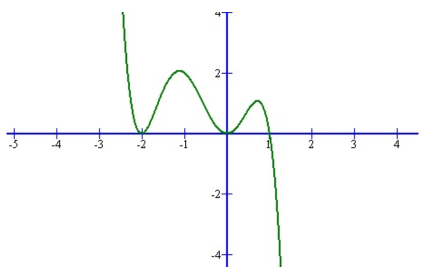
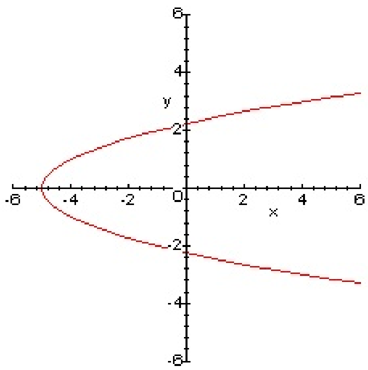
MATH 1300 A, Fall 2013
Solution Midterm 1

Show all work clearly and in order. Justify your answers whenever possible. No calculators or books are allowed. You have 50 minutes.

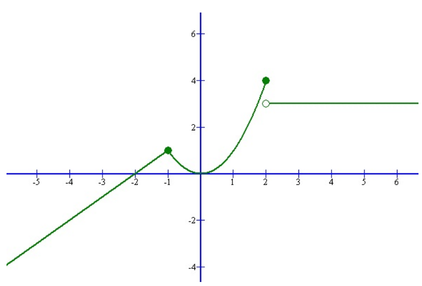
1. (*10 points*) For each of the following graphs determine if it is the graph of a function. Mark yes or not.



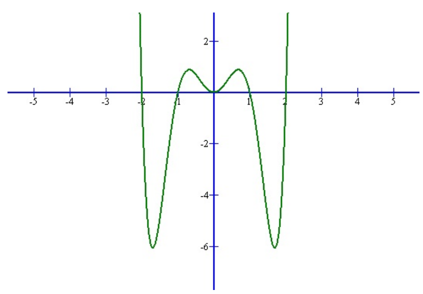
(a) **(yes)** or (not)



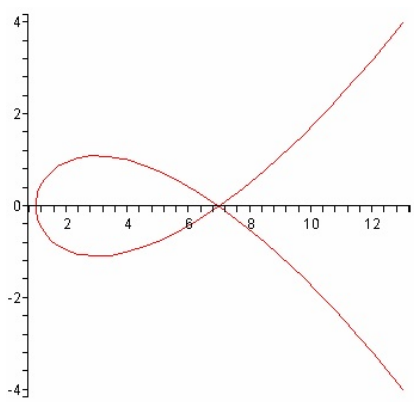
(b) (yes) or **(not)**



(c) **(yes)** or (not)



(d) **(yes)** or (not)



(e) (yes) or **(not)**

2. (*10 points*) Determine the largest domain for which the following formula determines a function:

$$f(x) = \frac{\sqrt{9-x^2}}{x^2-1}$$

The domain of f is the set of values of x for which $9 - x^2 \geq 0$ and $x^2 - 1 \neq 0$. This happens when $-3 \leq x \leq 3$ and $x \neq -1, 1$. Therefore

$$\text{Domain } f = [-3,-1) \cup (-1,1) \cup (1,3]$$

3. (*20 points*) Consider the function $f(x) = \frac{5x+4}{x+3}$.

(a) Is f one-to-one? **Yes**

The function f is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

$$\frac{5x_1+4}{x_1+3} = \frac{5x_2+4}{x_2+3} \quad \Rightarrow \quad \begin{aligned} (5x_1+4)(x_2+3) &= (5x_2+4)(x_1+3) \\ 5x_1x_2 + 15x_1 + 4x_2 + 12 &= 5x_1x_2 + 15x_2 + 4x_1 + 12 \\ 15x_1 + 4x_2 &= 15x_2 + 4x_1 \\ 11x_1 &= 11x_2 \\ x_1 &= x_2 \end{aligned}$$

(b) Find a formula for the inverse of f.

$$y = \frac{5x+4}{x+3} \quad \Rightarrow \quad \begin{aligned} (x+3)y &= 5x+4 \\ xy + 3y &= 5x + 4 \\ xy - 5x &= 4 - 3y \\ x(y-5) &= 4 - 3y \\ x &= \frac{4-3y}{y-5} \end{aligned}$$

Thus, $f^{-1}(y) = \frac{4-3y}{y-5}$. Changing the variables y for x, we obtain that the inverse of f is

$$f^{-1}(x) = \frac{4-3x}{x-5}$$

(c) Find the domain and the range of f^{-1}.

$$\text{Domain } f^{-1} = (-\infty, 5) \cup (5, \infty)$$

$$\text{Range } f^{-1} = \text{Domain } f = (-\infty, -3) \cup (-3, \infty)$$

4. (*20 points*) Evaluate each of the following:

(a) $\sin(\frac{2\pi}{3})$

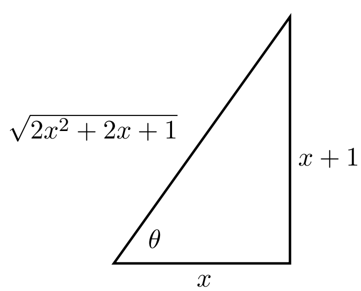
$\sin(\frac{2\pi}{3}) = \sin(\pi - \frac{2\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

(b) $\cos(\operatorname{arcsec} 4)$

If $\theta = \operatorname{arcsec} 4$, then $\sec\theta = \frac{1}{\cos\theta} = 4$. Then, $\cos\theta = \frac{1}{4}$. Thus

$$\cos(\operatorname{arcsec} 4) = \frac{1}{4}$$

(c) Assume $x > 0$. Find $\cos(\operatorname{arccot} \frac{x}{x+1})$.



