MATH 1300 A, Summer 2015 Name: ______________________

Midterm 2 Student number: ______________________

Solution

Show all work clearly and in order. Justify your answers whenever possible. No calculators, cell phones, head phones or books are allowed. You have 50 minutes.

1. (*15 points*) Use the definition of the derivative to find the derivative of $f(x) = x^2$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \\
&= \lim_{h\to 0} (2x + h) \\
&= 2x
\end{aligned}
$$

2. (*15 points*) Find the second derivative of $\arctan(2x^2)$

$$
(\arctan(2x^2))' = \frac{1}{1+(2x^2)^2} \cdot 4x = \frac{4x}{1+4x^4}
$$

$$
\begin{aligned}
(\arctan(2x^2))'' &= \frac{4(1+4x^4) - 4x(16x^3)}{(1+4x^4)^2} \\
&= \frac{4 + 16x^4 - 64x^4}{(1+4x^4)^2} \\
&= \frac{4 - 48x^4}{(1+4x^4)^2}
\end{aligned}
$$

3. (*15 points*) Find the equation of the normal line to the graph of the implicitly defined function $4xy = 4x^3 - xy^2 - 2y^2$ at the point $(1, -2)$.

$$4y + 4xy' = 12x^2 - (y^2 + 2xyy') - 4yy'$$

at the point $(1,-2)$ we get

$$-8 + \cancel{4y'} = 12 - 4 + \cancel{4y'} + 8y'$$

$$-16 = 8y'$$

$\boxed{-2 = y'}$ → slope $m = -2$ of the tangent line at $(1,-2)$

$\boxed{-\frac{1}{m} = \frac{1}{2}}$ → slope of the normal line at $(1,-2)$

Equation of the normal line:

$$y = y_0 - \frac{1}{m}(x - x_0)$$

$$\boxed{y = -2 + \frac{1}{2}(x-1)}$$

4. (*15 points*) Find the critical points of the function $f(x) = (x^2 + 2x - 8)^{\frac{2}{3}}$. Identify each of them as a local maximum, a local minimum or a non-local extremum.

$$f'(x) = \frac{2}{3} \cdot \frac{2x+2}{(x^2+2x-8)^{1/3}} = \frac{4}{3} \cdot \frac{x+1}{((x+4)(x-2))^{1/3}}$$

Critical points : $\boxed{x=-1}$, $\boxed{x=-4}$ and $\boxed{x=2}$

		-4		-1		2	
$x+1$	$-$		$-$		$+$		$+$
$x+4$	$-$		$+$		$+$		$+$
$x-2$	$-$		$-$		$-$		$+$
$f'(x)$	$-$		$+$		$-$		$+$
	↘		↗		↘		↗
		loc. min		loc max		loc min.	

$x = -4$ is a local min

$x = -1$ is a local max

$x = 2$ is a local min

5. (*15 points*) Sketch the graph of the function $f(x) = 3x^5 - 10x^3 + 15x - 8$.

$$
\begin{aligned}
f'(x) &= 15x^4 - 30x^2 + 15 \\
&= 15(x^4 - 2x^2 + 1) \\
&= 15(x^2-1)^2 \\
&= 15(x-1)^2(x+1)^2
\end{aligned}
$$

