Combinatorics of the zeta map on rational Dyck paths

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joint with Tom Denton and Christopher Hanusa

XX Coloquio Latinoamericano de Álgebra, Lima
Dec 8, 2014

Plan of the talk

1. Simultaneous core partitions & rational Dyck paths
2. Skew length
3. Conjugation
4. Zeta map

1. Simultaneous core partitions & rational Dyck paths

Simultaneous core partitions

Definition

Let $\lambda \vdash n$ be a partition of n

- say λ is an a-core if it has no cell with hook length a
- say λ is an (a, b)-core partition if it has no cell with hook length a or b

Example

A $(5, 8)$-core:

14	9	6	4	2	1
11	6	3	1		
9	4	1			
7	2				
6	1				
4					
3					
2					
1					

Simultaneous core partitions

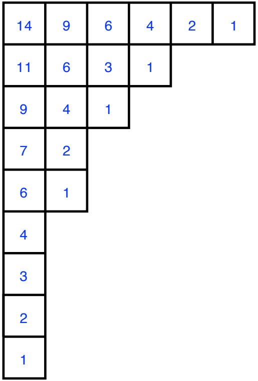
Theorem (Anderson 2002)

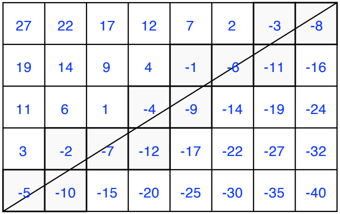
The number of (a, b)-cores is finite if and only if a and b are relatively prime, in which case they are counted by the rational Catalan number

$$C_{a,b} = \frac{1}{a+b}\binom{a+b}{a}$$

Simultaneous core partitions: Anderson's bijection

Beautiful bijection: (a, b)-cores $\longleftrightarrow$ Dyck paths in an $a \times b$ rectangle





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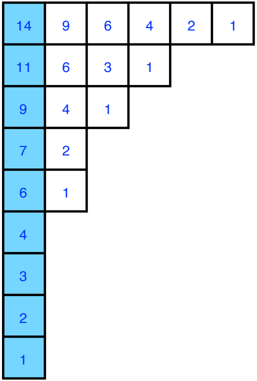

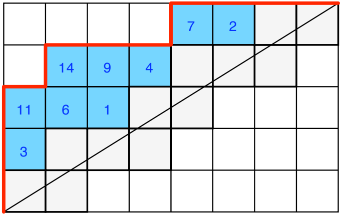

Rational q-Catalan

Define the q-analog of the (a, b)-Catalan number as

$$C_{a,b}(q) = \frac{1}{[a+b]} \begin{bmatrix} a+b \\ a \end{bmatrix}$$

obtained by replacing every number r by its q-analog

$$[r] = 1 + q + \cdots + q^{r-1}$$

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$$[r] = 1 + q + \cdots + q^{r-1}$$

Proposition

$C_{a,b}(q)$ is a polynomial if and only if a and b are relatively prime.

Rational q-Catalan and q, t-Catalan

Conjecture (Armstrong–Hanusa–Jones 2014)

$$C_{a,b}(q) = \sum q^{\mathrm{sl}(\kappa)+\mathrm{area}(\kappa)}$$

Conjecture (Armstrong–Hanusa–Jones 2014)

$$\sum q^{\mathrm{area}(\kappa)} t^{\mathrm{sl}'(\kappa)} = \sum q^{\mathrm{sl}'(\kappa)} t^{\mathrm{area}(\kappa)}$$

sums over all (a, b)-cores

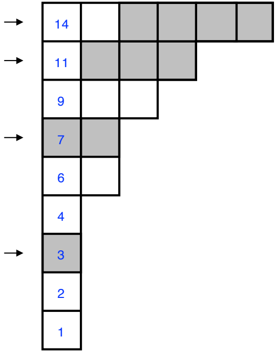
2. Skew length

Skew length

a-rows: largest hooks of each residue mod *a*
b-boundary: boxes with boxes with hooks less than *b*

