Exercise 1

Show that for integers $m \geq k \geq 0$,

$$\sum_{n \geq k} \binom{n+m-k}{m} z^n = \frac{z^k}{(1-z)^{m+1}}.$$

Exercise 2

Find a formula for the generating function $F(z) = \sum_{n \geq 0} f(n)$ of the following sequences:

(i) $f(n) = 3n + 1$.

(ii) $f(n) = \binom{n}{1} + 4\binom{n}{2} + \binom{n}{3}$.

(iii) The tribonacci sequence $0, 0, 1, 1, 2, 4, 7, 13, 24, \ldots$ determined by the recurrence

$$f(n+3) = f(n+2) + f(n+1) + f(n),$$

with initial values $f(0) = f(1) = 0$ and $f(2) = 1$.

Exercise 3

For each sequence in Exercise 2, find a formula for the generating function

$$F^\circ(z) := \sum_{n \geq 1} f^\circ(n) z^n,$$

where $f^\circ(n) := f(-n)$. Note that the sum starts at $n = 1$.

Exercise 4

Let $(f(n))_{\geq 0}$ be a sequence with initial values $f(0), f(1), \ldots, f(d-1)$, such that for every $n \geq 0$ it satisfies the linear recurrence

$$c_0 f(n+d) + c_1 f(n+d-1) + \cdots + c_d f(n) = 0,$$

for some $c_0, \ldots, c_d \in \mathbb{C}$ with $c_0, c_d \neq 0$.

(i) Show that

$$F(z) = \sum_{n \geq 0} f(n) = \frac{p(z)}{c_0 + c_1 z + \cdots + c_d z^d},$$

for some polynomial $p(z)$ of degree $< d$.

(ii) We can run the recurrence backwards and define $f^\circ(n) = f(-n)$ for $n \geq 1$. Show that

$$F^\circ(z) := \sum_{n \geq 1} f^\circ(n) z^n = -\frac{z^d p(\frac{1}{z})}{c_0 z^d + c_1 z^{d-1} + \cdots + c_d}.$$