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Combinatorial Reciprocity Theorems via Geometry
Exercise Sheet 11 (Potential Research Project)

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Exercise 1

Let $\mathbb{Q}[\mathbf{x}] := \mathbb{Q}[x_1, \dots, x_n]$ be the polynomial ring in n variables. A polynomial $f \in \mathbb{Q}[\mathbf{x}]$ is called $\mathfrak{S}_n$-invariant if $f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n)$ for every permutation $\sigma \in \mathfrak{S}_n$. Let I be the ideal generated by $\mathfrak{S}_n$-invariant polynomials with no constant term. Show that the quotient

$$\mathbb{Q}[\mathbf{x}]/I$$

is a finite dimensional space of dimension $n!$.

Exercise 2

Let $\mathbb{Q}[\mathbf{x}, \mathbf{y}] := \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring in two sets of n variables. The symmetric group $\mathfrak{S}_n$ acts “diagonally” in $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ via

$$(\sigma f)(\mathbf{x}, \mathbf{y}) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}, y_{\sigma(1)}, \dots, y_{\sigma(n)}).$$

Let I be the ideal generated by $\mathfrak{S}_n$-invariant polynomials with no constant term. Show that the quotient

$$\mathbb{Q}[\mathbf{x}, \mathbf{y}]/I$$

is a finite dimensional space of dimension $(n+1)^{(n-1)}$, the number of parking functions.

Exercise 3

Let $\mathbb{Q}[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r)}]$ be the polynomial ring in r sets of n variables, where $\mathbf{x}^i = [x_1^{(i)}, \dots, x_n^{(i)}]$. The symmetric group $\mathfrak{S}_n$ acts “diagonally” in $\mathbb{Q}[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r)}]$ similarly as above. Let I be the ideal generated by $\mathfrak{S}_n$-invariant polynomials with no constant term. Show that for a fixed n and arbitrary number of sets of variables r, the dimension

$$D_n(r) := \dim\left(\mathbb{Q}[\mathbf{x}, \mathbf{y}]/I\right)$$

is a polynomial in r.

Here are the first values of these polynomials for $n = 1, 2, 3, 4$:

$$D_1(r) = 1$$

$$D_2(r) = \binom{r+1}{1}$$

$$D_3(r) = \binom{r+1}{1} + 4\binom{r+1}{2} + \binom{r+1}{3}$$

$$D_4(r) = \binom{r+1}{1} + 22\binom{r+1}{2} + 56\binom{r+1}{3} + 40\binom{r+1}{4} + 11\binom{r+1}{5} + \binom{r+1}{6}$$

You are invited to double check that $D_n(1) = n!$ and $D_n(2) = (n+1)^{n-1}$ for these four polynomials.

Exercise 4

Open Problem by François Bergeron: Show that

$$D_n(-2) = (-1)^{n-1} C_{n-1},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.