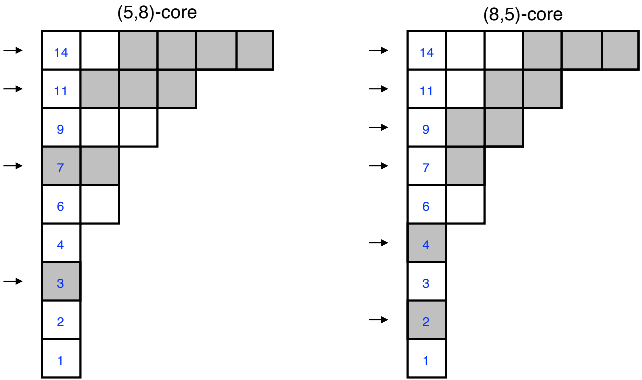
skew length: number of boxes in both the *a*-rows and *b*-boundary

(5,8)-core



sl = 4+3+2+1 = 10

Skew length



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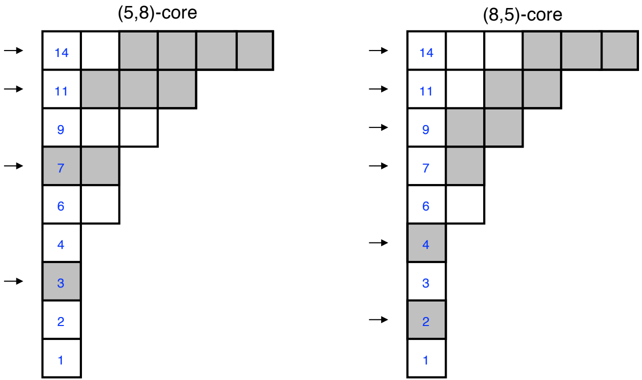

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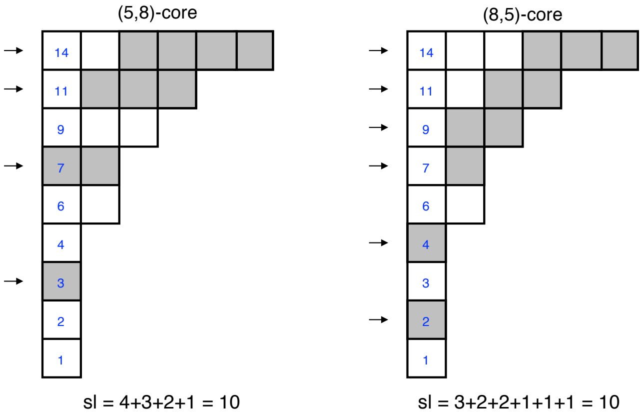
Skew length



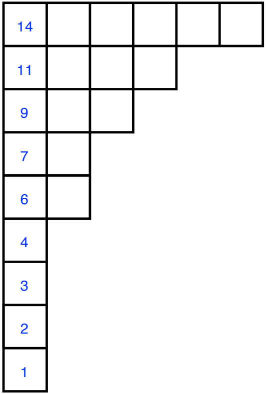
Theorem (C.–Denton–Hanusa)

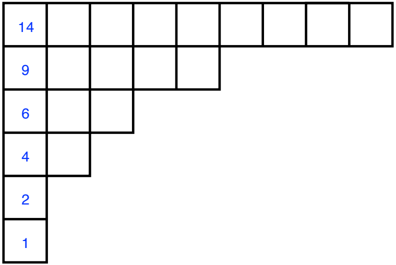
Skew length is independent of the ordering of a and b.

3. Conjugation

Conjugation on cores

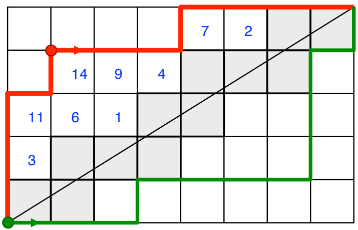
conjugation: reflect along a diagonal

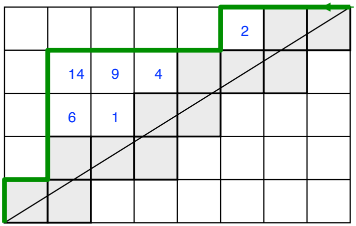




Conjugation on Dick paths

conjugation: cyclic rotation to get a path below the diagonal, rotate 180° degrees

