to minimizing the function q on $X_0 \times \cdots \times X_n$.

Theorem 2 *Assume that $X_0, \ldots, X_n$ are smooth submanifolds of $\mathbb{R}^d$. A curve in $\mathscr{P}^3$ makes the functional $F_{u_0}^{u_n}$ stationary within the set of $\mathscr{P}^3$-interpolants of $X_0, \ldots, X_n$ if and only if the following conditions are fulfilled:*

(a) $\Delta_{u_i} c'''$ is orthogonal to the tangent space $T_{c(u_i)} X_i$ for $i = 1, \ldots, n-1$,

(b) $c''(u_0) = c''(u_n) = 0$.

Proof: The proof proceeds by finding curves in $\mathscr{P}^3$ (i.e., variations of c within $\mathscr{P}^3$) such that the first variation of $F_{u_0}^{u_n}$ is nonzero. The proof imitates the proof of Lemma 1.

An admissible variation $s(u, v)$ must have $s(u_i, v) \in X_i$, and the curves $v = \text{const}$ have to be in $\mathscr{P}^3$. The first variation of $F_{u_0}^{u_n}$ (cf. Equ. 5 and its proof) simplifies to

$$\begin{aligned} \frac{1}{2} \frac{d}{dv} F_{u_0}^{u_n}(c_v)\Big|_{v=0} &= \left\langle V'(u_n), c''(u_n) \right\rangle - \left\langle V'(u_0), c''(u_0) \right\rangle \\ &- \left\langle V(u_n), c'''(u_n) \right\rangle + \left\langle V(u_0), c'''(u_0) \right\rangle + \sum_{i=1}^{n-1} \left\langle \Delta_{u_i} c''', V(u_i) \right\rangle, \end{aligned} \tag{19}$$

To show (b), we have to find an admissible variation with $V(u_i) = 0$, but $V'(u_0) = c''(u_0)$ and $V'(u_n) = c''(u_n)$. Then the derivative of F will be positive unless $c''(u_0) = c''(u_n)$. To find s, we choose curves $w_0(v)$ and $w_n(v)$ such that $dw_0/dv(0) = c''(u_0)$, and $dw_n/dv(0) = c''(u_n)$. Using Def. 2, we let

$$s(u,v) = \widetilde{c}(p_0,\dots,p_n,w_0(v),w_n(v))(u). \tag{20}$$

To show (a), we choose curves $p_i(v)$ in X_i such that $p_0(0) = p_0,\dots,p_n(0) = p_n$, and such that the tangent vectors $dp_0/dv(0),\dots,dp_n/dv(0)$ equal the orthogonal projections of the vectors $c'''(u_0)$, $\Delta_{u_1}c''',\dots,\Delta_{u_{n-1}}c'''$, $c'''(u_n)$ onto the respective tangent spaces $T_{c(u_i)}X_i$. Finally, we let

$$s(u,v) = c(p_0(v),\dots,p_n(v)). \tag{21}$$

By construction, the first variation is positive, unless (a) is fulfilled. □

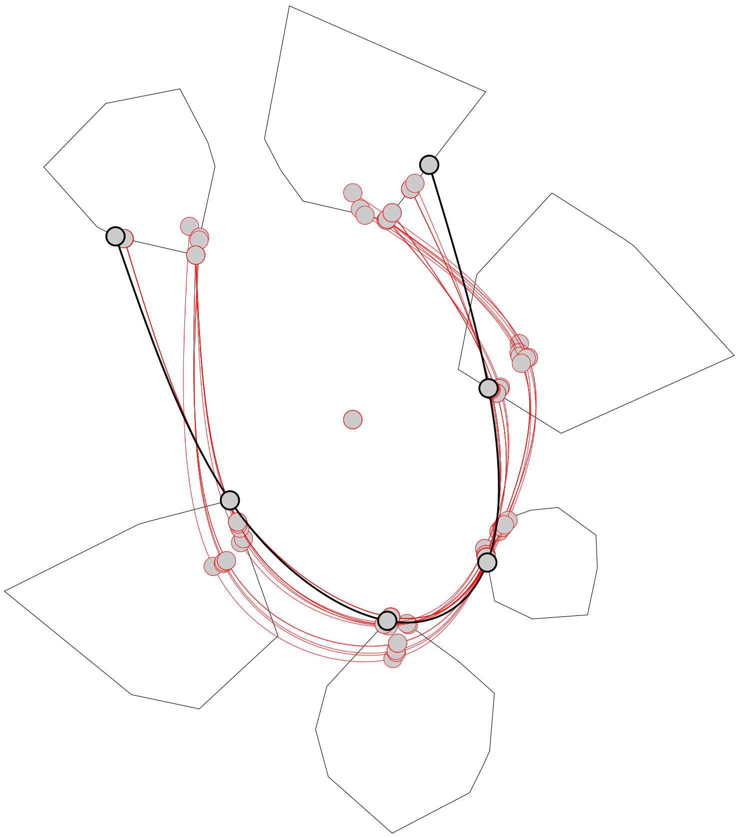

Figure 1: Interpolation of convex polygons in the plane

3.4 Existence and uniqueness of the minimizing interpolant

The scalar product $\langle\ ,\ \rangle$ in V defined by the quadratic form q is positive semidefinite. The set of isotropic vectors $p = (p_0,\dots,p_n)$ with $q(p) = 0$ coincides with the radical (those vectors p with $\langle p,x\rangle = 0$ for all x).

