Exercise 1

We say that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d \in \mathbb{Z}^d$ form a lattice basis of $\mathbb{Z}^d$ if every point in $\mathbb{Z}^d$ can be uniquely expressed as an integral linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$. Let $\mathbf{A}$ be the matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$. Show that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$ form a lattice basis if and only if $\det(\mathbf{A}) = \pm 1$.

Exercise 2

Let Δ be the convex hull of the origin and the d unit vectors in $\mathbb{R}^d$. Show that

$$\mathrm{ehr}_\Delta(n) = \binom{n+d}{d}.$$

More generally, show that $\mathrm{ehr}_\Delta(n) = \binom{n+d}{d}$ for every unimodular simplex Δ in $\mathbb{R}^d$.

Exercise 3

Show that a polyhedron $Q \subseteq \mathbb{R}^d$ is a polyhedral cone if and only if

$$Q = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \leq \mathbf{0}\}$$

for some matrix $\mathbf{A}$.

Exercise 4

Let $Q = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ be a nonempty polyhedron.

(i) Show that
$$\mathrm{rec}(Q) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \leq \mathbf{0}\}.$$

(ii) Infer that $\mathbf{p} + \mathrm{rec}(Q) \subseteq Q$ for all $\mathbf{p} \in Q$.

(iii) Show that Q is bounded if and only if $\mathrm{rec}(Q) = \{\mathbf{0}\}$.

Exercise 5

Let $Q = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ be a nonempty polyhedron.

(i) Show that
$$\mathrm{lineal}(Q) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} = \mathbf{0}\}.$$

(ii) Infer that $\mathbf{p} + \mathrm{lineal}(Q) \subseteq Q$ for all $\mathbf{p} \in Q$.