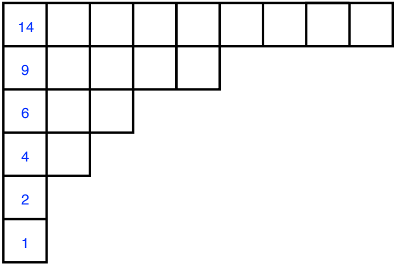


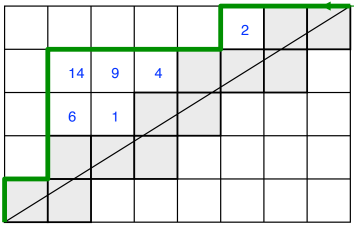


Conjugation

Theorem (C.–Denton–Hanusa)

Both conjugations coincide under Anderson's bijection

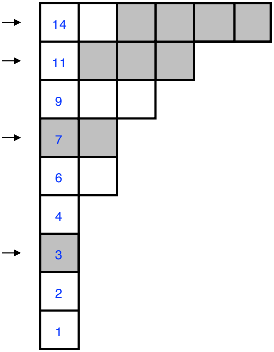




Conjugation

Theorem (C.–Denton–Hanusa)

Conjugations preserves skew length



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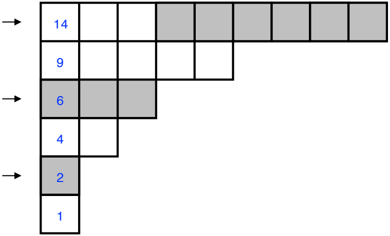
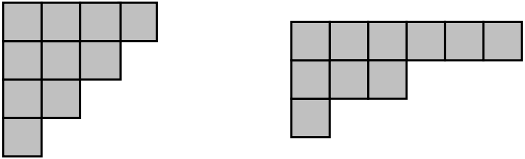


sl = 6+3+1 = 10

The shaded partitions determine two amazing maps called zeta and eta



statistics for q, t-enumeration of classical Dyck paths were famously difficult to find, but were nearly simultaneously discovered by Haglund (area and bounce) and Haiman (dinv and area). The zeta map sends

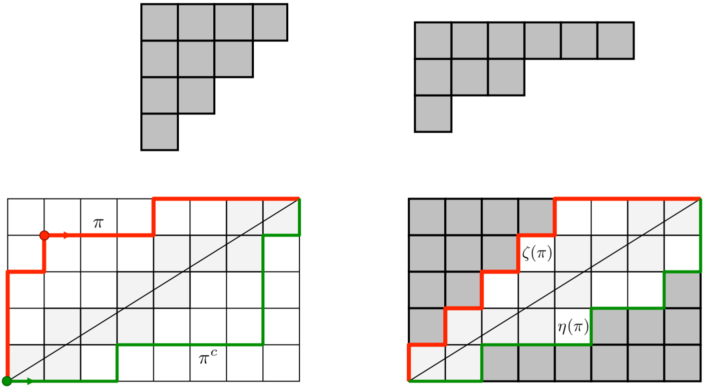
$$\begin{array}{ccc} \text{dinv} & \to & \text{area} \\ \text{area} & \to & \text{bounce} \end{array}$$

Drew Armstrong: generalized this zeta map to (a, b)-Dyck paths

4. Zeta map (and eta)

Zeta and eta on cores

Armstrong (zeta):
The bounded partitions of zeta and eta are the shaded partitions before



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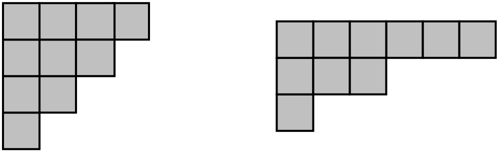
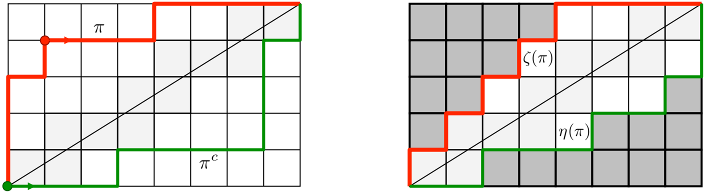




eta := zeta of the conjugate

Note: the map $\zeta(\pi) \to \eta(\pi)$ is an area preserving map

Zeta and eta

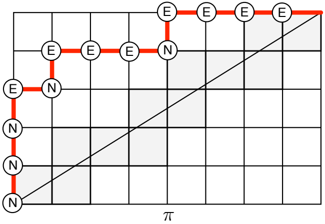
Exercise for the party tonight:
The shaded partitions fit above the main diagonal!

