Exercise 1

We say that a finite poset Π is *graded* if every maximal chain in Π has the same length r, which we call the *rank* of Π. The *length* $\ell_\Pi(x,y)$ of two elements $x \preceq y$ in Π is the length of a maximal chain in $[x,y]$. A graded poset that has a minimal element $\hat{0}$ and maximal element $\hat{1}$ is *Eulerian* if its Möbius function for $x \preceq y$ is

$$\mu_\Pi(x,y) = (-1)^{\ell_\Pi(x,y)}.$$

The zeta polynomial of Π is defined as

$$Z_\Pi(n) := \zeta^n(\hat{0},\hat{1}).$$

Show that if Π is a finite Eulerian poset of rank r then

$$Z_\Pi(-n) = (-1)^r Z_\Pi(n).$$

Exercise 2

Show that the face lattice $\Phi = \Phi(P)$ of a polytope P is an Eulerian poset of rank $\dim P$ and

$$\ell_\Phi(F,G) = \dim G - \dim F$$

for any two faces $F \preceq G$ in Φ.

Exercise 3

Let P be a d-dimensional polytope and $\Phi(P)$ be its face lattice. We define

$$\Delta Z_{\Phi(P)}(n) := Z_{\Phi(P)}(n+1) - Z_{\Phi(P)}(n).$$

(i) Show that $\Delta Z_{\Phi(P)}(n)$ equal to the number of multichains

$$\emptyset = F_0 \preceq F_1 \preceq \cdots \preceq F_n \prec P.$$

(ii) Show that the combinatorial reciprocity for Eulerian posets implies that

$$(-1)^d \Delta Z_{\Phi(P)}(-n) = \Delta Z_{\Phi(P)}(n-1).$$

(iii) Show that if P is simplicial, that is all of its proper faces are simplices, then

$$\Delta Z_{\Phi(P)}(n) = 1 + \sum_{\emptyset \prec F \prec P} n^{\dim(F)+1} = \sum_{k=0}^{d} f_{k-1}(P) n^k,$$

where $f_i(P)$ is the number of faces of i-th dimensional faces of P for $i \geq 0$, and we set $f_{-1}(P) = 1$ accounting for the empty face.

(iv) Deduce that for every simplicial polytope P and $0 \leq j \leq d$

$$f_{j-1}(P) = \sum_{k=j}^{d} (-1)^{d-k} \binom{k}{j} f_{k-1}(P).$$

These linear relations are known as the Dehn-Sommerville relations. The case $j = 0$ recovers the Euler-Poincaré formula.

Exercise 4

Let P be the d-dimensional *cross polytope*

$$P = \{\mathbf{x} \in \mathbb{R}^d : |x_1| + \dots |x_d| \leq 1\}.$$

If $1 \leq j \leq d$, show that $f_{j-1}(P) = \binom{d}{j} 2^j$. Verify that the Dehn-Sommerville relations are satisfied.