Lemma 5 *The radical $V^\perp$ of $(V, \langle\ ,\ \rangle)$ consists of vectors $(p_0, \ldots, p_n)$ of the form $p_i = a_0 + u_i a_1$. $\dim V^\perp = 2d$.*

Proof: $q(p) = 0$ means that the cubic minimizing C^2 interpolant of the points $p_1, \ldots, p_n$ is linearly parametrized. $\square$

The scalar product $\langle\ ,\ \rangle$ is *positive definite* on the factor space $V/V^\perp$, and projection parallel to $V^\perp$ onto any complementary subspace does not change scalar products. This can be used to give a general criterion concerning the solvability of the variational interpolation problem. Clearly, the natural mapping $V \to V/V^\perp$ is nothing but the projection mentioned above, but without the arbitrariness of choice of a complement.

Lemma 6 *If the image of $X_0 \times \cdots \times X_n$ under a projection parallel to the subspace $V^\perp$ into any complementary $d(n-1)$-dimensional subspace is closed, then the variational interpolation problem with input data $X_0, \ldots, X_n$ is solvable.*

Proof: If q is a positive definite bilinear form, it assumes its infimum on any closed subset (the closed q-unit balls are compact). Thus the variational interpolation problem has a solution if

$$X_0 \times \cdots \times X_n + V^\perp \subset V/V^\perp \tag{22}$$

is closed. This is equivalent to the statement concerning the projection. $\square$

Corollary 3 *If $X_0, \ldots, X_n$ are affine subspaces, the variational interpolation problem is solvable.*

Proof: Both products and projections of affine subspaces are affine subspaces, and therefore closed. $\square$

This immediately implies

Theorem 3 *Consider affine subspaces $X_0, \ldots, X_n$ of $\mathbb{R}^d$. Assume that $X_j = p_j + U_j$, with linear subspaces U_j. Let $W = V^\perp \cap (U_0 \times \cdots \times U_n)$. Then the variational interpolation problem with input data $X_0, \ldots, X_n$ is solvable. In the notation of Def. 2, the solutions is of the form $c(p_0, \ldots, p_n)$, where $(p_0, \ldots, p_n)$ ranges in an affine subspace parallel to W.*

Lemma 3 is a corollary of Th. 3. This becomes clear if we look at Lemma 5, which explicitely describes $V^\perp$.

As products and projections of compact subsets are compact, and therefore cosed, we have the following

Corollary 4 *If $X_0, \ldots, X_n$ are compact, the variational interpolation problem is solvable.*