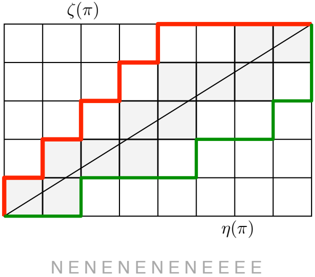
Conjecture (Armstrong)

The zeta map is a bijection on (a, b)-Dyck paths

Zeta and eta on Dyck paths

Armstrong–Loehr–Warrington, . . . :
Zeta: move diagonal up and record north and east steps as crossed





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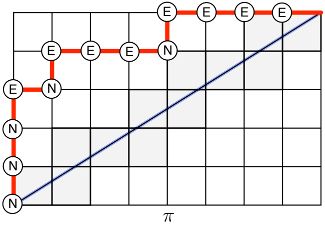

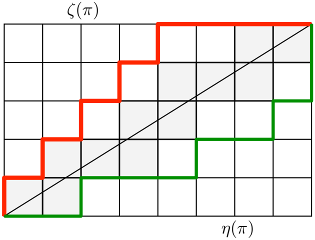

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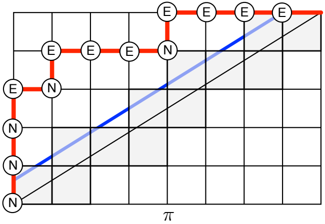

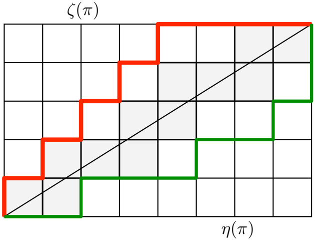

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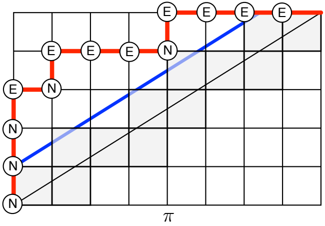

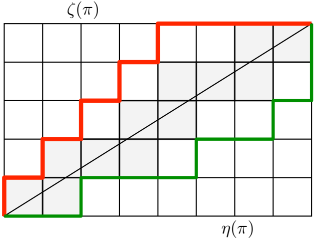

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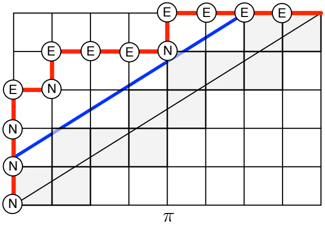

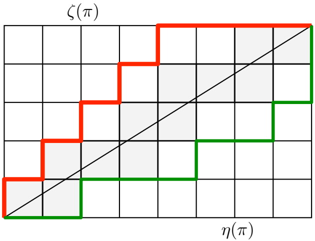

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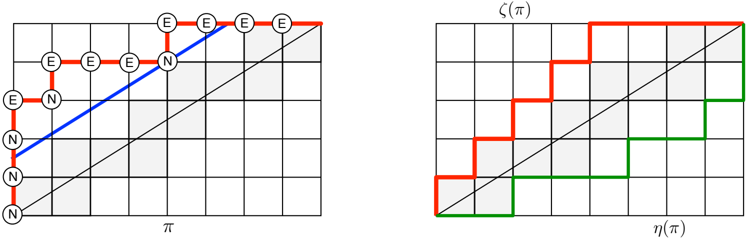

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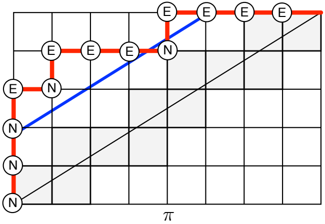