f': -1, 1

$++++ \quad -1 \quad ++++ \quad 1 \quad ++++$

$\nearrow \quad \nearrow \quad \nearrow$

Critical points: $x = -1$ a non local extremum

$x = 1$ a non local extremum

$$
\begin{aligned}
f''(x) &= 60x^3 - 60x \\
&= 60x(x-1)(x+1)
\end{aligned}
$$

f'': -1, 0, 1

$--- \quad -1 \quad ++++ \quad 0 \quad --- \quad 1 \quad ++++$

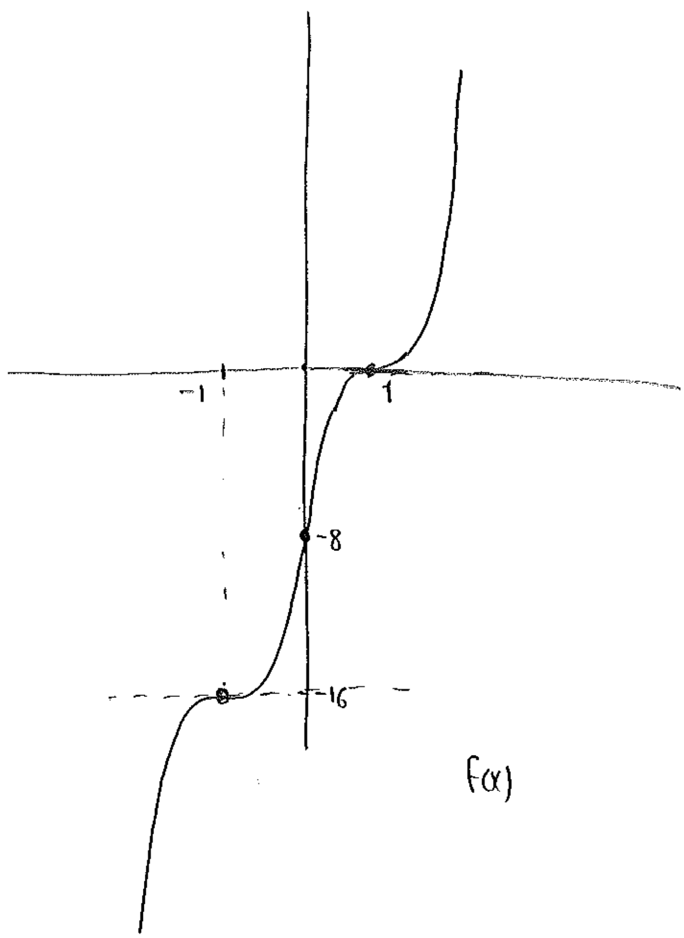
$\cap \quad \cup \quad \cap \quad \cup$

Inflection points: $x = -1$, $x = 0$, $x = 1$.

$f(0) = -8$

$f(1) = 0$

$f(-1) = -16$



6. (*15 points*) Find the following limits. You may use L'Hôpital's rule where appropriate.

(a) $\lim_{x \to 0} \frac{\sin x - x}{x^3}$ form $\frac{0}{0}$ $\rightarrow$ Apply L'Hôpital's rule.

$$= \lim_{x \to 0} \frac{\cos x - 1}{3x^2}$$ form $\frac{0}{0}$ $\rightarrow$ Apply rule again

$$= \lim_{x \to 0} \frac{-\sin x}{6x} \quad 1$$

$$= \boxed{-\frac{1}{6}}$$

(b) $\lim_{x \to 0} \frac{\tan x - x}{x^3}$ form $\frac{0}{0}$ $\rightarrow$ Apply L'Hôpital's rule.

$$= \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{\tan^2 x}{3x^2}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{3x^2 \cos^2 x} \quad 1 \quad 1$$

$$= \boxed{\frac{1}{3}}$$

7. (*15 points*) A hawk attacks a dove sitting on an egg. In the ensuing fight the egg falls to the ground. If the egg hits the ground at 49 meters per second, how high is the dove's nest?

$V(t) = -gt + V(0)$ $\quad$ $s(t) = -\frac{g}{2} t^2 + V(0)t + s(0)$ $\quad$ $V(0) = 0$ $\quad$ $s(0) = 0$

$V(t) = -gt$ $\quad$ $s(t) = -\frac{g}{2} t^2$

When the egg hits the ground:

$V(t) = -gt = -49$

$-9.8t = -49$

$\boxed{t = 5 \text{ sec}}$

$s(5) = -4.9(5)^2$

$\boxed{s(5) = -122.5 \text{ m}}$

Then, the dove's nest is 122.5 m high.