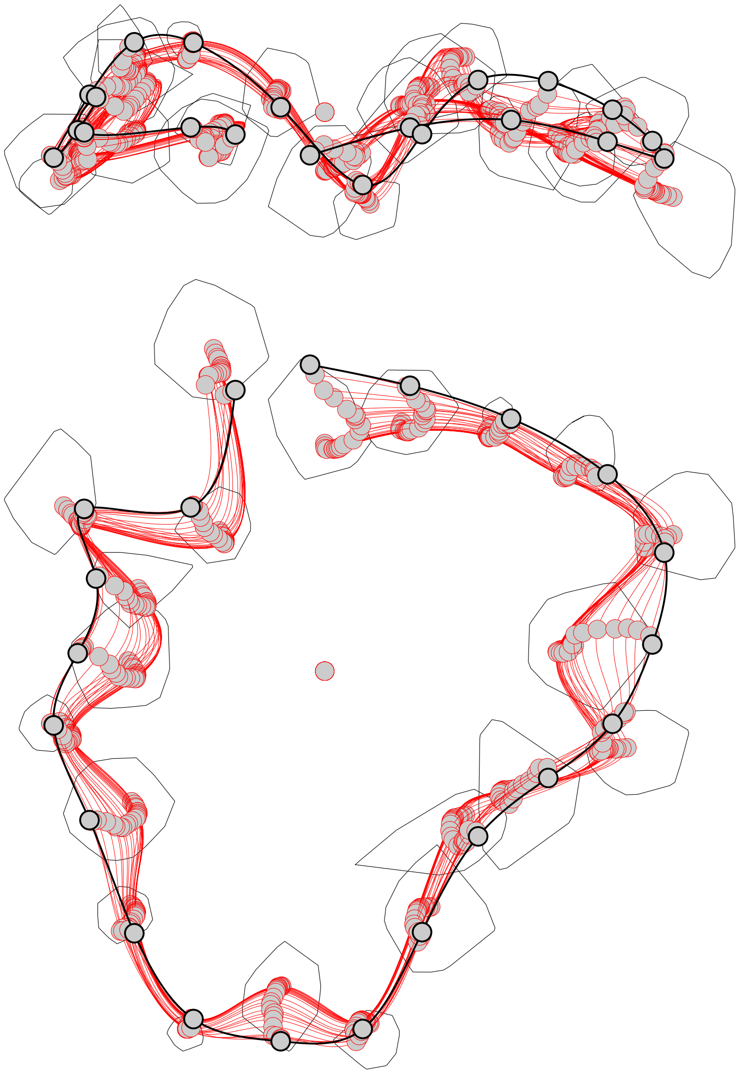

Figure 2: Interpolation of convex polytopes in three-space

4 Variational interpolation of convex polyhedra and algorithms

Another class of subsets of $\mathbb{R}^d$ which is closed under the operations of product and projections are the convex polyhedra with finitely many faces. They can be defined as convex hulls of a finite number of points and closed rays. An immediate consequence of Lemma 6 is the following

Corollary 5 *The variational interpolation problem for closed convex polyhedra is solvable. The set of solutions is of the form* $c(p_0,\ldots,p_n)$, *where* $(p_0,\ldots,p_n)$ *ranges in a set of the form* $((a_0,\ldots,a_n)+V^\perp)\cap(X_0\times\cdots\times X_n)$.

Proof: Lemma 6 implies the solvability of the problem. There is a unique minimum $a+V^\perp$ of q

in the subset $(X_0 \times \cdots \times X_n) + V^\perp$ of the factor space $V/V^\perp$, as the q-unit ball is strictly convex there. It follows that the set of solution curves has the form described above. □

An *algorithm* to compute such a minimizing interpolant is based on an algorithm of determining the point of a convex polyhedron closest to the origin with respect to some positive definite metric. Clearly we have to minimize q on the polyhedron $X_0 \times \cdots \times X_n)$, but q is not positive definite. Alternatively, we may minimize q in the polyhedron $X_0 \times \cdots \times X_n + V^\perp$ in the factor space $V/V^\perp$, where q is indeed positive definite.

The product polyhedron has a simple structure: Its faces are of the form $C_0 \times \cdots \times C_n$, where C_i is a face of X_i. Consider such a product face, and denote the affine span of C_i by A_i. The affine span of the product face is the product of spaces $A_0 \times \cdots \times A_n$.

It may happen that the projection of the product polyhedron into the factor space has the faces $C_0 \times \cdots \times C_n + V^\perp$, with the same dimension $\dim A_0 + \cdots + \dim A_n$. This is the case if and only if the product subspace $A_0 \times \cdots \times A_n$ is transverse to $V^\perp$, i.e., if the conditions of Lemma 3 are fulfilled for A_j instead of X_j.

4.1 Algorithms

In order to find a minimum of q in the set $X_0 \times \cdots \times X_n$, we may start form an interior point of this product polyhedron and follow the flow line of q's gradient which initiates in this interior point (the gradient being defined with respect to some positive definite metric).

If q were positive definite, we could use the gradient with respect to q, and the flow lines would be polygons. In the implementation which produces the figures in this paper we therefore chose the scalar product

$$q'(v,w) = q(v,w) + \varepsilon \langle v, w \rangle \,, \tag{23}$$

where ε is very small, and $\langle v, w \rangle$ is the standard scalar product in V. Then q' is positive definite. For all affine subspaces A in V, there is a unique $w \in A$ with $q'(w)$ minimal — w is found by orthogonally projecting the origin into A.

Now the flow lines of the q'-gradient of q' restricted to $X_0 \times \cdots \times X_n$ can be followed easily: Starting from an interior point (which is the product of a sequence of interior points of the single polytopes X_i), we march towards the origin until we meet the boundary of the product polytope. Then we follow the face we have hit until we arrive in a minimum of q' or we leave this face. In the latter case we have to decide wheter we can diminish q' by marching into an adjacent face, or by continuing in a face of lower dimension. The algorithm terminates when q' has a minimum in the interior of a face, or the face is already of dimension zero.

Figures 1 and 2 show some steps of this iteration together with the final minimal interpolant.

4.2 'Interval' interpolation

Interpolation of convex polytopes has as a special subcase interpolation of points whose coordinates are bounded by intervals — the polytopes in question are the products of these intervals. Fig. 3 shows a minimal interpolant of a sequence of rectangles.