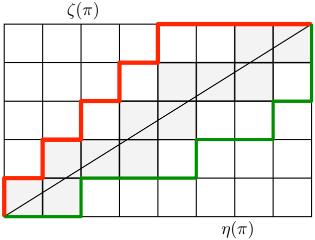

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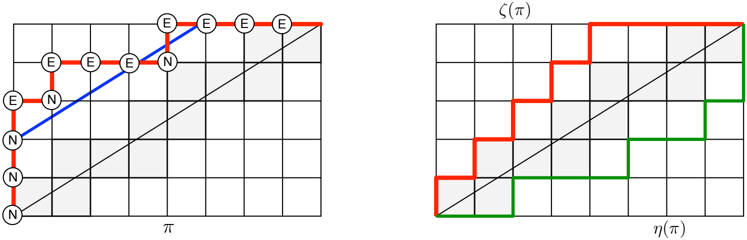

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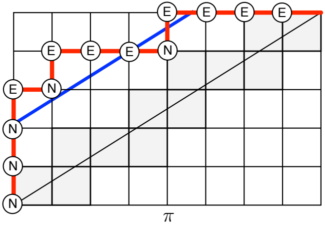

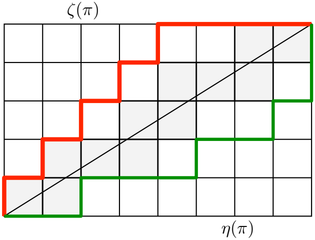

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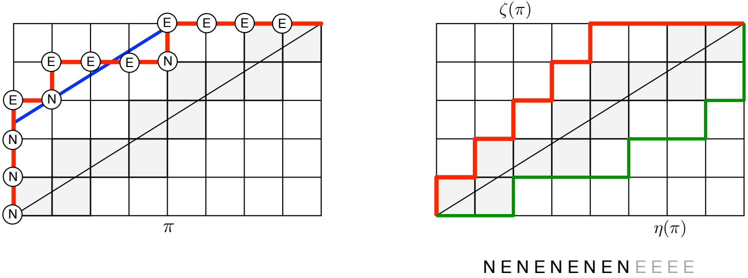

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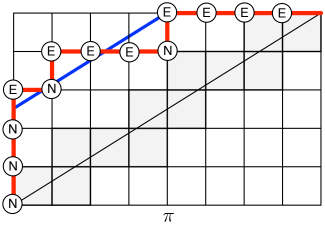

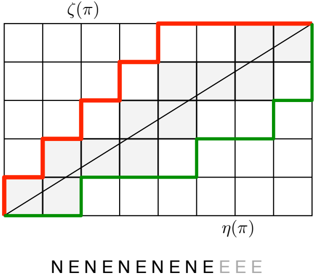

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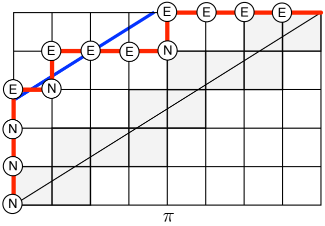

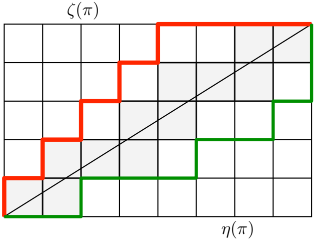

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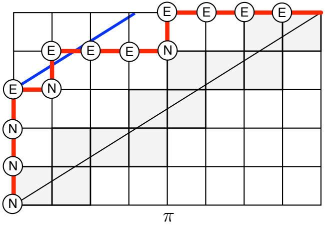

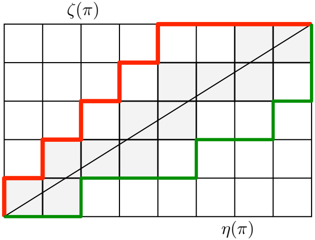

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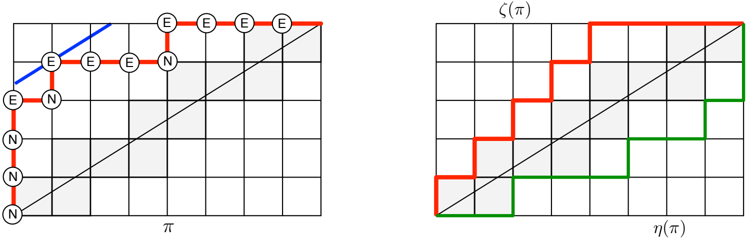