If $x > 0$ and $\theta = \operatorname{arccot} \frac{x}{x+1}$ then we can compute all trigonometric functions of θ from the triangle above. In particular, $\cos\theta = \frac{x}{\sqrt{2x^2+2x+1}}$. Thus,

$$\cos(\operatorname{arccot} \frac{x}{x+1}) = \frac{x}{\sqrt{2x^2 + 2x + 1}}$$

5. (*20 points*) Find the following limits:

(a) $\lim_{x\to\infty} \frac{1}{x}\cos x =$ **0**

$$-\frac{1}{x} \leq \frac{1}{x}\cos x \leq \frac{1}{x}$$

Since $\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$, by the pinching theorem we obtain

$$\lim_{x\to\infty} \frac{1}{x}\cos x = 0$$

(b) $\lim_{x\to 2} \frac{\sin(x^2-4)}{x^2-x-2} = \mathbf{\frac{4}{3}}$

$$\lim_{x\to 2} \frac{\sin(x^2-4)}{x^2-x-2} = \lim_{x\to 2}\left(\frac{\sin(x^2-4)}{x^2-4}\cdot\frac{x^2-4}{x^2-x-2}\right) = \lim_{x\to 2}\frac{\sin(x^2-4)}{x^2-4}\cdot\lim_{x\to 2}\frac{x^2-4}{x^2-x-2}$$

$$\lim_{x\to 2}\frac{\sin(x^2-4)}{x^2-4} = 1 \qquad \lim_{x\to 2}\frac{x^2-4}{x^2-x-2} = \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x+1)} = \frac{4}{3}$$

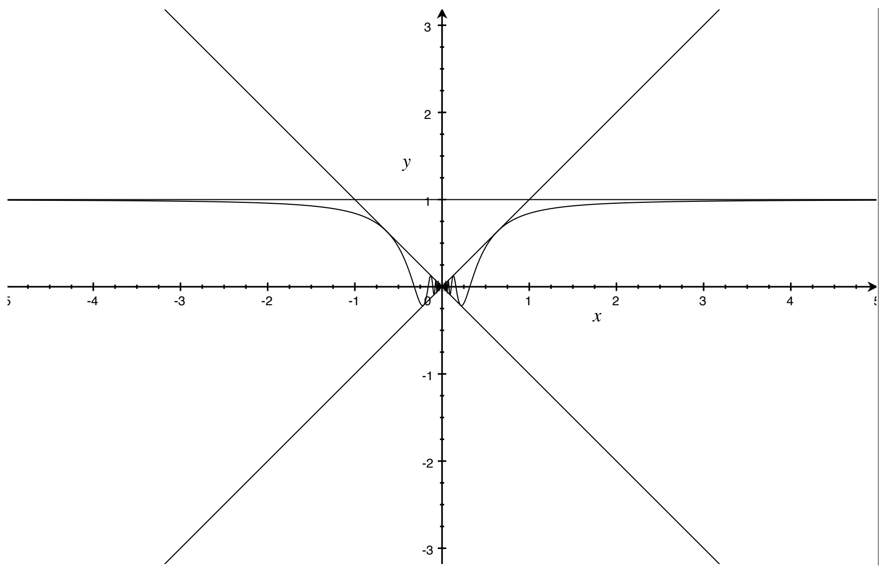
Thus

$$\lim_{x\to 2} \frac{\sin(x^2-4)}{x^2-x-2} = 1\cdot\frac{4}{3} = \frac{4}{3}$$

(c) $\lim_{x\to\infty} x\sin(\frac{1}{x}) =$ **1**

$$\lim_{x\to\infty} x\sin(\frac{1}{x}) = \lim_{x\to\infty}\frac{\sin(\frac{1}{x})}{\frac{1}{x}} = 1$$

6. (*10 points*) Sketch the graph of $f(x) = x\sin(\frac{1}{x})$



The graph of f lies between $y = -x$ and $y = x$. Moreover,

$$\lim_{x\to\infty} x\sin(\frac{1}{x}) = \lim_{x\to-\infty} x\sin(\frac{1}{x}) = 1,$$

so the graph approaches the line $y = 1$ as x goes to infinity.

7. (*10 points*) Start with the approximation $2 < \sqrt{7} < 3$. Apply the bisection method once to obtain a better approximation of $\sqrt{7}$.

Let $f(x) = x^2 - 7$. Note that $f(2) = -3 < 0$, and that $f(3) = 2 > 0$. Then $2 < \sqrt{7} < 3$. The midpoint of the interval $[2, 3]$ is $5/2$. Observe that

$$f(5/2) = \left(\frac{5}{2}\right)^2 - 7 = -\frac{3}{4} < 0.$$

Since f is continuous, the intermediate value theorem guaranties that the root $\sqrt{7}$ of f is between $5/2$ and 3:

$$5/2 < \sqrt{7} < 3$$