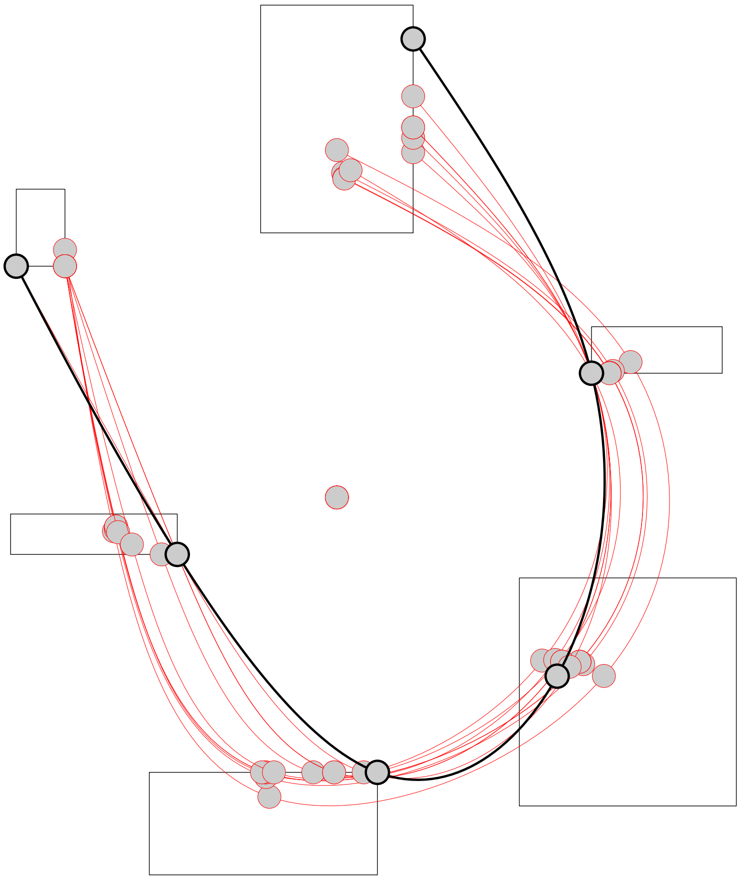

Figure 3: Interpolation of convex polygons in the plane

Appendix

To complete the proof of Lemma 1, we still have to show the following:

Lemma 7 *Assume that $X_0,\ldots,X_n$ are smooth submanifolds of a Riemannian manifold X, and that $u_0 < \ldots < u_n$ are real numbers. If $c : [u_0,u_n] \to X$ is a curve in $\mathscr{C}^4$ with $c(u_i) \in X_i$ and $W_i \in T_{c(u_i)}X_i$ are tangent vectors, then there is an admissible variation $s(u,v)$ with $s(u,0) = c$ and $\partial s/\partial v(u_i,0) = W_i$.*

Proof: Let $I = [u_0,u_n]$ and $c(u_i) = p_i$. For $i = 0,\ldots,n$, fix an open subset $J_i \subset I$ such that $u_i \in J_i$ and such that the J_i are disjoint. Choose a neighbourhood U_i of p_i in X such that there is a diffeomorphism $\phi_i : U_i \to \mathbb{R}^d$ which straightens $X_i \cap U_i$, i.e., $\phi_i(U_i \cap X_i) = \phi_i(U_i) \cap (R^{\dim X_i} \times 0) \subset \mathbb{R}^d$. Let $\widetilde{W}_i = d_{p_i}\phi_i(W_i)$. Without loss of generality, $\phi_i(p_i) = 0$, $\|\widetilde{W}_i\| = 1$, $\phi_i(U_i)$ contains a ball of radius 2ε, $c(J_i) \subset U_i$, and $\phi_i \circ c(J_i)$ is contained in a ball of radius ε, for a certain $\varepsilon > 0$.

Choose C^∞ functions $\psi_i : I \to \mathbb{R}$ with $\psi_i(u_i) = 1$ and $\operatorname{supp}(\psi_i) \subset J_i$. Define the variation $s(u,v)$ for $u \in I$ and $v \in (-\varepsilon,\varepsilon)$ by

$$s(u,v) \quad = \quad \phi_i^{-1}(\phi_i(c(u)) + v\psi_i(u)\widetilde{W}_i) \quad \text{for } u \in J_i$$

$$s(u,v) = c(u) \quad \text{else.}$$

By construction, $s(u,v) = c(u)$ if $u \in J_i \setminus \mathrm{supp}(\psi_i)$, so s is smoothly defined. Further, the partial derivative $\partial s/\partial v(u_i, 0) = d_0\phi^{-1}(\widetilde{W}_i) = W_i$. □

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