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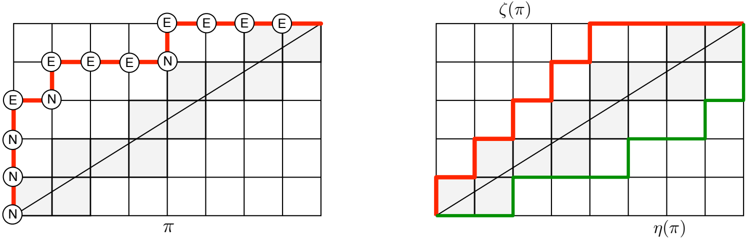

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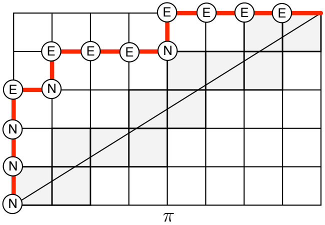
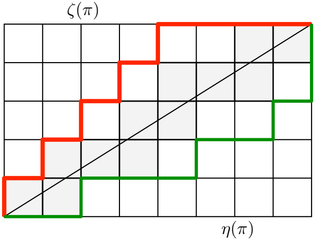




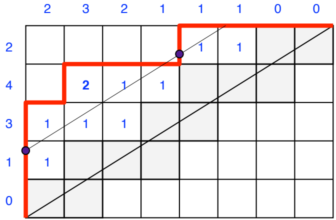
N E N E N E N E N E E E E

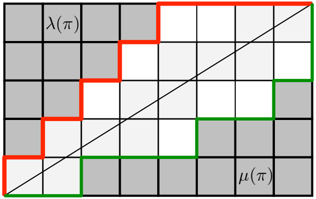
Eta: move diagonal down and record south and weast steps as crossed

Zeta and eta via lasers

Theorem (C.–Denton-Hanusa)

Description of zeta and eta in terms of a laser filling





$\lambda = (4, 3, 2, 1, 0)$
$\mu = (3, 2, 2, 1, 1, 1, 0, 0)$

Zeta and eta

Conjecture (Armstrong)

The zeta map is a bijection on (a, b)-Dyck paths

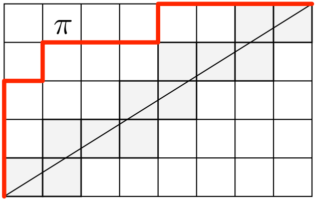
Zeta and eta

Conjecture (Armstrong)

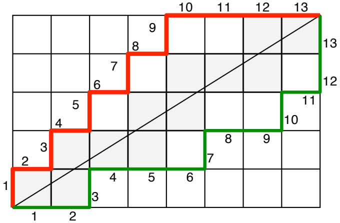
The zeta map is a bijection on (a, b)-Dyck paths

Lets construct the inverse!!
(knowing zeta and eta)

Zeta inverse knowing eta

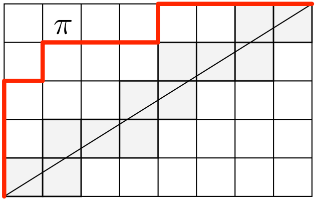


(N,N,N,**E**,N,**E,E,E**,N,**E,E,E,E)**



γ = (1,3,7,**12**,9,**13**,**11**,**8**,5,**10**,**6**,**4**,**2**)

Zeta inverse knowing eta



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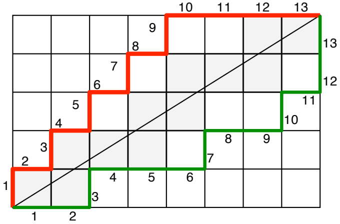

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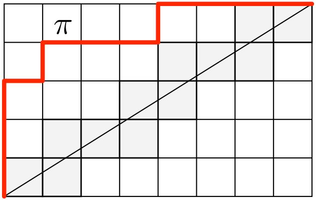
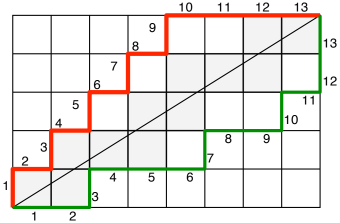
Theorem (C.–Denton–Hanusa)

- *γ is a cycle permutation.*
- *The east steps of π correspond to the descents of γ.*

Zeta inverse knowing eta



(N,N,N,**E**,N,**E**,**E**,**E**,N,**E**,**E**,**E**,**E**)



