Exercise 1

Consider the poset Π of d pairwise incomparable elements, and its lattice of order ideals $\mathcal{J}(\Pi) \cong B_d$.

(i) Show that the Möbius function of the Boolean lattice B_d is given by:

$$\mu_{B_d}(A, B) = \begin{cases} (-1)^{|B \setminus A|} & \text{if } A \subseteq B \subseteq [d], \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Verify that

$$\Omega_\Pi(-n) = \mu^n_{\mathcal{J}(\Pi)}(\emptyset, \Pi).$$

Exercise 2

Consider the poset $\Pi = [k]$ and its lattice of order ideals $\mathcal{J}(\Pi) \cong [k+1]$.

(i) Show that the Möbius function of $[k+1]$ is given by:

$$\mu_{[k+1]}(A, B) = \begin{cases} (-1)^{|B \setminus A|} & \text{if } A \subseteq B \subseteq [k+1] \text{ and } |B \setminus A| \in \{0, 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Verify that

$$\Omega_\Pi(-n) = \mu^n_{\mathcal{J}(\Pi)}(\emptyset, \Pi).$$

Exercise 3

For two posets $(\Pi_1, \preceq_1)$ and $(\Pi_2, \preceq_2)$, we define their direct product as the poset with underlying set $\Pi_1 \times \Pi_2$ and partial order

$$(x_1, x_2) \preceq (y_1, y_2) \quad :\Longleftrightarrow x_1 \preceq_1 y_1 \text{ and } x_2 \preceq_2 y_2.$$

(i) Show that every interval $[(x_1, x_2), (y_1, y_2)]$ of $\Pi_1 \times \Pi_2$ is of the form $[x_1, y_1] \times [x_2, y_2]$.

(ii) Show that $\mu_{\Pi_1 \times \Pi_2}((x_1, x_2), (y_1, y_2)) = \mu_{\Pi_1}(x_1, y_1)\mu_{\Pi_2}(x_2, y_2)$.

Exercise 4

The Möbius function in number theory is the function $\mu : \mathbb{Z}_{>0} \to \mathbb{Z}$ defined as:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is not squarefree,} \\ (-1)^r & \text{if } n \text{ is the product of } r \text{ distinct primes.} \end{cases}$$

Consider the poset D_n whose elements are the divisors of n ordered by divisibility.

(i) Show that $\mu(n) = \mu_{D_n}(1, n)$.