γ = (1,3,7,**12**,9,**13**,**11**,**8**,5,**10**,**6**,**4**,**2**)

Theorem (C.–Denton–Hanusa)

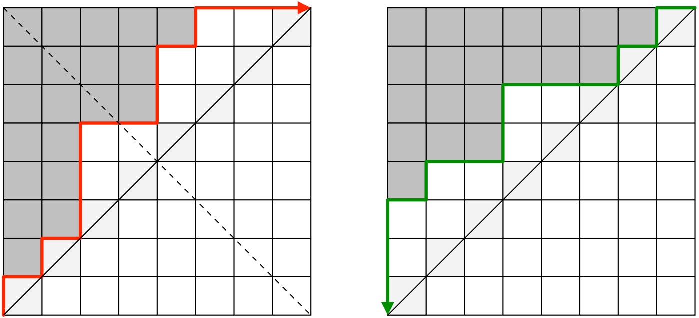
- *γ is a cycle permutation.*
- *The east steps of π correspond to the descents of γ.*

missing: combinatorial description of the area preserving involution

Square case

Theorem (C.–Denton-Hanusa)

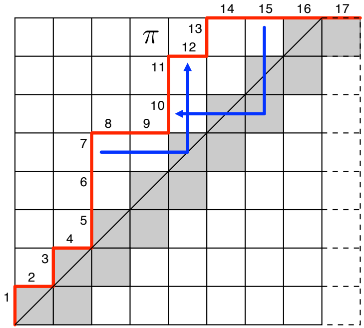
Area preserving involution: reverse the path



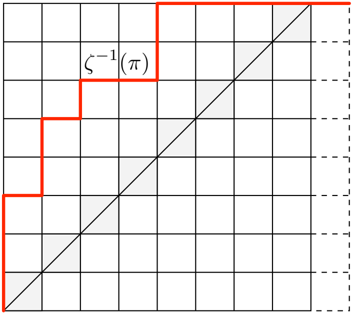
Square case

Corollary (C.–Denton-Hanusa)

Inverse: descents of γ are the east steps of the inverse



γ = (1,3,5,**9**,6,10,**15**,11,**16**,**12**,7,13,**17**,**14**,**8**,**4**,**2**)

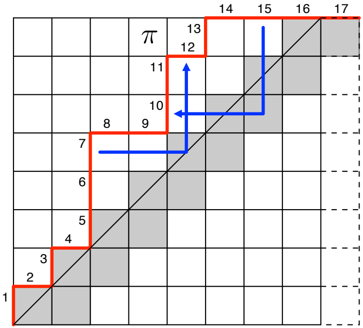


(N,N,N,**E**,N,N,**E**,N,**E**,**E**,N,N,**E**,**E**,**E**,**E**,**E**)

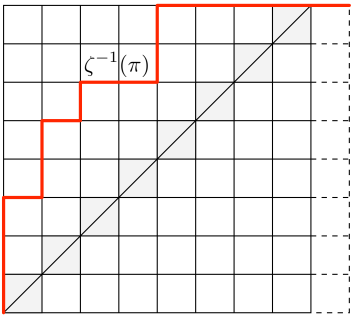
Square case

Corollary (C.–Denton-Hanusa)

Inverse: descents of γ are the east steps of the inverse



γ = (1,3,5,**9**,6,10,**15**,11,**16**,**12**,7,13,**17**,**14**,**8**,**4**,**2**)



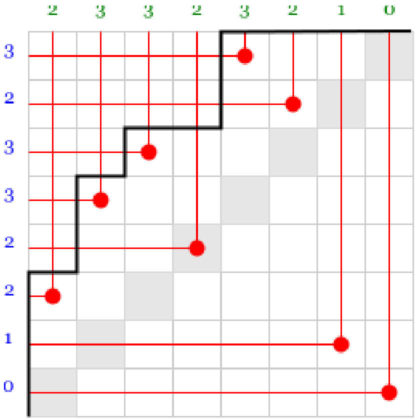
(N,N,N,**E**,N,N,**E**,N,**E**,**E**,N,N,**E**,**E**,**E**,**E**,**E**)

Different from the known inverse description using "bounce paths"!

Square case

Theorem (C.–Denton–Hanusa)

Co-skew length is equal to the dinv *statistic*



Thank you!